Green's Function for an Oblate Bloid using cone method PhL 2.3.10 - 2.9.10

In Retrospect (2.28.10) 1

THE CONE APPROACH TO THE OBLATE BLOID PROBLEMS 2

TRY THE SIMPLE SMYTHE FORM FOR THE OPEN BLOID PROBLEM 5

The Smythian Form. 5

Continuity Checks. 6

Green's charge pillbox. 7

$P_n^m(\xi)$ Orthogonality and finding the F_{nm} coefficients. 8

The SL Problem Interpretation. 10

The large ζ limit of the result matches Smythe's cone result. 10

Summary of the solution to the Green's Function problem for an Oblate Hyperboloid. 13

What makes the potential be a real number? 16

Construction of Theorems A and A1. 18

Theorem A: 18

Theorem A1: 19

The complex nature of $j\, Q_n^m(j\zeta_0)\, P_n^m(j\zeta)]$ for general real n. 20

A Paradox to Worry About. 22

The Hole in Plate Limit 23

(a) The spectrum of n. 23

In Retrospect (2.28.10). In this doc, I came up with what I thought was a solution to the bloid Green's problem that met all the required boundary conditions, and I was puzzled that the result was coming out being a complex-valued potential. Today for the first time I think I see what I did wrong here. I assumed a certain form (***) below,

$$V(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} F_{nm}\, P_n^m(j\zeta_<)\, Q_n^m(j\zeta_>)\, P_n^m(\xi) \cos(m\varphi) \qquad \text{// upper } \zeta \geq 0$$

$$V(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} G_{nm}\, P_n^m(j\zeta_<)\, Q_n^m(j\zeta_>)\, P_n^m(\xi) \cos(m\varphi) \qquad \text{// lower } \zeta \geq 0$$

Then by studying things in the neck, I concluded that we must have $G_{nm} = (-1)^m F_{nm}$. I went on to do the pillbox in the upper region to find an expression for F_{nm} and then I just assumed the lower region had G_{nm} as just stated. However, I now see that had I applied the pillbox in the lower region, I would have obtained $G_{nm} = 0$, since there is no Green's charge down there. This would have caused a contradiction since we have $G_{nm} = 0$, $F_{nm} \neq 0$, and $G_{nm} = (-1)^m F_{nm}$. Therefore, the assume Smythian form is no good! So this "complex solution" to the problem is not really a solution at all! The paradox goes away.

Motivation for the Cone Method. As I think about the jigger problem (walking to Smith's) I can sort of see the problem with trying to get the $Q_n^m(j\zeta)$ functions to decay far from the neck, and then come in for a landing with $Q_n^m(j0+) = 0$. This is sort of like asking r^{-n-1} to come in and be finite at the origin in a spherical coordinates problem. So one idea I got was to put another boundary surface at $\zeta = \zeta_0$. This would be a spheroidal surface. Above it you would really have just Q functions, and below it you would

try a linear combination of Q and P functions. [This in fact is what I will be doing below.] But then in the upper bloid we have now 4 regions to worry about, and two sets of matching to do. I could imagine doing a lot of work in this direction and then finding in the end that it won't fly. [But I now think two spheroidal regions might work.]

Then on that same walk today, I got what I think is the right idea. I think my Smythe form really is non-viable. This form uses a certain linear combination trick to get V = 0 on the bloid surface, similar to the trick used in the spheroid problem. We have integer m, and then the spectrum of n is controlled by the ζ SL problem. But in my form, I could not get a viable spectrum! I remembered how I did the *cone* Green's problem in spherical coordinates, and I see now that that is probably the right thing to do here. You then take your n spectrum from the $P_n^m(\xi)$ functions by requiring n to be the zeros of $P_n^m(\xi_1) = 0$. Although our shape here is not a cone, this equation makes V = 0 on the bloid, the difference is ξ versus z. I guess I missed this point. The bigger reason for this approach is a winner: somehow as you move the Green's charge farther away, the jigger solution has to morph into the cone solution! And in the cone solution we used this method of getting the n spectrum.

So I am basically proposing that we "start over" and find a new Smythian form that uses this method of getting V = 0 on the metal bloid, then process that and see where it leads. Basically, my original Smythian form was "not capable of" providing a solution to this problem. It was a linear combination of the correct atoms, but I could not make it meet the BC's.

The Cone Approach to the Oblate Bloid Problems.

What would such a new Smythian form look like? We certainly want to use the continuous-in-the-hole coordinates. I think we would then want our upper region "boundary" to be a spheroid surface $\zeta = \zeta_0$, not a bloid surface. Ironically, this is how I *originally* started this whole problem, though I destroyed the record because I quickly went to the bloid boundary idea. So maybe we would have *something like this*,

$$V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} \left[P_n^m(j\zeta_1)Q_n^m(j\zeta_0) - Q_n^m(j\zeta_1)P_n^m(j\zeta_0)\right] Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi} \quad \text{no good!}$$
$$V_i(\zeta,\xi) = \Sigma_{nm} E_{nm} \left[P_n^m(j\zeta_1) Q_n^m(j\zeta) - Q_n^m(j\zeta_1)P_n^m(j\zeta)\right] Q_n^m(j\zeta_0) P_n^m(\xi) e^{im\varphi}$$

where this is just my spheroid Green's form with the m labels added everywhere. This form already matches on the spheroid $\zeta = \zeta_0$. On the outside we have our Q decay, and on the inside we have a lincomb of P and Q which was not available to me in the previous Smythe form I was using.

But this form gives us $V_i = 0$ on the *spheroid* $\zeta = \zeta_1$ which is obviously not desirable in our bloid problem, so this form cannot be right. So let's try to concoct something else:

$$V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$
$$V_i(\zeta,\xi) = \Sigma_{nm} \left[A_{nm}Q_n^m(j\zeta) + B_{nm}P_n^m(j\zeta)\right] P_n^m(\xi) e^{im\varphi}$$

The matching condition would then be

$$E_{nm} Q_n^m(j\zeta_0) = A_{nm}Q_n^m(j\zeta_0) + B_{nm}P_n^m(j\zeta_0)$$

which then suggests something like this:

$$V_o(\zeta,\xi) = \Sigma_{nm} [A_{nm}Q_n^m(j\zeta_0) + B_{nm}P_n^m(j\zeta_0)] / Q_n^m(j\zeta_0) * Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [A_{nm}Q_n^m(j\zeta) + B_{nm}P_n^m(j\zeta)] P_n^m(\xi) e^{im\varphi}$$

Then mult both through by $Q_n^m(j\zeta_0)$ to get

$$V_o(\zeta,\xi) = \Sigma_{nm} [A_{nm}Q_n^m(j\zeta_0) + B_{nm}P_n^m(j\zeta_0)] Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi} \qquad \text{(*) fancy}$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [A_{nm}Q_n^m(j\zeta) + B_{nm}P_n^m(j\zeta)] Q_n^m(j\zeta_0) P_n^m(\xi) e^{im\varphi}$$

So now the matching at $\zeta = \zeta_0$ is "manifest". It does look Smythian! The downside is that there seem to be two sets of coefficients to worry about. In both regions we have V = 0 at the bloid due to $P_n^m(\xi)$.

<u>Here is another more restrictive starting point.</u> Suppose we just assume we can make do with a P in the inside region. Then we get

$$V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$
$$V_i(\zeta,\xi) = \Sigma_{nm} F_{nm} P_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$

This form worked for the cone, which gives it at least a little gravitas. Here we cannot use (r/a) powers so we still have two sets of coefficients. The boundary match would then require that

$$E_{nm} Q_n^m(j\zeta_0) = F_{nm} P_n^m(j\zeta_0)$$

so we could repeat our sequence above. First replace the F:

$$V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$
$$V_i(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta_0)/ P_n^m(j\zeta_0) * P_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$

Then multiply through by $P_n^m(j\zeta_0)$ to get

$$V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta) P_n^m(j\zeta_0) P_n^m(\xi) e^{im\varphi} \qquad \text{(**) simpler}$$
$$V_i(\zeta,\xi) = \Sigma_{nm} E_{nm} Q_n^m(j\zeta_0) P_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$

which certainly has some cosmetic appeal and gives a simple < and > form. So this makes the match manifest as above. So this is a "single set of coefficients" form attempt.

