Green's for Bloid by the non-cone method, v2 PhL 3.7.10

See comments at the end of this document. In Part I we look back at the previous non-cone-method attempt at the bloid Green's problem, and we see that the pillbox in the lower region forces $V \equiv 0$ in the entire lower region, so the Smythian form is invalid. In Part II we try to "repair" this problem with a more general "four region" Smythian form, but it leads to intractable non-diagonalizable equations for the coefficients assumed in the Smythian form. The last section has some general comments.

PART I: REVIEW OF MY PREVIOUS NON-CONE METHOD ATTEMPT. 1

1. The Picture..... 1

2. Smythian form..... 2

3. The Pillbox in the upper region. 2

3. The Pillbox in the lower region. 3

PART II: A FOUR-REGION NON-CONE METHOD ATTEMPT. 3

1. The Four Region Model..... 3

2. The ζ Direction Pillbox Upper Bloid Region. 4

Note added next day: 7

3. The ζ Direction Pillbox Lower Bloid Region. 7

4. The ξ Direction Pillbox Upper Bloid Region. 8

5. The ξ Direction Pillbox Lower Bloid Region. 9

6. Conditions at the Neck..... 10

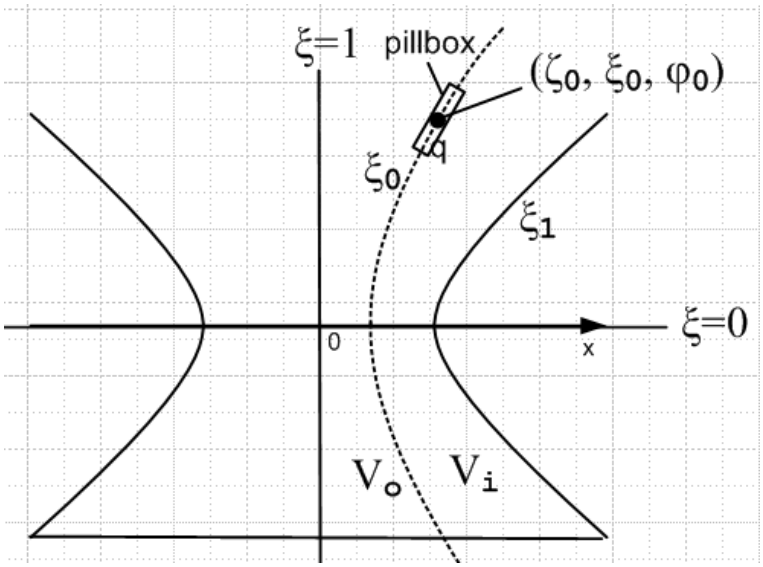
(a) continuity at the neck:..... 10

(b) slope continuity at the neck: 12

7. Comments:..... 13

PART I: Review of my previous non-cone method attempt.

1. The Picture. Let's just take another look at this method. Here is our picture:



If we think of the upper part of the dotted line as $\xi_0 > 0$ and the lower part as $\xi_0 < 0$ (which is not the usual choice I make) then each spheroid has a single $\zeta > 0$ label. The reason I would make this choice is only this: we now have ξ in the full range (-1,1) and we then use the ξ system to quantize n to be the usual integers n = 0,1,2. since that is one of the result I want to see.

2. Smythian form. What is our most general allowed form here? If we require a form that is continuous in either of our two regions, then we are really forced to the $Q_n^m(j\zeta)$ function since both regions go off to infinity. This produces then the following form (let's say just for the upper region)

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m(\xi) Q_n^m(j\zeta) e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0) Q_n^m(j\zeta) e^{im\varphi}$$

or

$$V(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_<) - Q_n^m(\xi_1) P_n^m(\xi_<)] P_n^m(\xi_>) Q_n^m(j\zeta) e^{im\varphi}$$

The V_i form when we set $\xi = \xi_1$ is manifestly Vi = 0. And when we set $\xi = \xi_0$, $V_i = V_0$. So this is the ample motivation for this form. For the moment, we shall just ignore the lower region. If I were starting over on this problem, I would probably use the cos(mφ) form instead of the $e^{im\varphi}$ form, but we can fix that later if we find any kind of success here.

3. The Pillbox in the upper region. You can read the details in the v1 doc. We first get this equation

$$(q/\varepsilon) (h_1/h_3h_2) \delta(\varphi-\varphi_0) \delta(\zeta-\zeta_0) = (1-\xi_0^2)^{-1} \Sigma_{nm} (-1)^m [(n+m)! / (n-m)!] E_{nm} Q_n^m(j\zeta) P_n^m(\xi_1) e^{im\varphi}$$

and then finally

$$(q/\varepsilon) (h_1/h_3h_2) e^{-im\varphi 0} \delta(\zeta-\zeta_0) = [2\pi / (1-\xi_0^2)] \Sigma_n (-1)^m [(n+m)! / (n-m)!] E_{nm} P_n^m(\xi_1) Q_n^m(j\zeta)$$

We are then faced with the major problem of how to extract the E_{nm} coefficients. If we think of the n-spectrum as being integers and being determined by the ξ dimension, then we are pretty sure there is no

simple Q function orthogonality that we just don't know about, for n and m both integers. So we are saying here that the ξ subsystem is our SL problem, not the ζ subsystem. We have m and n quantized, that so the ζ subsystem has to just "go along for the ride" (my emitter follower comment in other docs).

So this is where my effort using this method ground to a halt. One could try, as I did, to have the ζ subsystem somehow determine the spectrum, and maybe use the nicer parameterization method so that we don't hit the value $\xi = -1$. I think this method does actually give values of n that are integral. But let's put this on hold for a moment, and move to the next section which I probably never did before.

3. The Pillbox in the lower region. If we assume the same form, but perhaps with F_{nm} coefficients, this lower pillbox will say that: (just set q = 0)

$$0 = \Sigma_{nm} (-1)^m [(n+m)! / (n-m)!] F_{nm} Q_n^m(j\zeta) P_n^m(\xi_1) e^{im\varphi}$$

Since the $e^{im\varphi}$ are a complete set, we somehow must have the following for each m:

$$0 = \Sigma_n (-1)^m f(n,m) F_{nm} Q_n^m(j\zeta) P_n^m(\xi_1)$$

We could just redefine the coefficients to be F' and then we have

$$0 = \Sigma_n F'_{nm} Q_n^m(j\zeta)$$

The F' cannot be functions of ζ, they are constants. The only solution here is $F_{nm} = 0$ IMHO so we then end up with V = 0 in the entire lower region which we know is wrong.