The jigger problem would require that $V_i = 0$ at $\zeta = 0$ which says this:

$$0 = \Sigma_{nm} E_{nm} Q_n^m(j\zeta_0) P_n^m(j0+) P_n^m(\xi) e^{im\varphi}$$

and this in turn would require that $P_n^m(j0+) = 0$. We know that

$$P_\nu^m(z=0\pm) = 2^m\sqrt{\pi}\ e^{\mp i m\pi/2} / [\ \Gamma(1/2 - \nu/2 - m/2)\ \Gamma(1 + \nu/2 - m/2)\ ] \qquad \text{// p 126 (22)}$$

so having these things vanish would require that ν+m or ν-m be certain integer values, but this will not be the case since ν = n will be whatever weird values we get from $P_n^m(\xi_1) = 0$.

Therefore, this simple Smythian form (**) is not going to fly for the jigger problem! We would then have to revert to form (*) and then have this hole condition instead:

$$0 = \Sigma_{nm} [A_{nm}Q_n^m(j0+) + B_{nm}P_n^m(j0+)] Q_n^m(j\zeta_0) P_n^m(\xi) e^{im\varphi}$$

and this is a condition we could meet by choosing A and B so that

$$A_{nm}Q_n^m(j0+) + B_{nm}P_n^m(j0+) = 0$$

As usual, we then solve this for B and write for (*)

$$V_o(\zeta,\xi) = \Sigma_{nm} [A_{nm}Q_n^m(j\zeta_0) - \{ A_{nm}Q_n^m(j0+)/ P_n^m(j0+)\}P_n^m(j\zeta_0)] Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [A_{nm}Q_n^m(j\zeta) - \{ A_{nm}Q_n^m(j0+)/ P_n^m(j0+)\}P_n^m(j\zeta)] Q_n^m(j\zeta_0) P_n^m(\xi) e^{im\varphi}$$

or

$$V_o(\zeta,\xi) = \Sigma_{nm} A_{nm} [Q_n^m(j\zeta_0) - \{ Q_n^m(j0+)/ P_n^m(j0+)\}P_n^m(j\zeta_0)] Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$
$$V_i(\zeta,\xi) = \Sigma_{nm} A_{nm} [Q_n^m(j\zeta) - \{ Q_n^m(j0+)/ P_n^m(j0+)\}P_n^m(j\zeta)] Q_n^m(j\zeta_0) P_n^m(\xi) e^{im\varphi}$$

giving now a viable form for the jigger problem. We can then make use of Smythe's fancy orthogonality of the $P_n^m(\xi)$ for our special spectrum of n, together with φ orthogonality, and maybe we could grind this thing through to a solution!

As for the open bloid problem, we might still get away with the simpler form (**) since we don't have to impose any condition on the hole disk. But this problem does involve both halves of the bloid, which brings in extra issues as we well know. Maybe we need to write the forms separately for the lower region.

So, I think I have maybe resuscitated both these problems at least for a while. I will work on them next time.

Note added 2.10.10. I have now developed my "real atoms" for the ζ problem which I call $p_n^m(\zeta)$ and $q_n^m(\zeta)$. Since these functions are real for real n,m, ζ (which is not the case for $P_n^m(j\zeta)$ and $Q_n^m(j\zeta)$), we are then assured of having a real potential when we are done. Rather then edit everything above, just imagine that we make global replacements $P_n^m(j\zeta) \rightarrow p_n^m(\zeta)$ and $Q_n^m(j\zeta) \rightarrow q_n^m(\zeta)$. As long as we don't take any derivatives, nothing else has to be changed. We have the following facts to work with, found in my new P and Q document

$$q_n^m(\zeta) \equiv (-i)^{2m+1\mp n} Q_n^m(i\zeta)$$

$$p_n^m(\zeta) \equiv [(\mp i)^{-m/2} P_n^m(i\zeta) + (\pm i)^{-m/2} P_n^m(-i\zeta)]$$

$$= (\pm i)^{-m/2} [\{ (\pm i)^m + (\mp i)^{2n} \} P_n^m(i\zeta) - (-i)^{2m} (2/\pi) \sin\pi(n+m) Q_n^m(i\zeta)]$$

$$W[p_n^m(\zeta), q_n^m(\zeta)] = 2 \cos[(n+m/2)\pi/2] [\Gamma(1+m+n)/ \Gamma(1-m+n)] * 1/(1+\zeta^2)$$

I will started doing the edits from here on:

Try the Simple Smythe Form for the Open Bloid Problem

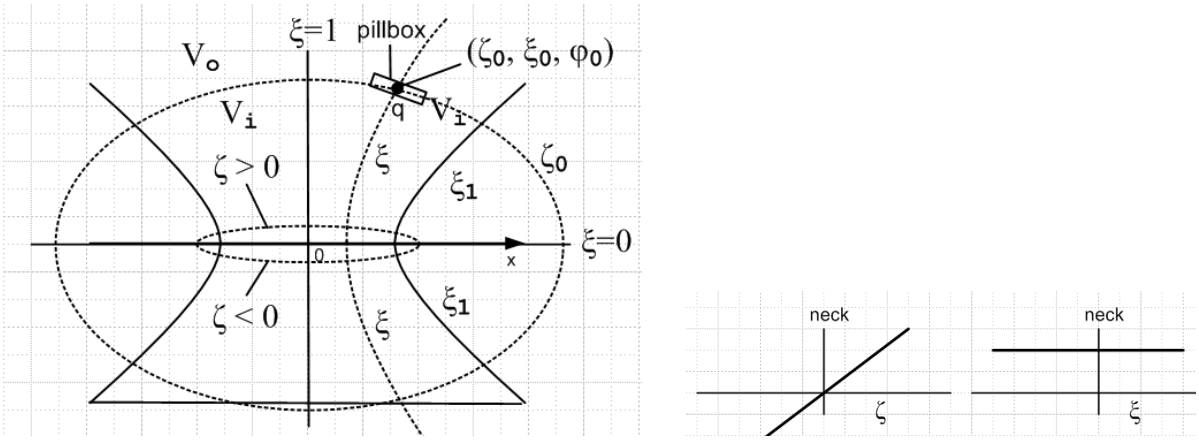
The Smythian Form.

That form is this for the upper bloid area (same with E'_{nm} coefficients in lower area)

$V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} q_n^m(\zeta) p_n^m(\zeta_0) P_n^m(\xi) e^{im\varphi}$ // modified below, see (***)

$V_i(\zeta,\xi) = \Sigma_{nm} E_{nm} q_n^m(\zeta_0) p_n^m(\zeta) P_n^m(\xi) e^{im\varphi}$

and the picture is this: [we use param. with ξ in (0,1) and $-\infty < \zeta < \infty$]



We can see that the potential "matches" along the boundary $\zeta = \zeta_0$.

How do we know that the normal gradient matches along this same boundary? If it does not, they we have some specious surface charge out in space. The answer to this is that below we will force the quantity $\partial_\zeta V_i(\zeta_0,\xi,\varphi) - \partial_\zeta V_o(\zeta_0,\xi,\varphi)$ to be proportional to $\delta(\xi-\xi_0)\delta(\varphi-\varphi_0)$, so this makes that gradient match everywhere except at the point charge.