This is just a typical indication that our Smythian form is "not general enough" to handle this problem.

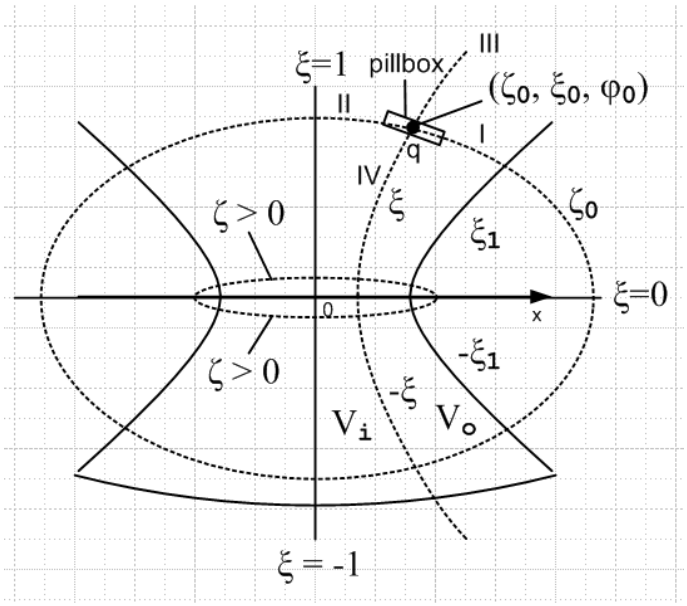
PART II: A four-region non-cone method attempt.

1. The Four Region Model. I can just tell from my experience now that the above Smythian form is wrong because it involves only the $Q_n^m(j\zeta)$ function, and I know it has to involved the P as well for the close-in solution. So we could draw in the dotted spheroid and break things into four regions. The two outer regions would be what we show above (but I now use the p and q functions in ζ and switch to the cosine φ form, and I show this result just for the upper region, and $\varphi_0 = 0$)

upper region: [later I added A_{nm} in front of $Q_n^m(\xi_1)$]

$$V_{o,out}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi_0)–Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi) [B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta) \cos(m\varphi)$$
$$V_{o,int}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi_0)–Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi) [B_{nm}p_n^m(\zeta)+C_{nm}q_n^m(\zeta)] q_n^m(\zeta_0) \cos(m\varphi)$$

$$V_{i,out}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi)–Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) [B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta) \cos(m\varphi)$$
$$V_{i,int}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi)–Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) [B_{nm}p_n^m(\zeta)+C_{nm}q_n^m(\zeta)] q_n^m(\zeta_0) \cos(m\varphi)$$



I pondered this four-region form back when I was doing this, but it seemed just too messy, and I was eager to do the cone method. But I think one might take this through to a successful conclusion. There are now really four dotted boundaries (all in the upper bloid region) across which you need to match, and I label each with a Roman numeral. The full words out and in are with respect to the spheroid, so that "out" always has a pure $q_n^m(\zeta)$ function to get decay. Here are the region matches:

I	$V_{o,in}(\zeta_0,\xi,\varphi) = V_{o,out}(\zeta_0,\xi,\varphi)$	manifestly met
II	$V_{i,in}(\zeta_0,\xi,\varphi) = V_{i,out}(\zeta_0,\xi,\varphi)$	manifestly met
III	$V_{i,out}(\zeta,\xi_0,\varphi) = V_{o,out}(\zeta,\xi_0,\varphi)$	manifestly met
IV	$V_{i,in}(\zeta,\xi_0,\varphi) = V_{o,in}(\zeta,\xi_0,\varphi)$	manifestly met

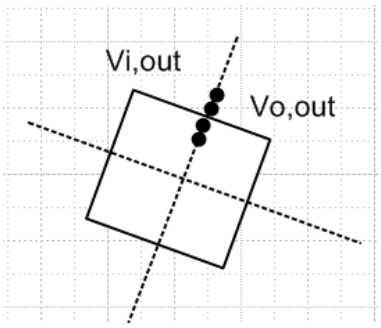
Things are pretty complicated now because we have now to consider:

- pillbox conditions in two directions, and this for both upper and lower bloids
- various neck conditions for the two "int" potentials

The idea here is that we assume the lower bloid has primed B and C coefficients to start with and we just see where this all takes us. Notice that the picture assumes the $\zeta > 0$ parameterization so we have the full range of ξ and we get n = integers from the ξ SL problem.

The pillbox is now at the intersection of four regions, so more attention is needed! This is something completely new, I have only done 2-region pillboxes before. So this will be an adventure should I choose to go forward.

2. The ζ Direction Pillbox Upper Bloid Region. Consider this picture which shows a blow-up of the pillbox in the picture above.



Suppose we are doing a pillbox condition relative to the ζ_0 near-horizontal boundary. We want the derivative ∂_ζ of the potential just above box center. Since both functions have the exact same value at all black dots, it seems to me that the ζ derivative would be the same if you used one function or the other to compute it.

Side question: in general, how do we know that we have continuity of the perp derivative at any of these four boundaries? Well, this is exactly what we require when we state our pillbox condition. We have something like $\delta(...)$ = expression, and away from the Green's charge, $\delta = 0$ so LHS = 0.