$\partial_\zeta V_o(\zeta,\xi) = \Sigma_{nm} E_{nm} q_n^m{}'(\zeta) p_n^m(\zeta_0) P_n^m(\xi) e^{im\varphi}$ // modified below, see (***)

$\partial_\zeta V_i(\zeta,\xi) = \Sigma_{nm} E_{nm} q_n^m(\zeta_0) p_n^m{}'(\zeta) P_n^m(\xi) e^{im\varphi}$

NOTE ADDED: Let's modify the form to make comparison with Smythe's cone problem easier, and also perhaps this will simplify the result. We shall set $\varphi_0 = 0$ and use $\cos(m\varphi)$ for expansion, knowing that the solution is even in φ in this case. Then we take note of these properties from Leg prop doc:

$[\ \Gamma(\nu+m+1)/\ \Gamma(\nu-m+1)\ ]\ P_\nu^{-m}(z) = P_\nu^m(z)$

$[\ \Gamma(\nu+m+1)/\ \Gamma(\nu-m+1)\ ]\ Q_\nu^{-m}(z) = Q_\nu^m(z)$

Looking at our p and q definitions above

Note that we are using the $P_n^m(\xi)$ functions off-the cut, and let's say we approach from above the cut for our real ξ arguments (I may change this later). As I show in my note in the cone solution doc, you can fold the negative m terms into positive m and redefine the coefficients, due to the symmetries just noted. When this is done, we end up with this starting form for the upper bloid region

$$V_o(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} Q_n^m(j\zeta) P_n^m(j\zeta_0) P_n^m(\xi) \cos(m\varphi) \qquad (***)$$
$$V_i(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} Q_n^m(j\zeta_0) P_n^m(j\zeta) P_n^m(\xi) \cos(m\varphi)$$

and the same form with coefficients G_{nm} in the lower bloid region.

NOTE ADDED: The $P_n^m(\xi)$ functions are the on-the-cut functions, while $P_n^m(j\zeta)$ are off-the-cut!

Now right off the bat, which $Q_n^m(j\zeta)$ are we implying here? Let's try $Q_n^m(j\zeta)_{0,0L}$ so it decays on all rays. This is our Bateman friend. We set our n spectrum by requiring $P_n(\xi_1) = 0$, and notice that in our choice to the right above, this picks up the entire hyperboloid and our range is ξ in (ξ_1, 1). More details on these n a little later, I know they exist. So our entire spectrum is now determined.

Now let's run down the BC's, including the hidden ones. The non-hidden include ray behavior, already present and accounted for. And V = 0 on the bloid is accounted for. And matching at our spheroidal boundary is accounted for.

Now we need to face the "neck" boundary conditions.

Continuity Checks.

(a) continuity of V at the neck. Since this must be true for all ξ and φ points in the neck disk, only V_i is involved and we require that

$$V_i(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} Q_n^m(j\zeta_0) P_n^m(j\zeta) P_n^m(\xi) \cos(m\varphi) \qquad \text{// lower bloid has } G_{nm}$$

$$F_{nm} Q_n^m(j\zeta_0) P_n^m(j0+) = G_{nm} Q_n^m(j\zeta_0) P_n^m(j0-)$$

$$\Rightarrow \qquad G_{nm}/F_{nm} = P_n^m(j0+)/ P_n^m(j0-)$$

Our Leg prop doc says

$$P_n^m(z=0\pm) = 2^m\sqrt{\pi}\ e^{\mp i n m/2} / [\ \Gamma(1/2 - n/2 - m/2)\ \Gamma(1 + n/2 - m/2)\ ] \qquad \text{// p 126 (22)}$$

so that

$$P_n^m(j0+)/ P_n^m(j0-) = e^{-i n m/2}/ e^{+i n m/2} = e^{-i n m} = (-1)^m$$

This tells us that

$$G_{nm} = F_{nm}(-1)^m$$

NOTE ADDED: Before doing gradients, bring in this info from the older bloid Green's doc:

$h_1 = c_1 \sqrt{\xi^2+\zeta^2} / \sqrt{1-\xi^2}$ $\quad\quad$ $q_1, q_2, q_3 = \xi, \zeta, \varphi$

$h_2 = c_1 \sqrt{\xi^2+\zeta^2} / \sqrt{1+\zeta^2}$

$h_3 = c_1 \sqrt{1+\zeta^2}\ \sqrt{1-\xi^2}$ $\quad\quad$ $\partial_n = (1/h_1)\,\partial_\xi$ etc.

(b) continuity of $\partial_\zeta V$ across the neck. I pick this direction first since I think it is the most dangerous. Remember that it is the $\partial_n V_i$ that must be continuous, so we are going to involve the scale factors. For the ζ coordinate, our scale factor is h_2 which is just $h_2 = c_1\xi$ at the neck. Since we work at a fixed ξ location, this is just a constant that will be the same on both sides of our balance, so we can ignore it. Then we have

$$\partial_\zeta V_i(\zeta,\xi,\varphi) = j \Sigma_n \Sigma_{m=0}^{\infty} F_{nm}\ Q_n^m(j\zeta_0)\ P_n^m{}'(j\zeta)\ P_n^m(\xi)\cos(m\varphi) \quad\quad \text{// with } G_{nm} \text{ below}$$

I can now pull up more Leg prop doc results,

$$P_n^m{}'(0\pm) = \mp\, i\, P_n^{m+1}(0\pm) \quad\quad m \geq 0 \text{ and integer}$$

and our requirement is now (ie, this is our balance, just mentioned above)

$$F_{nm}\ Q_n^m(j\zeta_0)\ P_n^m{}'(j0+) = G_{nm}\ Q_n^m(j\zeta_0)\ P_n^m{}'(j0-)$$

or

$$G_{nm}/ F_{nm} = P_n^m{}'(j0+)/ P_n^m{}'(j0-) = - i\, P_n^{m+1}(0+) \ / \ [+ i\, P_n^{m+1}(0-)]$$

$$= - P_n^{m+1}(0+) / P_n^{m+1}(0-) = - (-1)^{m+1} = (-1)^m$$

Wheww!!! So the same G_{nm}/ F_{nm} ratio establishes continuity of both the potential AND the derivative of the potential at the hole plane!

(c) continuity of $\partial_\xi V$ across a bloid ξ This is a non condition! A point on the bloid ξ is either in the V_o region or it is in the V_i region, just look at the picture. In either region, the expression for V is the same on both sides of the bloid, so everything matches across the bloid dotted line -- exact same function.

(d) continuity of $\partial_\varphi V$ across an azimuthal plane φ. Same comment as (c).

Are there any other possible hidden BC's ? !! I don't see any. So, let us attempt to proceed!

Green's charge pillbox. Our pillbox situation is the same as in the oblate spheroid Green's problem. So I now have that up for viewing, and we just try to imitate what we did there.

$$\partial_\zeta V_o(\zeta,\xi,\varphi) = j \Sigma_n \Sigma_{m=0}^{\infty} F_{nm}\ Q_n^m{}'(j\zeta)\ P_n^m(j\zeta_0)\ P_n^m(\xi)\cos(m\varphi)$$

$$\partial_\zeta V_i(\zeta,\xi,\varphi) = j \Sigma_n \Sigma_{m=0}^{\infty} F_{nm}\ Q_n^m(j\zeta_0)\ P_n^m{}'(j\zeta)\ P_n^m(\xi)\cos(m\varphi)$$

$$\partial_\zeta V_i(\zeta,\xi,\varphi) - \partial_\zeta V_o(\zeta,\xi,\varphi) = j \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} P_n^m(\xi) \cos(m\varphi) [\ Q_n^m(j\zeta_0) P_n^m{}'(j\zeta) - Q_n^m{}'(j\zeta) P_n^m(j\zeta_0)\ ]$$

$$\partial_\zeta V_i(\zeta_0,\xi,\varphi) - \partial_\zeta V_o(\zeta_0,\xi,\varphi) = j \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} P_n^m(\xi) \cos(m\varphi) [\ Q_n^m(j\zeta_0) P_n^m{}'(j\zeta_0) - Q_n^m{}'(j\zeta_0) P_n^m(j\zeta_0)]$$

$$= -j \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} P_n^m(\xi) \cos(m\varphi) \ W[P_n^m(j\zeta_0), Q_n^m(j\zeta_0)]$$

$$W[\ P_n^m(j\zeta_0), Q_n^m(j\zeta_0)] = (-1)^m [\Gamma(1+m+n)/\ \Gamma(1-m+n)] \ * \ 1/(1+\zeta_0^2) \qquad \text{ok}$$

=>

$$\partial_\zeta V_i(\zeta_0,\xi,\varphi) - \partial_\zeta V_o(\zeta_0,\xi,\varphi) = (1+\zeta_0^2)^{-1} (-j) \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} (-1)^m [\Gamma(1+m+n)/\ \Gamma(1-m+n)] \ P_n^m(\xi) \cos(m\varphi)$$

where the Wronskian comes from "the Smythe method for Green's...doc" and here n are not integers so we don't use the factorial form.