So I pick $V_{i,out}$ arbitrarily and find that

$$V_{i,out}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi)-Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) [B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta) \cos(m\varphi)$$
$$\partial_\zeta V_{i,out}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi)-Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) [B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta)' \cos(m\varphi)$$

Then on the opposite edge, I again pick the "i" function and say

$$V_{i,int}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi)-Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) [B_{nm}p_n^m(\zeta)+C_{nm}q_n^m(\zeta)] q_n^m(\zeta_0) \cos(m\varphi)$$
$$\partial_\zeta V_{i,int}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi)-Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) [B_{nm}p_n^m(\zeta)'+C_{nm}q_n^m(\zeta)'] q_n^m(\zeta_0) \cos(m\varphi)$$

As usual we construct the difference, which was for the spheroid called $\partial_\zeta V_i(\zeta,\xi) - \partial_\zeta V_o(\zeta,\xi)$, but here (we put the "int or i " first)

$$\partial_\zeta V_{i,int}(\zeta,\xi,\varphi) - \partial_\zeta V_{i,out}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi)-Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) \cos(m\varphi) *$$

$$\{ [B_{nm}p_n^m(\zeta)'+C_{nm}q_n^m(\zeta)'] q_n^m(\zeta_0) - [B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta)' \}$$

We then set $\zeta = \zeta_0$ and, again as usual, the C_{nm} terms cancel and we get only the B_{nm} stuff:

$$= - \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi)-Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) \cos(m\varphi) B_{nm} W[p_n^m(\zeta_0), q_n^m(\zeta_0)]$$

$$= - \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi)-Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) \cos(m\varphi) B_{nm} W[p_n^m(\zeta_0), q_n^m(\zeta_0)]$$

And we call upon

$$W[p_n^m(\zeta), q_n^m(\zeta)] = \pm 2\cos[(n+m)\pi/2]f(n,m) * 1/(1+\zeta^2)$$

where we use the + sign since in the upper bloid region we have $\zeta > 0$, so we get

$\partial_\zeta V_{i,int}(\zeta,\xi,\varphi) - \partial_\zeta V_{i,out}(\zeta,\xi,\varphi) \quad =$

$$- \; 2/(1+\zeta^2)\, \Sigma_{nm}\, B_{nm} \cos[(n+m)\pi/2] f(n,m)\, [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0) \cos(m\varphi)$$

Then we go on to claim that, as in our oblate spheroid doc, where really ζ is ζ_0 since that is where the pillbox is located,

$$\partial_\zeta V_{i,int}(\zeta,\xi,\varphi) - \partial_\zeta V_{i,out}(\zeta,\xi,\varphi) \quad = \quad q/[\varepsilon c_1(1+\zeta^2)] \quad \delta(\varphi-\varphi_0)\, \delta(\xi - \xi_0)$$

Comparing, we get our standard type result:

$$q/[2\varepsilon c_1] \;\; \delta(\varphi-\varphi_0)\, \delta(\xi - \xi_0) \;\; =$$

$$- \; \Sigma_{nm}\, B_{nm} \cos[(n+m)\pi/2] f(n,m)\, [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0) \cos(m\varphi)$$

Not all the cosine factors vanish, so we at least have something non-zero on the right. We then use our usual result which says

$$\int_0^{2\pi} d\varphi \cos(m'\varphi) \cos(m\varphi) \;\; = \delta_{m,m'} 2\pi/(2-\delta_{m,0})$$

and we apply this to both sides to get (and $\varphi_0 = 0$)

$$q/[2\varepsilon c_1]\, \delta(\xi - \xi_0) \;\; =$$
$$- \; 2\pi/(2-\delta_{m\ ,0}) \Sigma_n\, B_{nm} \cos[(n+m)\pi/2] f(n,m)\, [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0)$$

$$(2-\delta_{m,0}) q/[4\pi\varepsilon c_1]\, \delta(\xi - \xi_0) \;\; = -\Sigma_n\, B_{nm} \cos[(n+m)\pi/2] f(n,m)\, [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0)$$

Now we are "stuck" because to clear out the δ on the left, we have to worry about the $Q(\xi)$ on the right which does not fit into our orthogonality plan.

So we can now go back and make the "other" choice and choose "o" instead of "i", then our result above comes out being this friendlier result: (I have here added A_{nm} as described below)

$$(2-\delta_{m,0}) q/[4\pi\varepsilon c_1]\, \delta(\xi - \xi_0) \;\; = -\Sigma_n\, B_{nm} \cos[(n+m)\pi/2] f(n,m)\, [P_n^m(\xi_1) Q_n^m(\xi_0) - A_{nm} Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m(\xi)$$

It is friendlier because we can now apply [this is where we need ξ in full range (-1,1)]

$$\int_{-1}^{1} d\xi\, P_n^m(\xi) P_{n'}^m(\xi) \;\; = \; 2\delta_{n,n'} \;\; (2n+1)^{-1}\, (n+m)! \,/\, (n-m)! \qquad\qquad n,k = m,\, m+1,\, m+2 \;\ldots\ldots\; \infty$$

$$= 2\delta_{n,n'} \;\; (2n+1)^{-1}\, f(n,m)$$

This gives us:

$$(2-\delta_{m,0}) q/[4\pi\varepsilon c_1]\, P_n^m(\xi_0) \;\; =$$

$$-2\,(2n+1)^{-1}B_{nm}\cos[(n+m)\pi/2]\,f(n,m)^2\,[P_n^m(\xi_1)Q_n^m(\xi_0)-A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)]$$

so we can regard this as telling us something about this combination:

$$B_{nm}\,[P_n^m(\xi_1)Q_n^m(\xi_0)-A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)] \;=\; \text{whatever}$$

Comment: it certainly is dubious that one choice in the pillbox let's you continue and one does not.

Note added next day: I now think you really should exactly average the "i" and "o" results to get a more accurate pillbox application. In this case we would replace the above result with this:

$$(2-\delta_{m,0})q/[4\pi\varepsilon c_1]\;\delta(\xi-\xi_0) \;=\; -\Sigma_n\,B_{nm}\cos[(n+m)\pi/2]f(n,m)\;*$$

$$(1/2)\;\{\;[P_n^m(\xi_1)Q_n^m(\xi_0)-A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi)\;+(1/2)\;[P_n^m(\xi_1)Q_n^m(\xi)-A_{nm}Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0)$$

$$=\;-\Sigma_n\,B_{nm}\cos[(n+m)\pi/2]f(n,m)\;*$$

$$\{\;(1/2)P_n^m(\xi_1)\;[\;Q_n^m(\xi_0)\;P_n^m(\xi)+Q_n^m(\xi)\;P_n^m(\xi_0)]\;-\;A_{nm}Q_n^m(\xi_1)\;P_n^m(\xi_0)\;P_n^m(\xi)\;\}$$

$$=\;-\Sigma_n\,B_{nm}\cos[(n+m)\pi/2]f(n,m)\;*$$

$$\{\;\;[(1/2)P_n^m(\xi_1)Q_n^m(\xi_0)-A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)]\;P_n^m(\xi)\;+[(1/2)P_n^m(\xi_1)\;P_n^m(\xi_0)]Q_n^m(\xi)\;\}$$

Not surprisingly, this has a $P_n^m(\xi)$ and a $Q_n^m(\xi)$ component. This latter component "defies" the use of orthogonality to remove the second term. If we attempt our usual orthogonality method, we end up with this integral on the RHS of {}, where all are on-cut functions,

$$\int_{-1}^{1} d\xi\;Q_n^m(\xi)P_{n'}^m(\xi) \;=\; (-1)^m\;[\;1-(-1)^{n+n'}(n+m)!\;]\;/\;[\;(n'-n)(n'+n+1)(n-m)!\;]$$ GR p 794

This is not a $\delta_{n,n'}$ thing so it does not remove the n sum. On the other hand, it does seem to have its largest value when n = n', which value I think is ∞. So even in our "simple" pillbox direction, we get ugly stuff and we cannot deduce the B_{nm} coefficients.

3. The ζ Direction Pillbox Lower Bloid Region.

First, we need an overall model for the potential in the lower region. I will go with this model. Here I show the actual negative values for ξ_0 and ξ_1 , but I leave the ξ argument as is. Also, all three constant sets are assumed primed.

lower region: ξ in (0,-1)

$$V_{o,out} = \Sigma_{nm}\,[P_n^m(-\xi_1)Q_n^m(-\xi_0)-A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi_0)]P_n^m(\xi)\;[B'_{nm}p_n^m(\zeta_0)+C'_{nm}q_n^m(\zeta_0)]\;q_n^m(\zeta)\cos(m\varphi)$$

$$V_{o,int} = \Sigma_{nm}\,[P_n^m(-\xi_1)Q_n^m(-\xi_0)-A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi_0)]\;P_n^m(\xi)\;[B_{nm}p_n^m(\zeta)+C_{nm}q_n^m(\zeta)]\;\;q_n^m(\zeta_0)\cos(m\varphi)$$

$V_{i,out} = \Sigma_{nm} [P_n^m(-\xi_1)Q_n^m(\xi) - A'_{nm}Q_n^m(-\xi_1)P_n^m(\xi)]P_n^m(-\xi_0) [B'_{nm}p_n^m(\zeta_0)+C'_{nm}q_n^m(\zeta_0)] q_n^m(\zeta) \cos(m\varphi)$
$V_{i,int} = \Sigma_{nm} [P_n^m(-\xi_1)Q_n^m(\xi) - A'_{nm}Q_n^m(-\xi_1)P_n^m(\xi)]P_n^m(-\xi_0) [B'_{nm}p_n^m(\zeta)+C'_{nm}q_n^m(\zeta)] q_n^m(\zeta_0) \cos(m\varphi)$

Just applying the same logic as above seems to tell us that

$(2-\delta_{m,0})q/[4\pi\varepsilon c_1] P_n^m(-\xi_0) =$

$$-2 (2n+1)^{-1}B'_{nm} \cos[(n+m)\pi/2] f(n,m)^2 [P_n^m(-\xi_1)Q_n^m(-\xi_0) - A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi_0)]$$

But of course there is no Green's charge in the lower region, so set q = 0 to get

$$0 = B'_{nm} \cos[(n+m)\pi/2] f(n,m)^2 [P_n^m(-\xi_1)Q_n^m(-\xi_0) - A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi_0)]$$

At least we are "still alive" with this condition imposed. Two easy ways to satisfy it, and we suspect that the correct way is $[P_n^m(-\xi_1)Q_n^m(-\xi_0) - A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi_0)] = 0$ which then sets the A_{nm}' coefficients.

4. The ξ Direction Pillbox Upper Bloid Region.

Here I will try to make the right choice at the start. I choose the "out" forms, so we have

$V_{i,out}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi) - Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) [B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta) \cos(m\varphi)$
$\partial_\xi V_{i,out}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi)' - Q_n^m(\xi_1)P_n^m(\xi)']P_n^m(\xi_0) [B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta) \cos(m\varphi)$

$V_{o,out}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi_0) - Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi) [B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta) \cos(m\varphi)$
$\partial_\xi V_{o,out}(\zeta,\xi,\varphi) = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi_0) - Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi)' [B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta) \cos(m\varphi)$

As usual we construct the difference, which was for the bloid was called $\partial_\xi V_i(\zeta,\xi) - \partial_\xi V_o(\zeta,\xi)$,

$\partial_\xi V_{i,out}(\zeta,\xi,\varphi) - \partial_\xi V_{o,out}(\zeta,\xi,\varphi) = \Sigma_{nm}[B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta) \cos(m\varphi) *$

$$\{[P_n^m(\xi_1)Q_n^m(\xi)' - Q_n^m(\xi_1)P_n^m(\xi)']P_n^m(\xi_0) - [P_n^m(\xi_1)Q_n^m(\xi_0) - Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi)'\}$$

At this point, I shall now expose a new set of coefficients that has been invisible in all the work shown above. We want to make this replacement globally above:

$[P_n^m(\xi_1)Q_n^m(\xi) - Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0) \rightarrow [P_n^m(\xi_1)Q_n^m(\xi) - A_{nm}Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0)$
$[P_n^m(\xi_1)Q_n^m(\xi_0) - Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi) \rightarrow [P_n^m(\xi_1)Q_n^m(\xi_0) - A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi)$

I am hoping this has no effect on anything above except we need to insert the A_{nm} in our computation of the B_{nm} at the end of the previous section. We could add a 4th set of coefficients, but that can be absorbed into the other three sets. You see that I have chosen A_{nm} to go with the term that cancels out below. So our result above is now rewritten as:

$\partial_\xi V_{i,out}(\zeta,\xi,\varphi) - \partial_\xi V_{o,out}(\zeta,\xi,\varphi) = \Sigma_{nm}[B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)] q_n^m(\zeta) \cos(m\varphi) *$

$$\{[P_n^m(\xi_1)Q_n^m(\xi)'-A_{nm}Q_n^m(\xi_1)P_n^m(\xi)']P_n^m(\xi_0) - [P_n^m(\xi_1)Q_n^m(\xi_0)- A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi)'\}$$