Note added: this Wronskian involves off the cut P and Q for which Smythe and Bateman are the same. Bateman shows that $(-1)^m$ factor there for off the cut functions. Smythe page 152 shows this same phase factor in his Wronskian.

Now the pillbox argument tells us that

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon) (h_2/h_3h_1) \delta(\varphi-\varphi_0) \delta(\xi - \xi_0) = (q/\varepsilon) [(1+\zeta_0^2) c_1]^{-1} \delta(\varphi-\varphi_0) \delta(\xi - \xi_0)$$

so combining this with the above and cancelling the obvious, we get

$$(q/[c_1\varepsilon]) \delta(\varphi-\varphi_0) \delta(\xi - \xi_0) = (-j) \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} (-1)^m [\Gamma(1+m+n)/\ \Gamma(1-m+n)] \ P_n^m(\xi) \cos(m\varphi)$$

NOW, we roll out our special cosine orthogonality rule after multiplying both sides by cos(m'φ),

$$\int_0^{2\pi} d\varphi \cos(m'\varphi) \cos(m\varphi) = \delta_{m,m'} 2\pi/(2-\delta_{m,0})$$

to get

$$(2-\delta_{m,0}) (q/[2\pi c_1\varepsilon]) \delta(\xi - \xi_0) = (-j) \Sigma_n F_{nm} (-1)^m [\Gamma(1+m+n)/\ \Gamma(1-m+n)] \ P_n^m(\xi)$$

Notice that we have $\varphi_0 = 0$ so our $\cos(m\varphi_0)$ on the LHS becomes just 1.

$P_n^m(\xi)$ Orthogonality and finding the F_{nm} coefficients.

Now, thank goodness, we have an assist from Smythe (p 156) which I have already heavily quoted in "the Smythe method for Green's...doc" . Here I will just state the result: [also true if you replace P with Q] [notice no asterisks]

$$\int_{\xi 1}^{1} d\xi P_n^m(\xi) P_{n'}^m(\xi) = \delta_{n,n'} K_n(\xi_1)$$

where [since there are two P functions, these P's could be both Bateman or both Smythe]

$$K_n(\xi_1) = -[(1-\xi_1^2)/(2n+1)]\, P_n^m{}'(\xi_1)\, \partial_n P_n^m(\xi_1)$$

where as usual the prime means ∂_ξ. So the tricky part here is that you need derivatives of the P function with respect to the degree n! Bateman's entire Legendre chapter says nothing about this. I think you would do this somehow using an integral representation of $P_n^m(z)$ [Smythe says integral expression] . So let's put off this K_n computation problem and press ahead.

Note added: According to my Leg prop doc, Smythe's P and Q agree with Bateman for off cut functions, BUT for on the cut P and Q functions, there is a $(-1)^m$ extra phase on each. Now, in the above orthog formula, there are two on-the-cut P functions on both sides, so whether we write this orthog in Smythe or Bateman, it looks the same.

We have: (I snuck in primes on the n's)

$$(2-\delta_{m,0})\,(q/[2\pi c_1\varepsilon])\,\delta(\xi-\xi_0) = (-j)\sum_{n'} F_{n'm}\,(-1)^m f(n',m)\, P_{n'}^m(\xi)$$

Mult LHS by $P_n^m(\xi)$ and do the above integral, and we get

$$(2-\delta_{m,0})\,(q/[2\pi c_1\varepsilon])\, P_n^m(\xi_0) = (-j)\, K_n(\xi_1)\, F_{nm}\,(-1)^m f(n,m)$$

So we now have our coefficients:

$$F_{nm} = j(2-\delta_{m,0})\,(q/[2\pi c_1\varepsilon\, K_n(\xi_1)])\, P_n^m(\xi_0)\,(-1)^m f(n,-m) \qquad G_{nm} = (-1)^m F_{nm}$$

Here then is the (supposed!) solution to our problem:

$$V_o(\zeta,\xi) = \sum_n \sum_{m=0}^{\infty} F_{nm}\, Q_n^m(j\zeta)\, P_n^m(j\zeta_0)\, P_n^m(\xi)\cos(m\varphi) \qquad (***)$$
$$V_i(\zeta,\xi) = \sum_n \sum_{m=0}^{\infty} F_{nm}\, Q_n^m(j\zeta_0)\, P_n^m(j\zeta)\, P_n^m(\xi)\cos(m\varphi)$$

$$V_o(\zeta,\xi) = j\,(q/[2\pi c_1\varepsilon])\sum_n \sum_{m=0}^{\infty}(2-\delta_{m,0})\,[(-1)^m f(n,-m)/K_n(\xi_1)]\, Q_n^m(j\zeta)\, P_n^m(j\zeta_0)\, P_n^m(\xi_0) P_n^m(\xi)\cos(m\varphi)$$
$$V_i(\zeta,\xi) = j\,(q/[2\pi c_1\varepsilon])\sum_n \sum_{m=0}^{\infty}(2-\delta_{m,0})\,[(-1)^m f(n,-m)/K_n(\xi_1)]\, Q_n^m(j\zeta_0)\, P_n^m(j\zeta)\, P_n^m(\xi_0) P_n^m(\xi)\cos(m\varphi)$$

where I use my old notation that $f(n,-m) = [\Gamma(1-m+n)/\Gamma(1+m+n)]$. We can write this in the usual fancy notation as follows:

$$\zeta_> = \max(\zeta,\zeta_0) \qquad \zeta_< = \min(\zeta,\zeta_0)$$

$$V(\zeta,\xi) = j\,(q/[2\pi c_1\varepsilon])\sum_n \sum_{m=0}^{\infty}(2-\delta_{m,0})\,[(-1)^m f(n,-m)/K_n(\xi_1)]\, Q_n^m(j\zeta_>)\, P_n^m(j\zeta_<)\, P_n^m(\xi_0) P_n^m(\xi)\cos(m\varphi)$$

$$= g(\zeta,\xi,\varphi|\,\zeta_0,\xi_0,\varphi_0) \qquad \text{for bloid with label } \xi_1$$

$f(n,-m) = [\Gamma(1-m+n)/\Gamma(1+m+n)]$ use $G_{nm} = (-1)^m F_{nm}$ for lower

$K_n(\xi_1) = -[(1-\xi_1^2)/(2n+1)]\, P_n^m{}'(\xi_1)\, \partial_n P_n^m(\xi_1)$

sum is over n which are zeros of $P_n(\xi_1) = 0$

This is, once again, **the Final Result.** I expect this result to explode any minute, but right now we have a result after a long time. [I have left in the $(-1)^m$ even though m is even so it can be removed.]

The SL Problem Interpretation. Look back now at

$$\int_{\xi 1}^{1} d\xi\, P_n^m(\xi)\, P_{n'}^m(\xi) = \delta_{n,n'}\, K_n(\xi_1)$$

Perhaps we can somehow interpret this as a Sturm-Liouville problem on the interval (ξ_1, 1) with our Legendre ODE. The BC at $\xi = 1$ I guess is just that P be finite, since this is a singular point of the ODE. The BC at $\xi = \xi_1$ is that $P_n^m(\xi_1) = 0$. This is pretty clearly a "homo BC". So, we end up with a set of orthogonal eigenfunctions in the usual way, and the spectrum is just the n such that $P_n^m(\xi_1) = 0$.

In my previous solution attempts, I kept doing SL problems in ζ , not ξ .

NOTE: We need only consider the $n_i > 0$ which make $P^m_{ni}(\xi_1) = 0$ because $P^m_\nu(z) = P^m_{-\nu-1}(z)$. If we included the negative values, they in the SL problem we are including each spectral point twice. I think this argument can be cleanly made.

The large ζ limit of the result matches Smythe's cone result.

If we go very far away, I think this might approach the cone result, because somehow the detail in the neck region should be unimportant. Smythe's form for this problem was [for off the cut P and Q functions, Smythe is the same as Bateman]

r<a $\quad V_i = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r/a)^n \quad (-1)^m P_n^m(z) \cos(m\varphi)$

r>a $\quad V_e = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r/a)^{-n-1} (-1)^m P_n^m(z) \cos(m\varphi)$ ok

I have added $(-1)^m$ above because these are on the cut P functions and Smythe has that phase difference.