We then set $\xi = \xi_0$ and, again as usual, the A_{nm} terms cancel and we get

$$= + \Sigma_{nm}[B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)]\, q_n^m(\zeta)\cos(m\varphi)\, P_n^m(\xi_1)\, W[P_n^m(\xi_0), Q_n^m(\xi_0)]$$

We now use Bateman's on-cut Wronskian

$$W[P_n^m(\xi_0)_{cut}, Q_n^m(\xi_0)_{cut}] = (1-\xi_0^2)^{-1}\,[(n+m)!/(n-m)!] = (1-\xi_0^2)^{-1}\, f(n,m)$$

and we then arrive here:

$$\partial_\xi V_{i,out}(\zeta,\xi,\varphi) - \partial_\xi V_{o,out}(\zeta,\xi,\varphi) =$$

$$+ (1-\xi_0^2)^{-1}\,\Sigma_{nm}[B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)]\, q_n^m(\zeta)\cos(m\varphi)\, P_n^m(\xi_1)\, f(n,m)$$

Then we use

$$\partial_\xi V_i - \partial_\xi V_o = q/[\varepsilon c_1 \quad (1-\xi^2)]\,\delta(\varphi-\varphi_0)\,\delta(\zeta-\zeta_0)$$

stolen from another doc and we equate to find that

$$q/[\varepsilon c_1(1-\xi^2)]\,\delta(\varphi-\varphi_0)\,\delta(\zeta-\zeta_0) =$$

$$+ (1-\xi_0^2)^{-1}\,\Sigma_{nm}[B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)]\, q_n^m(\zeta)\cos(m\varphi)\, P_n^m(\xi_1)\, f(n,m)$$

which becomes

$$q/[\varepsilon c_1]\,\delta(\varphi-\varphi_0)\,\delta(\zeta-\zeta_0) = + \Sigma_{nm}[B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)]\, q_n^m(\zeta)\cos(m\varphi)\, P_n^m(\xi_1)\, f(n,m)$$

The same φ stuff gives us then

$$(2-\delta_{m,0})q/[4\pi\varepsilon c_1]\,\delta(\zeta-\zeta_0) = + \Sigma_n[B_{nm}p_n^m(\zeta_0)+C_{nm}q_n^m(\zeta_0)]\, q_n^m(\zeta)\, P_n^m(\xi_1)\, f(n,m)$$

and once again, we face or " Q orthogonality problem". Before worrying about this, we continue.

5. The ξ Direction Pillbox Lower Bloid Region.

If we had a Green's charge down south, we would have (one minus sign added)

$$(2-\delta_{m,0})q/[4\pi\varepsilon c_1]\,\delta(\zeta-\zeta_0) = + \Sigma_n[B'_{nm}p_n^m(\zeta_0)+C'_m q_n^m(\zeta_0)]\, q_n^m(\zeta)\, P_n^m(-\xi_1)\, f(n,m)$$

but now we set q = 0 and this becomes

$$0 = \Sigma_n[B'_{nm}p_n^m(\zeta_0)+C'_{nm}q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(-\xi_1)\ f(n,m)$$

This must be true on any spheroid in the lower region, that is, for any value of ζ, so we would conclude that this must be true:

$$0 = [B'_{nm}p_n^m(\zeta_0)+C'_{nm}q_n^m(\zeta_0)]$$

6. Conditions at the Neck.

(a) continuity at the neck:

Normally I do these first and then the pillbox, but the pillbox was "something new" so I did it first instead. We seem to "still be alive" with our Smythe form through both pillboxes up and down, but now we may have sudden death. Our functions of interest are these for the neck.

upper: ξ in (0,1) ζ > 0

$$V_{o,int} = \Sigma_{nm}\ [P_n^m(\xi_1)Q_n^m(\xi_0)-A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi)\ [B_{nm}p_n^m(\zeta)+C_{nm}q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\cos(m\varphi)$$
$$V_{i,int} = \Sigma_{nm}\ [P_n^m(\xi_1)Q_n^m(\xi)-A_{nm}Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0)\ [B_{nm}p_n^m(\zeta)+C_{nm}q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\cos(m\varphi)$$

lower: ξ in (0,-1) ζ > 0

$$V_{o,int} = \Sigma_{nm}\ [P_n^m(-\xi_1)Q_n^m(-\xi_0)-A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi_0)]P_n^m(\xi)[B'_{nm}p_n^m(\zeta)+C'_{nm}q_n^m(\zeta)]q_n^m(\zeta_0)\cos(m\varphi)$$
$$V_{i,int} = \Sigma_{nm}\ [P_n^m(-\xi_1)Q_n^m(\xi)-A'_{nm}Q_n^m(-\xi_1)P_n^m(\xi)]P_n^m(-\xi_0)[B'_{nm}p_n^m(\zeta)+C'_{nm}q_n^m(\zeta)]\ q_n^m(\zeta_0)\cos(m\varphi)$$

We break these into the two pairs of interest, where the first line in each pair is the "upper".

"i" :

$$V_{i,int} = \Sigma_{nm}\ [P_n^m(\xi_1)Q_n^m(\xi)-A_{nm}Q_n^m(\xi_1)P_n^m(\xi)]P_n^m(\xi_0)\ [B_{nm}p_n^m(\zeta)+C_{nm}q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\cos(m\varphi)$$
$$V_{i,int} = \Sigma_{nm}\ [P_n^m(-\xi_1)Q_n^m(\xi)-A'_{nm}Q_n^m(-\xi_1)P_n^m(\xi)]P_n^m(-\xi_0)[B'_{nm}p_n^m(\zeta)+C'_{nm}q_n^m(\zeta)]\ q_n^m(\zeta_0)\cos(m\varphi)$$

"o":

$$V_{o,int} = \Sigma_{nm}\ [P_n^m(\xi_1)Q_n^m(\xi_0)-A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)]\quad P_n^m(\xi)\ [B_{nm}p_n^m(\zeta)+C_{nm}q_n^m(\zeta)]q_n^m(\zeta_0)\cos(m\varphi)$$
$$V_{o,int} = \Sigma_{nm}\ [P_n^m(-\xi_1)Q_n^m(-\xi_0)-A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi_0)]P_n^m(\xi)[B'_{nm}p_n^m(\zeta)+C'_{nm}q_n^m(\zeta)]q_n^m(\zeta_0)\cos(m\varphi)$$

Regular continuity of the potential at ζ = 0 ? The p and q do not vanish at ζ = 0,

$$p_n^m(0) = \ 2^{m+1}\sqrt{\pi}\ \ /\ [\ \Gamma(1/2 - n/2 - m/2)\ \Gamma(1 + n/2 - m/2)\ ]$$
$$q_n^m(0)\ = (-1)^m\ \sqrt{\pi}\ 2^{m-1}\ [\ \Gamma(1/2 + n/2+ m/2)\ /\ \Gamma(1 + n/2- m/2)]$$

Now to allow comparison on the two sides of the neck, we for the moment set ξ → - ξ in the "lower" forms so for the moment everything is ξ in (0,1). So at the neck we write these things one more time:

"i" :

$$V_{i,int} = \Sigma_{nm}\ [P_n^m(\xi_1)Q_n^m(\xi)-A_{nm}Q_n^m(\xi_1)P_n^m(\xi)]\quad P_n^m(\xi_0)\ [B_{nm}p_n^m(0)+C_{nm}q_n^m(0)]\ \ q_n^m(\zeta_0)\cos(m\varphi)$$
$$V_{i,int} = \Sigma_{nm}\ [P_n^m(-\xi_1)Q_n^m(-\xi)-A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi)]P_n^m(-\xi_0)[B'_{nm}p_n^m(0)+C'_{nm}q_n^m(0)]\ q_n^m(\zeta_0)\cos(m\varphi)$$

"o":

$V_{o,int} = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi_0) - A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)] \quad P_n^m(\xi) [B_{nm}p_n^m(0)+C_{nm}q_n^m(0)] q_n^m(\zeta_0) \cos(m\varphi)$

$V_{o,int} = \Sigma_{nm} [P_n^m(-\xi_1)Q_n^m(-\xi_0) - A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi_0)]P_n^m(-\xi)[B'_{nm}p_n^m(0)+C'_{nm}q_n^m(0)]q_n^m(\zeta_0) \cos(m\varphi)$

At this point we need these facts from leg prop doc

$P_\nu^\mu(-x) = P_\nu^\mu(x) \cos\pi(\nu+\mu) - (2/\pi) \sin\pi(\nu+\mu) Q_\nu^m(x)$

$Q_\nu^\mu(-x) = - Q_\nu^\mu(x) \cos\pi(\nu+\mu) - (\pi/2) P_\nu^\mu(x) \sin\pi(\nu+\mu)$

But since we have integer m and n these become (same as Bateman page 145 (19)

$P_n^m(-x) = P_n^m(x) (-1)^{n+m}$

$Q_n^m(-x) = - Q_n^m(x) (-1)^{n+m}$

so we are going to replace:

$[P_n^m(-\xi_1)Q_n^m(-\xi) - A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi)]P_n^m(-\xi_0)$

$= [(-1)^{n+m} P_n^m(\xi_1) (-1)^{n+m+1} Q_n^m(\xi) - A'_{nm}(-1)^{n+m+1}Q_n^m(\xi_1) (-1)^{n+m} P_n^m(\xi)] (-1)^{n+m} P_n^m(\xi_0)$

$= [P_n^m(\xi_1) (-1)^{n+m+1} Q_n^m(\xi) - A'_{nm}(-1)^{n+m+1}Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0)$

$= (-1)^{n+m+1} [P_n^m(\xi_1) Q_n^m(\xi) - A'_{nm}Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0)$

and similarly we replace

$[P_n^m(-\xi_1)Q_n^m(-\xi_0) - A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi_0)]P_n^m(-\xi)$

$= (-1)^{n+m+1} [P_n^m(\xi_1) Q_n^m(\xi_0) - A'_{nm}Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m(\xi)$

We can then rewrite our four forms above as

"i" :

$V_{i,int} = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi) - A_{nm}Q_n^m(\xi_1)P_n^m(\xi)] P_n^m(\xi_0) [B_{nm}p_n^m(0)+C_{nm}q_n^m(0)]q_n^m(\zeta_0) C (m\varphi)$

$V_{i,int} = \Sigma_{nm}(-1)^{n+m+1}[P_n^m(\xi_1) Q_n^m(\xi) - A'_{nm}Q_n^m(\xi_1) P_n^m(\xi)]P_n^m(\xi_0)[B'_{nm}p_n^m(0)+C'_{nm}q_n^m(0)]q_n^m(\zeta_0)C(m\varphi)$

"o":

$V_{o,int} = \Sigma_{nm} [P_n^m(\xi_1)Q_n^m(\xi_0) - A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)] P_n^m(\xi) [B_{nm}p_n^m(0)+C_{nm}q_n^m(0)] q_n^m(\zeta_0) C (m\varphi)$

$V_{o,int} = \Sigma_{nm}(-1)^{n+m+1}[P_n^m(\xi_1) Q_n^m(\xi_0) - A'_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi)[B'_{nm}p_n^m(0)+C'_{nm}q_n^m(0)]q_n^m(\zeta_0) C(m\varphi)$

Since ξ is a variable in the "i" equations, we seem to require there that $A'_{nm} = A_{nm}$ for starters. Then at the same time we need this:

$[B_{nm}p_n^m(0)+C_{nm}q_n^m(0)] = (-1)^{n+m+1} [B'_{nm}p_n^m(0)+C'_{nm}q_n^m(0)]$

This same set of conditions works also in the "o" region. So we have then these conditions on the coefficients to make the potential be continuous in the neck:

$$A'_{nm} = A_{nm}$$
$$[B_{nm}p_n^m(0)+C_{nm}q_n^m(0)] = (-1)^{n+m+1}\,[B'_{nm}p_n^m(0)+C'_{nm}q_n^m(0)]$$

with

$$p_n^m(0) = 2^{m+1}\sqrt{\pi}\ /\ [\,\Gamma(1/2 - n/2 - m/2)\,\Gamma(1 + n/2 - m/2)\,]$$
$$q_n^m(0) = (-1)^m\sqrt{\pi}\,2^{m-1}\,[\,\Gamma(1/2 + n/2 + m/2) / \Gamma(1 + n/2 - m/2)]$$