His coefficients were [again, I add a $(-1)^m$ to convert $P_n^m(\cos\beta)$ to a Bateman P function]

$$A_{mn} = -(2-\delta_{m,0})\,(2\pi\varepsilon a)^{-1}\,(-1)^m\, P_n^m(\cos\beta)\, \{\sin^2\alpha\ \partial_z P_n^m(\cos\alpha)\, \partial_n P_n^m(\cos\alpha)\}^{-1}$$

where $\cos\beta$ = location of his point charge = ξ_0 for me
$\cos\alpha$ = his cone angle = ξ_1 for me.
a = his distance of the point charge from the cone origin
q = 1 = his point charge value

Question: Am I doing something wrong here? We have

$$V_i = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r/a)^n \; P_n^m(z)_{Smythe} \cos(m\varphi)$$

$$A_{mn} = -(2-\delta_{m,0}) (2\pi\varepsilon a)^{-1} P_n^m(\cos\beta)_{Smythe} \; \{ \sin^2\alpha \; \partial_z P_n^m(\cos\alpha) \; \partial_n P_n^m(\cos\alpha)\}^{-1}$$

I can then rewrite both as:

$$V_i = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r/a)^n \; (-1)^m P_n^m(z) \cos(m\varphi)$$

$$A_{mn} = -(2-\delta_{m,0}) (2\pi\varepsilon a)^{-1} (-1)^m P_n^m(\cos\beta) \; \{ \sin^2\alpha \; \partial_z P_n^m(\cos\alpha) \; \partial_n P_n^m(\cos\alpha)\}^{-1}$$

then upon insertion the two factors of $(-1)^m$ cancel. Basically again we have two P functions so either P function gives the same expansion. So I think I have done this right.

So his coefficients are then

$$A_{mn} = -(2-\delta_{m,0}) (2\pi\varepsilon a)^{-1} (-1)^m P_n^m(\xi_0)\{ (1-\xi_1^2) P_n^m{}'(\xi_1) \; \partial_n P_n^m(\xi_1) \}^{-1}$$

I can identify

$$-\{ (1-\xi_1^2) P_n^m{}'(\xi_1) \; \partial_n P_n^m(\xi_1)\} \;\; = K_n(\xi_1) (2n+1)$$

so his coefficients are then (Bateman P here)

$$A_{mn} = (2-\delta_{m,0}) (2\pi\varepsilon a)^{-1} (-1)^m P_n^m(\xi_0) / [(2n+1)K_n(\xi_1)]$$

Now we know that when we go far away, we get $\zeta = r/c_1$ so then $\zeta_0 = a/c_1$ so replace $a = c_1\zeta_0$ and his coefficient then becomes

$$A_{mn} = (2-\delta_{m,0}) (2\pi\varepsilon c_1 \zeta_0)^{-1} \; (-1)^m P_n^m(\xi_0) / [(2n+1)K_n(\xi_1)]$$

Meanwhile, *my* coefficient is this

$$F_{nm} = \; j (2-\delta_{m,0}) (q/[2\pi c_1\varepsilon K_n(\xi_1)]) P_n^m(\xi_0) (-1)^m f(n,-m)$$

If I set my q = 1 then we have

$$A_{mn} = \; (2-\delta_{m,0}) (2\pi\varepsilon c_1 K_n(\xi_1))^{-1} \; (-1)^m P_n^m(\xi_0) \qquad (1/\zeta_0) \; */ (2n+1)$$

$$F_{nm} = \; j (2-\delta_{m,0}) (2\pi c_1\varepsilon K_n(\xi_1))^{-1} P_n^m(\xi_0) \qquad (-1)^m f(n,-m)$$

Now, let's compare once again our "forms" . His cone deal is this (confirming $\varphi_0 = 0$)

r<a $\qquad V_i = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r/a)^n \qquad (-1)^m P_n^m(z) \cos(m\varphi)$

r<a $\qquad V_e = \Sigma_{m=0}^{\infty} \Sigma_n \; A_{mn} (r/a)^{-n-1} \; (-1)^m P_n^m(z) \cos(m\varphi)$

or

$$V = \Sigma_{m=0}^{\infty} \Sigma_n \; A_{mn} (r_</a)^n (r_>/a)^{-n-1} \; (-1)^m P_n^m(z) \cos(m\varphi)$$

where $r_> = \max(r,a)$ $r_< = \min(r,a)$

whereas my bloid form was this:

$$V_o(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} \; Q_n^m(j\zeta) \; P_n^m(j\zeta_0) P_n^m(\xi) \cos(m\varphi) \qquad (***)$$
$$V_i(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} \; Q_n^m(j\zeta_0) P_n^m(j\zeta) \; P_n^m(\xi) \cos(m\varphi)$$

or

$$V(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} \; Q_n^m(j\zeta_>) P_n^m(j\zeta_<) P_n^m(\xi) \cos(m\varphi)$$

If we are *far away*, we regard both ζ and ζ_0 as being large, so we need these limits which I have in Leg prop doc which are relevant for our situation here:

$$e^{-i\pi\mu} Q_\nu^\mu(z) = \sqrt{\pi} \; [\Gamma(1+\nu+\mu)/\Gamma(\nu+3/2)] \; (2z)^{-\nu-1}$$
$$P_\nu^\mu(z) = (2^\nu/\sqrt{\pi}) \, [\Gamma(\nu+1/2)/\Gamma(1+\nu-\mu)] \, z^\nu$$

Then we have:

$$(-1)^m Q_n^m(j\zeta) \; P_n^m(j\zeta_0) = \sqrt{\pi} \; [\Gamma(1+n+m)/\Gamma(n+3/2)] \; (2[j\zeta])^{-n-1} \; (2^n/\sqrt{\pi}) \, [\Gamma(n+1/2)/\Gamma(1+n-m)] \, [j\zeta_0]^n$$

$$= [\Gamma(1+n+m)/\Gamma(n+3/2)] \; 2^{-1}[j\zeta]^{-n-1} \, [\Gamma(n+1/2)/\Gamma(1+n-m)] \, [j\zeta_0]^n$$

$$= [\Gamma(1+n+m)/(n+1/2)] \; 2^{-1}[j\zeta]^{-n-1} \, [1/\Gamma(1+n-m)] \, [j\zeta_0]^n$$

$$= f(n,m) / (2n+1) \; * [j\zeta]^{-n-1} \, [j\zeta_0]^n$$

$$= -j \, f(n,m) / (2n+1) \; * [\zeta_>]^{-n-1} \, [\zeta_<]^n$$

Then we have for this grouping,

$$F_{nm} \; Q_n^m(j\zeta_>) P_n^m(j\zeta_<) = j (2-\delta_{m,0}) \, (2\pi c_1 \varepsilon K_n(\xi_1))^{-1} \, P_n^m(\xi_0) \; f(n,-m)$$
$$\times \; -j \; f(n,m) / (2n+1) \; * [\zeta_>]^{-n-1} \, [\zeta_<]^n$$

$$= (2-\delta_{m,0}) \, (2\pi c_1 \varepsilon K_n(\xi_1))^{-1} \, P_n^m(\xi_0)/(2n+1) \; * [\zeta_>]^{-n-1} \, [\zeta_<]^n$$

Now, as usual we know that $\zeta = r/a$ so I guess we would say $\zeta_> = r_>/c_1$ and the reverse. Then we have

$$F_{nm} \; Q_n^m(j\zeta_>) P_n^m(j\zeta_<) = (2-\delta_{m,0}) (2\pi c_1 \varepsilon K_n(\xi_1))^{-1} \, P_n^m(\xi_0)/(2n+1) \; * [r_>/c_1]^{-n-1} \, [r_</c_1]^n$$

If we now compare our two forms for the potential

$$V = \Sigma_n \Sigma_{m=0}^{\infty} A_{mn} (r_</a)^n (r_>/a)^{-n-1} \; (-1)^m P_n^m(z) \cos(m\varphi) \qquad // \text{ his}$$