(b) slope continuity at the neck: The derivative of interest here is ∂_ζ . We copy down the above and do derivatives before setting arg to 0 to get

"i" :
$$= \Sigma_{nm}\quad [P_n^m(\xi_1)Q_n^m(\xi) - A_{nm}Q_n^m(\xi_1)P_n^m(\xi)]\ P_n^m(\xi_0)\,[B_{nm}p_n^m(0)'+C_{nm}q_n^m(0)']q_n^m(\zeta_0)\,C\,(m\varphi)$$
$$= \Sigma_{nm}(-1)^{n+m+1}[P_n^m(\xi_1)\,Q_n^m(\xi) - A'_{nm}Q_n^m(\xi_1)\,P_n^m(\xi)]P_n^m(\xi_0)[B'_{nm}p_n^m(0)'+C'_{nm}q_n^m(0)']q_n^m(\zeta_0)C(m\varphi)$$

"o":
$$= \Sigma_{nm}\quad [P_n^m(\xi_1)Q_n^m(\xi_0) - A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)]\ P_n^m(\xi)\,[B_{nm}p_n^m(0)'+C_{nm}q_n^m(0)']\,q_n^m(\zeta_0)\,C\,(m\varphi)$$
$$=\Sigma_{nm}(-1)^{n+m+1}[P_n^m(\xi_1)\,Q_n^m(\xi_0) - A'_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)]P_n^m(\xi)[B'_{nm}p_n^m(0)'+C'_{nm}q_n^m(0)']q_n^m(\zeta_0)\,C(m\varphi)$$

There is a tricky sign issue here since ζ grows in both directions up and down. So maybe our conditions then become, (remember that from either side, we have $\zeta > 0$, so only care about 0+ on both sides)

$$[B_{nm}p_n^m(0+)'+C_{nm}q_n^m(0+)'] = -\,(-1)^{n+m+1}\,[B'_{nm}p_n^m(0+)'+C'_{nm}q_n^m(0+)']$$

Assuming we set $A'_{nm} = A_{nm}$, this same condition satisfies both pairs of equations. And we know,

$$p_n^m\,'(0+) = 0$$
$$q_n^m\,'(0+) = -\sqrt{\pi}\,2^m\ [\,\Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

so our condition is now

$$[C_{nm}q_n^m(0+)'] = -\,(-1)^{n+m+1}\,[C'_{nm}q_n^m(0+)']$$

$$C_{nm} = C'_{nm}\,(-1)^{n+m}$$

So we might then gather all our "neck conditions" as follows:

$$A'_{nm} = A_{nm} \qquad 1$$
$$[B_{nm}p_n^m(0)+C_{nm}q_n^m(0)] = (-1)^{n+m+1}\,[B'_{nm}p_n^m(0)+C'_{nm}q_n^m(0)] \qquad 2$$

$$C_{nm} = C'_{nm}\,(-1)^{n+m} \qquad 3$$

There are also **tangential neck conditions** to consider, but usually these just replicate other conditions or are trivially true, so I will ignore them here and hope that doesn't cost me down the road.

The lower bloid region pillbox conditions were these

$$[P_n^m(-\xi_1)Q_n^m(-\xi_0) - A'_{nm}Q_n^m(-\xi_1)P_n^m(-\xi_0)] = 0 \qquad 4$$

$$[B'_{nm}p_n^m(\zeta_0) + C'_{nm}q_n^m(\zeta_0)] = 0 \qquad 5$$

and our two pillbox upper region conditions are these:

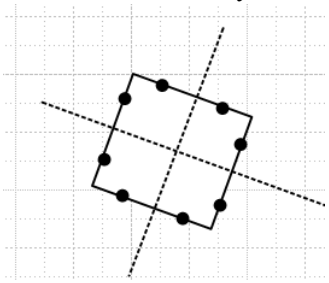
$$(2-\delta_{m,0})q/[4\pi\varepsilon c_1]\ P_n^m(\xi_0) = -2\ (2n+1)^{-1}B_{nm}\cos[(n+m)\pi/2]\ f(n,m)^2\ [P_n^m(\xi_1)Q_n^m(\xi_0) - A_{nm}Q_n^m(\xi_1)P_n^m(\xi_0)] \qquad 6$$

$$(2-\delta_{m,0})q/[4\pi\varepsilon c_1]\ \delta(\zeta - \zeta_0) = +\Sigma_n[B_{nm}p_n^m(\zeta_0) + C_{nm}q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi_1)\ f(n,m) \qquad 7$$

Conclusions: (1) I don't know what to do with this δ thing, (2) but if we count it as a condition, we seem to have one too many conditions!

7. Comments:

(1) I am of course not happy with the way I did the pillbox. There is really only one pillbox and in theory you should consider its 8 difference "faces" somehow, I did a rough job on this and that probably invalidates everything. Here is what I mean by the 8 faces: each pair of two points uses one of the four regional functions. But even this may not be good enough.



On the other hand, maybe my two thin pillboxes in the two directions was OK after all.

(2) I end up stuck with this for one of the thin pillbox conditions

$$(2-\delta_{m,0})q/[4\pi\varepsilon c_1]\ \delta(\zeta - \zeta_0) = +\Sigma_n[B_{nm}p_n^m(\zeta_0) + C_{nm}q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi_1)\ f(n,m)$$

This is "non-diagonalizable", see below.

(3) My approach to this problem today was to use the horrible parameterization only because it then forces us to have n = 0,1,2,3... which I was looking for (at least today). The other approach is to use only

the (0,1) ξ range and put the SL problem into the ζ coordinate. I tried this once, as shown in my "non-cone-method" document, but I ended up with V = 0 in the neck which was no good.

(4) Another way to summarize the pillbox situation. Probably in each direction, a thin pillbox is correct, and if we use the averaging method I show above, you get something that is probably right. However, the results although perhaps right are "non-diagonalizable" and you really just end up with a horrible matrix equation of this form:

$$(2-\delta_{m,0})q/[4\pi\varepsilon c_1] = \Sigma_n f(\xi_1,\xi_0)_{mn} B_{nm} = [f\,B]_{mm} = M_{mm}(\xi_1,\xi_0) \qquad m = \text{any integer}$$

We only know the diagonal elements of M, so we cannot matrix-solve for B. In any event, it is a complete mess. So maybe our Smythe form is in some sense "OK", but we cannot solve for the coefficients which appear in the form, and we would be hard pressed if we were handed solution coefficients to show that they really satisfied the above equation.