$V(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} Q_n^m(j\zeta_>) P_n^m(j\zeta_<) \quad P_n^m(\xi) \cos(m\varphi)$ // mine

my limit would agree with his cone solution if we could show that

$$(-1)^m A_{mn} (r_</a)^n (r_>/a)^{-n-1} = F_{nm} Q_n^m(j\zeta_>) P_n^m(j\zeta_<)$$

$$A_{mn} = (2-\delta_{m,0}) (2\pi\varepsilon c_1 \zeta_0)^{-1} (-1)^m P_n^m(\xi_0) / [(2n+1)K_n(\xi_1)]$$

which means

$$(2-\delta_{m,0}) (2\pi\varepsilon c_1 K_n(\xi_1))^{-1} P_n^m(\xi_0) (1/\zeta_0) \; */ (2n+1) (r_</a)^n (r_>/a)^{-n-1}$$

$$= (2-\delta_{m,0})(2\pi c_1\varepsilon K_n(\xi_1))^{-1} P_n^m(\xi_0) / (2n+1) * [r_>/c_1]^{-n-1} [r_</c_1]^n$$

or

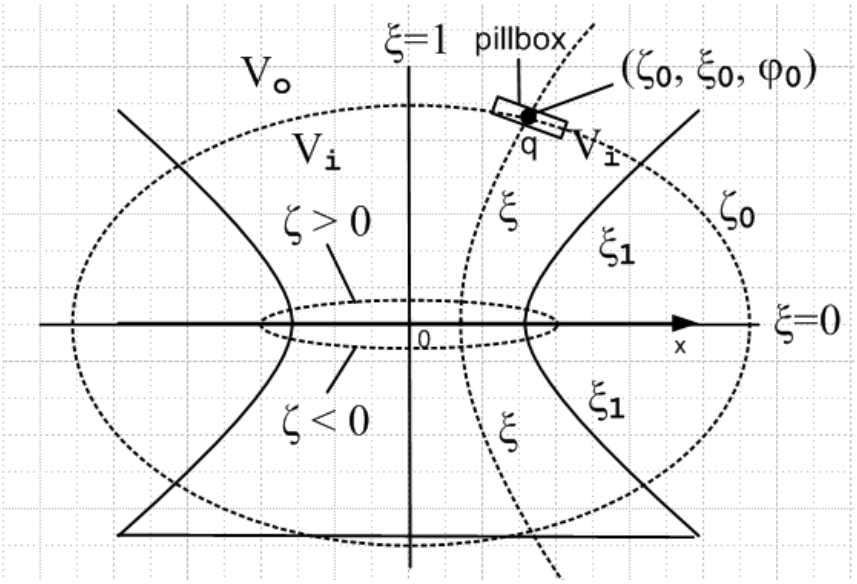
$$(1/\zeta_0) (r_</a)^n (r_>/a)^{-n-1} = [r_>/c_1]^{-n-1} [r_</c_1]^n$$

$$1 = 1$$

I got this finally to work at 10AM Feb 4. So the limit of my solution exactly matches the Smythe Cone solution.

Summary of the solution to the Green's Function problem for an Oblate Hyperboloid.

First, we put the Green's point charge at location $(\xi_0,\zeta_0,\varphi_0)$. Our picture is this:



The picture shows our convention choosing for the oblate coordinates. Although not shown in the picture, the hyperboloids ξ are confocal with focal distance c_1 , which is the symbol Smythe uses.

We start with the following Smythian Form where we set $\varphi_0 = 0$ during our work, and later reset it to a general value:

$$V(\zeta,\xi) = \Sigma_{m=0}^{\infty} \Sigma_n F_{nm} Q_n^m(j\zeta_>) P_n^m(j\zeta_<) P_n^m(\xi) \cos(m\varphi) \qquad \text{// upper bloid region}$$
$$V(\zeta,\xi) = \Sigma_{m=0}^{\infty} \Sigma_n G_{nm} Q_n^m(j\zeta_>) P_n^m(j\zeta_<) P_n^m(\xi) \cos(m\varphi) \qquad \text{// lower bloid region}$$

where $\zeta_> = \max(|\zeta|,\zeta_0)$ $\zeta_< = \min(|\zeta|,\zeta_0)$.

Within each of our two regions (upper and lower) we have a further division into a V_o and V_i subregion as shown. In V_o then we have $Q_n^m(j\zeta) P_n^m(j\zeta_o)$ while in V_i we have $Q_n^m(j\zeta_0) P_n^m(j\zeta)$. On the boundary between these two regions we then have $Q_n^m(j\zeta_o) P_n^m(j\zeta_o)$ so $V_o = V_i$ on the boundary (except at the point charge location). This is the motivation for the Smythian form shown.

The point charge is assumed to be in the upper bloid region where we have F_{nm} coefficients, and we assume that the other side has G_{nm} coefficients. We find that if $G_{nm} = (-1)^m F_{nm}$, then both V and $\partial_n V$ are continuous across the neck disk of the bloid. So this condition basically sets the G_{nm} once we know the F_{nm}. In a sense, the entire lower bloid region is just the tail on the dog, and all the action is in the upper region.

The $Q_n^m(j\zeta_>)$ function decays at large argument, which explains its position in the Smythian forms.

The P and Q functions of imaginary argument used here are off-the-cut Bateman P and Q functions on the usual 0,0L principle sheet, while the $P_n^m(\xi)$ functions are on-the-cut Bateman functions. Smythe uses the same functions for imaginary argument, but for his on-the-cut P and Q functions, Smythe uses the extra $(-1)^m$ non-Condon-Shortley phase factor. In our work here, this factor is never an issue since the $P_n^m(\xi)$ functions always appear bilinearly.

In order to have V = 0 on the hyperboloid ξ_1 , our Σ_n is a sum over the infinite set of discrete n values which cause $P_n^m(\xi_1) = 0$ for each given m. We could enumerate these as $n = n_i(\xi_1,m)$, $i = 1...\infty$. Thus, we should always show the sum in the form $\Sigma_{m=0}^{\infty} \Sigma_n$. (I did not do this in the details above, but realize now that this is the correct way to show it.) These n_i values are the eigenvalues of a Sturm Liouville Legendre ODE problem on the interval ξ in $(\xi_1, 1)$ where 1 is a regular singular point of the ODE, and where ξ_1 is a regular point where we apply the homogeneous boundary condition $P_n^m(\xi_1) = 0$. From SL theory, we know that this produces a set of eigenvalues n_i and orthogonal eigenfunctions. The orthogonality condition is this:

$$\int_{\xi 1}^{1} d\xi\, P_n^m(\xi)\, P_{n'}^m(\xi) = \delta_{n,n'}\, K_n(\xi_1)$$

where $K_n(\xi_1) = -[(1-\xi_1^2)/(2n+1)]\, \partial_\xi P_n^m(\xi_1)\, \partial_n P_n^m(\xi_1)$

which is proven in Smythe, and which proof I followed carefully. It must be that the product of derivatives shown is always negative. I don't have a way yet to compute the $\partial_n P_n^m(\xi_1)$ values, but think using an integral representation for P would be the way to go. There are eigenvalues $n_i < 0$ which we ignore since they are redundant due to the general rule $P_\nu^\mu(\xi) = P^\mu_{-\nu-1}(\xi)$, so all our $n_i(\xi_1,m)$ are > 0.

The eigenvalues m = integer are of course required by the φ nature of the problem. We could include the negative m values in the sum, but using symmetry properties such as f(ν,m) $Q_\nu^{-m}(z) = Q_\nu^m(z)$, the negative m contributions to the sum can be folded into the corresponding positive terms, and then the F_{nm} coefficient accounts for both terms and then only a sum on $m \geq 0$ is required.

By applying the pillbox Gauss's Law at the point charge, we find that

$$(q/[c_1\varepsilon])\, \delta(\varphi-\varphi_0)\, \delta(\xi - \xi_0) = (-j) \Sigma_n \Sigma_{m=0}^{\infty} F_{nm} (-1)^m f(n,m)\ P_n^m(\xi) \cos(m\varphi)$$

where $f(n,m) \equiv [\Gamma(1+m+n)/\Gamma(1-m+n)]$

We then extract the F_{nm} coefficients by using two orthogonality conditions:

$$\int_0^{2\pi} d\varphi \cos(m'\varphi) \cos(m\varphi) = \delta_{m,m'} 2\pi/(2-\delta_{m,0})$$

$$\int_{\xi 1}^{1} d\xi\, P_n^m(\xi)\, P_{n'}^m(\xi) = \delta_{n,n'}\, K_n(\xi_1)$$

and we find that

$$F_{nm} = j(2-\delta_{m,0})\, (q/[2\pi c_1 \varepsilon\, K_n(\xi_1)])\, P_n^m(\xi_0)\, (-1)^m f(n,-m) \qquad G_{nm} = (-1)^m F_{nm}$$

We can then install these coefficients into our Smythian forms to get, for the upper region,

$$V(\zeta,\xi,\varphi) = \Sigma_{m=0}^{\infty}\ \Sigma_n F_{nm}\ Q_n^m(j\zeta_>)\, P_n^m(j\zeta_<)\, P_n^m(\xi) \cos(m\varphi) \qquad \text{// upper bloid region}$$

$$= \Sigma_{m=0}^{\infty}\ \Sigma_n [j(2-\delta_{m,0})\, (q/[2\pi c_1 \varepsilon\, K_n(\xi_1)])\, P_n^m(\xi_0)\, (-1)^m f(n,-m)]\ Q_n^m(j\zeta_>)\, P_n^m(j\zeta_<)\, P_n^m(\xi) \cos(m\varphi)$$

$$= (jq/[2\pi c_1\varepsilon]) \Sigma_{m=0}^{\infty} (2-\delta_{m,0}) \cos(m\varphi) \Sigma_n [P_n^m(\xi_0)\, f(n,-m)]\ Q_n^m(j\zeta_>)\, P_n^m(j\zeta_<)\, P_n^m(\xi)/K_n(\xi_1)$$

Then we finally restore our $\varphi_0 \neq 0$ and our result is then

$$V(\xi,\zeta,\varphi) = (jq/[2\pi c_1\varepsilon]) \Sigma_{m=0}^{\infty} (2-\delta_{m,0})\, (-1)^m \cos(m[\varphi-\varphi_0])\ x \qquad \text{// upper region}$$
$$\Sigma_n f(n,-m)\, Q_n^m(j\zeta_>)\, P_n^m(j\zeta_<)\, P_n^m(\xi_0)\, P_n^m(\xi)/K_n(\xi_1)$$

$$V(\xi,\zeta,\varphi) = (jq/[2\pi c_1\varepsilon]) \Sigma_{m=0}^{\infty} (2-\delta_{m,0}) \cos(m[\varphi-\varphi_0])\ x \qquad \text{// lower region}$$
$$\Sigma_n f(n,-m)\, Q_n^m(j\zeta_>)\, P_n^m(j\zeta_<)\, P_n^m(\xi_0)\, P_n^m(\xi)/K_n(\xi_1)$$

Taken together, we have $V(\xi,\zeta,\varphi) = g(\xi,\zeta,\varphi \mid \xi_0,\zeta_0,\varphi_0)$. I am using the ξ,ζ,φ ordering since this is 1,2,3 in Smythe, though I would normally use a different ordering. Notice that the lower region is missing the factor $(-1)^m$ shown in the upper region, otherwise the formulas are the same.

The above result has the following ancillary information:

c_1 = focal distance of the hyperboloid ξ_1 which we assume is held at V = 0

ε = the constant in V = q/[4$\pi\varepsilon$r] for a point charge (choose your units as you like)
q = the point charge size
$\zeta_>$ = max(|ζ|,ζ_0) and $\zeta_<$ = min(|ζ|,ζ_0)
$K_n(\xi_1)$ = - [(1-ξ_1^2) /(2n+1)] $\partial_\xi P_n^m(\xi_1)\ \partial_n P_n^m(\xi_1)$
f(n,-m) = Γ(1-m+n)/ Γ(1+m+n)
Σ_n is the sum over n = $n_i(\xi_1,m)$ i=1..∞ which are the positive values which make $P_n^m(\xi_1) = 0$

We showed above that when ζ and ζ_0 are very large, so we are far from the neck of the hyperboloid, the hyperboloid is indistinguishable from a cone, and our solution them exactly matches Smythe's spherical-coordinates solution for the cone Green's function (which solution I verified in detail). That cone solution is this: (Smythe 2nd Ed 1950, page 156)

r<a $\quad V_i = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r/a)^n \quad P_n^m(z) \cos(m[\varphi-\varphi_0])$
r<a $\quad V_o = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r/a)^{-n-1} P_n^m(z) \cos(m[\varphi-\varphi_0])$

$$A_{mn} = -(2-\delta_{m,0})(2\pi\varepsilon a)^{-1} P_n^m(\cos\beta)\ \{\sin^2\alpha\ \partial_z P_n^m(\cos\alpha)\ \partial_n P_n^m(\cos\alpha)\}^{-1}$$

where $\cos\beta$ = location of his point charge = ξ_0 for me
$\cos\alpha$ = his cone angle = ξ_1 for me.
z = $\cos\theta$
a = his distance of the point charge from the cone origin
q = 1 = his point charge value
$\zeta = r/c_1$ is the relation between ζ and r when both are large.

In terms of our symbols, then, we have

$$A_{mn} = (2-\delta_{m,0})\,[2\pi\varepsilon c_1 \zeta_0 K_n(\xi_1)(2n+1)]^{-1}\, P_n^m(\xi_0)$$

The cone limit is verified by taking large z limits of the P and Q functions of jζ .

What makes the potential be a real number?

$$V(\xi,\zeta,\varphi) = (jq/[2\pi c_1\varepsilon]) \Sigma_{m=0}^{\infty}(2-\delta_{m,0})(-1)^m \cos(m[\varphi-\varphi_0])\ x \quad \text{// upper region}$$
$$\Sigma_n f(n,-m)\, Q_n^m(j\zeta_>)\, P_n^m(j\zeta_<)\, P_n^m(\xi_0)\, P_n^m(\xi)/K_n(\xi_1)$$

The claim probably is that j $Q_n^m(j\zeta_>)\ P_n^m(j\zeta_<)$ is real for all n,m, but how would I show that was true?

Well, first suppose we write each in terms of F(a,b,c,z^2). We know this will be real for z = $\pm i\zeta$. For Q, we use 134 (4) and we have

$$(-1)^m Q_n^m(j\zeta_>) = \text{real stuff} * e^{+i\pi(m-n-1)/2} (-\zeta_>^2 - 1)^{-m/2}$$
$$+ \quad \text{real stuff} * e^{+i\pi(m-n)/2} (i\,\zeta_>) (-\zeta_>^2 - 1)^{-m/2}$$

$$= \text{real} * (i)^{m-n-1} (-i)^m + \text{real'} * (i)^{m-n} i^1 (-i)^m$$

$= \text{real} * (i)^{m-n-1} (i)^{-m} + \text{real}' * (i)^{m-n} i^{1} (i)^{-m}$

$= \text{real} * (i)^{-n-1} + \text{real}' * (i)^{-n+1}$

$= (i)^{-n-1} [\text{real} + \text{real}' * (i)^{2}] = (i)^{-n-1} * \text{real}''$ // so real/imag is a function of n.

For P we use 126 (22) which says

$P_n^m(j\zeta_<) = \text{real} * (i)^{-m} + \text{real}' * i\,(i)^{-m}$

$= (i)^{-m} [\text{real} + i\,\text{real}'] =$ complex for sure, for any m,

So I have arrived at this conclusion:

$j\,Q_n^m(j\zeta_>)\,P_n^m(j\zeta_<) = i\,(i)^{-n-1} * \text{real}'' * (i)^{-m} [\text{real} + i\,\text{real}']$

So the answer is, $j\,Q_n^m(j\zeta_>)\,P_n^m(j\zeta_<)$ is not in general real. Here is a Maple supporting example:

```
I * LegendreQ(2.3,3,1*I)* LegendreP(2.3,3,3*I);
                                  -62.36240814 - .1209166466 I
```

So magically, it must only be the sum that is real, not the individual terms? I find this to be very mysterious and dubious. This casts a dark shadow on my little project here.