So my conclusion is that the "four region" approach leads to (intractable equations for the coefficients) which arise from the pillbox conditions.

(5) I think I know the solution to this problem using the cone method. The n-spectrum in that solution is non-integral and is a superposition of "oblate spheroidal atoms". It seems somehow unlikely that there exists some OTHER solution to the same problem, also constructed from these atoms, which has an n spectrum which is integral. But maybe this is possible.

(6) The motivation for the current document was this: I wanted to try to subtract off the potential of the point charge in the cone-method solutions. But the only form I had for the oblate coordinates potential of a point charge involves an integral n sum (Smythe). Therefore I was looking for an integral-n solution to this problem. I thought maybe "subtracting off" would give a simpler result for the oblate bloid problem, the potential only of the bloid's induced charge. Had that potential been simple, maybe I could have computed the "sticky charge density" from it.

(7) [obscure remarks here]. Somehow the spectrum of n has a **group theoretic** underpinning. For the normal spherical coordinates world, n = integers results from the SO(3) Lie Algebra spectrum for $\boldsymbol{\ell}$. The Laplacian ∇^2 is SO(3) invariant and we get this n = $\boldsymbol{\ell}$ as a good quantum number. The half integer n are thrown out on physical grounds. When you view ∇^2 in oblate spheroidal coordinates and do the spheroid problem, somehow we still have the n = integer spectrum, though "something" seems to break SO(3). When we do the bloid problem, however, maybe there is some other "group" associated with the problem that has the non-integral n spectrum. Maybe some other Lie algebra, maybe SO(2,1) instead of SO(3). I have seen how certain groups are associated with certain choices of coordinates. This is from a certain Helmholtz + group theory PDF I have. It shows that for oblate spheroidal, the good quantum number operators are no longer J^2 and J_3 at least if the Helmholtz value $\omega^2 \neq 0$.

Table 14 Operators and Separable Coordinates for $(\Delta_3+\omega^2)\Psi=0$ $((x_1,x_2,x_3)=(x,y,z))$

	Commuting operators S_1, S_2	Separable coordinates
1	P_2^2, P_3^2	Cartesian x, y, z
2	J_3^2, P_3^2	Cylindrical $x=r\cos\varphi,$ $y=r\sin\varphi,\ z=z$
3	$\{J_3, P_2\}, P_3^2$	Parabolic cylindrical $x=(\xi^2-\eta^2)/2,$ $y=\xi\eta,\ z=z$
4	$J_3^2+d^2P_1^2, P_3^2,$ $d>0$	Elliptic cylindrical $x=d\cosh\alpha\cos\beta,$ $y=d\sinh\alpha\sin\beta,\ z=z$
5	$\mathbf{J}\cdot\mathbf{J}, J_3^2$	Spherical $x=\rho\sin\theta\cos\varphi,$ $y=\rho\sin\theta\sin\varphi,\ z=\rho\cos\theta$
6	$\mathbf{J}\cdot\mathbf{J}-a^2(P_1^2+P_2^2), J_3^2,$ $a>0$	Prolate spheroidal $x=a\sinh\eta\sin\alpha\cos\varphi$ $y=a\sinh\eta\sin\alpha\sin\varphi$ $z=a\cosh\eta\cos\alpha$
7	$\mathbf{J}\cdot\mathbf{J}+a^2(P_1^2+P_2^2), J_3^2,$ $a>0$	Oblate spheroidal $x=a\cosh\eta\sin\alpha\cos\varphi$ $y=a\cosh\eta\sin\alpha\sin\varphi$ $z=a\sinh\eta\cos\alpha$
8	$\{J_1, P_2\}-\{J_2, P_1\}, J_3^2$	Parabolic $x=\xi\eta\cos\varphi,$ $y=\xi\eta\sin\varphi,\ z=(\xi^2-\eta^2)/2$
9	$J_3^2-c^2P_3^2+c(\{J_2, P_1\}+\{J_1, P_2\}),$ $c(P_2^2-P_1^2)+\{J_2, P_1\}-\{J_1, P_2\}$	Paraboloidal $x=2c\cosh\alpha\cos\beta\sinh\gamma$ $y=2c\sinh\alpha\sin\beta\cosh\gamma$ $z=c(\cosh 2\alpha+\cos 2\beta-\cosh 2\gamma)/2$
10	$P_1^2+aP_2^2+(a+1)P_3^2+\mathbf{J}\cdot\mathbf{J},$ $J_2^2+a(J_1^2+P_3^2),$ $a>1$	Ellipsoidal $x=\left[\frac{(\mu-a)(\nu-a)(\rho-a)}{a(a-1)}\right]^{1/2}$ $y=\left[\frac{(\mu-1)(\nu-1)(\rho-1)}{1-a}\right]^{1/2}$ $z=\left[\frac{\mu\nu\rho}{a}\right]^{1/2}$
11	$\mathbf{J}\cdot\mathbf{J}, J_1^2+bJ_2^2,$ $1>b>0$	Conical $x=r\left[\frac{(b\mu-1)(b\nu-1)}{1-b}\right]^{1/2}$ $y=r\left[\frac{b(\mu-1)(\nu-1)}{b-1}\right]^{1/2},\ z=r[b\mu\nu]^{1/2}$

The spectrum of n I suppose is continuous on birth and is only quantized by the particular boundary conditions of your problem. The cone method quickly gets you to a spectrum, whereas the four-region method seems to give an integer spectrum. I really cannot say the integer spectrum result is impossible, it might be "just another expansion" of the same solution, in the sense that you can write various expansions which add up π, say.

The above paper later goes onto the Laplace equation ($\omega^2 = 0$) and I think identifies 10 symmetries of the oblate system. I am electing to not get involved in that subject right now, though I admit it is sort of in my general alley of interest. My group theory is so rusty right now, it would take several months to come up to speed, rereading various books I have. But I am still on the Stakgold problem right now!