Let's try 126 (23) instead for the P, which has $1/z^2$ as the F arg.

$P_n^m(j\zeta_<) = \text{real}\,(i)^{-n+m-1} (i)^{-m} + \text{real}'\,(i)^{n+m} (i)^{-m}$

$= \text{real}\,(i)^{-n-1} + \text{real}'\,(i)^{n}$

$= (i)^{-n-1} [\text{real} + \text{real}'\,(i)^{2n-1}]$

Suppose n = 2. This then says $(i)^{-2-1} [\text{real} + \text{real}'\,(i)^{4-1}] = i^{-3} [\text{real} + \text{real}'\,i^{3}]$

This keeps making the P(iζ) functions look complex, but the Smythe listings shown them all as being pure:

$$P_1(j\zeta) = j\zeta$$

$$P_1^1(j\zeta) = j(1 + \zeta^2)^{\frac{1}{2}}$$

$$P_2(j\zeta) = -\tfrac{1}{2}(3\zeta^2 + 1)$$

$$P_2^1(j\zeta) = -3(1 + \zeta^2)^{\frac{1}{2}}\zeta$$

$$P_2^2(j\zeta) = -3(1 + \zeta^2)$$

But if m+n = integer ≥ 0, that kills off the first term so if n = 2, you get real'. So you tend to get complex values when you take n away from simple integers, as in our current problem!

Can we show that the charge density on the bloid surface is real?

$$V(\xi,\zeta,\varphi) = (jq/[2\pi c_1\varepsilon]) \Sigma_{m=0}^{\infty}(2-\delta_{m,0}) (-1)^m \cos(m[\varphi-\varphi_0]) \text{ x} \quad \text{// upper region}$$
$$\Sigma_n f(n,-m) Q_n^m(j\zeta_>) P_n^m(j\zeta_<) P_n^m(\xi_0) P_n^m(\xi)/K_n(\xi_1)$$

$$\partial_\xi V(\xi,\zeta,\varphi) = (jq/[2\pi c_1\varepsilon]) \Sigma_{m=0}^{\infty}(2-\delta_{m,0}) (-1)^m \cos(m[\varphi-\varphi_0]) \text{ x} \quad \text{// upper region}$$
$$\Sigma_n f(n,-m) Q_n^m(j\zeta_>) P_n^m(j\zeta_<) P_n^m(\xi_0) P_n^{m\,\prime}(\xi)/K_n(\xi_1)$$

$$\partial_\xi V(\xi_1,\zeta,\varphi) = (jq/[2\pi c_1\varepsilon]) \Sigma_{m=0}^{\infty}(2-\delta_{m,0}) (-1)^m \cos(m[\varphi-\varphi_0]) \text{ x} \quad \text{// upper region}$$
$$\Sigma_n f(n,-m) Q_n^m(j\zeta_>) P_n^m(j\zeta_<) P_n^m(\xi_0) P_n^{m\,\prime}(\xi_1)/K_n(\xi_1)$$

Same mystery. Looks complex to me, not real.

Construction of Theorems A and A1. Suppose we have an ODE EV problem where the coefficient functions are all real. This is true, for example, of the Legendre equation in z, and also in ζ where z = iζ. Here it is in z for example

$$L = (1-z^2) D_z^2 - 2z D_z + [\, n(n+1) - m^2/(1-z^2)] \qquad Lu = 0 \qquad \text{Bateman p 121}$$

You can see that replacing z = iζ keeps everything real in the ζ world. So, suppose we find a solution ψ of this equation which is a complex function, such as $e^{im\varphi}$ is a complex solution of $-D^2u = m^2u$, Then we write ψ = Re(ψ) + i Im(ψ). We have $L = \Sigma a_i(z) \partial_z^i$ where a_i are real. then

$$L\psi = \lambda\psi \quad \Rightarrow \quad L[\mathrm{Re}(\psi) + i\,\mathrm{Im}(\psi)] = \lambda\,[\mathrm{Re}(\psi) + i\,\mathrm{Im}(\psi)]$$

Now it must be that L[Re(ψ)] = real. That is, if a function g(z) = Re(ψ(z)) is real for all z, then all derivatives are real, etc. Therefore, we can split the ODE above into these two parts (λ we know real)

$$L[\mathrm{Re}(\psi)] = \lambda\,[\mathrm{Re}(\psi)]$$
$$L[\mathrm{Im}(\psi)] = \lambda\,[\mathrm{Im}(\psi)]$$

Theorem A: If f(z) solves ODE Lu = λu and L is "real", then the Re(f) and Im(f) also solve Lu = λu .

But: If f(iz) solves ODE Lu = λu and L is "real", *then* what?

Then $Lf(iz) = \lambda f(iz) \Rightarrow L(\mathrm{Re}[f(iz)] + i\,\mathrm{Im}[f(iz)]) = \lambda(\mathrm{Re}[f(iz)] + i\,\mathrm{Im}[f(iz)])$, this much is true.

Now what can we say about L(Re[f(iz)] ? It is NOT real because first derivatives create a factor of i. For example here is a "real L" acting on Re[f(iz)],

$$(D^2 + D + 2)\,\mathrm{Re}[f(iz)] = -\mathrm{Re}[f(iz)]'' + i\,\mathrm{Re}[f(iz)]' + 2\,\mathrm{Re}[f(iz)]$$

It is true that Re[f(iz)]" is real, and so is Re[f(iz)]' and so is Re[f(iz)], but you see the i sitting there, so for sure we cannot say L(Re[f(iz)]) is real if L is real. Thus, we *cannot* "partition" our equation

$$L(\text{Re}[f(iz)] + i\,\text{Im}[f(iz)]) = \lambda(\text{Re}[f(iz)] + i\,\text{Im}[f(iz)])$$

into two parts, one of which is $L(\text{Re}[f(iz)]) = \lambda(\text{Re}[f(iz)])$ and the other with Im.

Theorem A1: If f(iz) solves ODE Lu = λu and L is "real", then if L includes a first derivative, we will find that Re[f(iz)] and Im[f(iz)] do NOT solve Lu = λu.

Example: We know that $L_\zeta\, Q_n^m(j\zeta) = n(n+1)\, Q_n^m(j\zeta)$ where L_ζ is real. Whether this is an EV problem which determines the spectrum of n , or whether it is just a cathode follower and n is some real non-integral value, in either case we will find that $\text{Re}[Q_n^m(j\zeta)]$ and $\text{Im}[Q_n^m(j\zeta)]$ do NOT satisfy the equation $L_\zeta u = n(n+1)u$. Therefore, these things are not factors in possible oblate "atoms".

Nevertheless, if some complex V(ζ) is C^1 on some interval, then so is ReV(ζ). And if V = 0 on some surface, then ReV = 0 as well on that surface.

In our pillbox stuff, everything was manifestly real due to the Wronskian simplification. But I would not want to put Re(P) and Re(Q) in that Wronskian. I would rather do everything I did above.

So how would you modify the "program" above? I started with this:

$$V_o(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} F_{nm}\, Q_n^m(j\zeta)\, P_n^m(j\zeta_0)\, P_n^m(\xi)\cos(m\varphi) \qquad (***)$$
$$V_i(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} F_{nm}\, Q_n^m(j\zeta_0)\, P_n^m(j\zeta)\, P_n^m(\xi)\cos(m\varphi)$$

and I found some F_{nm} that worked, and they are pure imaginary.

$$F_{nm} = j(2-\delta_{m,0})\,(q/[2\pi c_1 \varepsilon\, K_n(\xi_1)])\, P_n^m(\xi_0)\,(-1)^m f(n,-m)$$

So lets write $F_{nm} = j\,H_{nm}$ where the H coefficients are purely real. Then we have

$$V_o(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} H_{nm}\,[j\,Q_n^m(j\zeta)\, P_n^m(j\zeta_0)]\, P_n^m(\xi)\cos(m\varphi) \qquad (***)$$
$$V_i(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} H_{nm}\,[j\,Q_n^m(j\zeta_0)\, P_n^m(j\zeta)]\, P_n^m(\xi)\cos(m\varphi)$$

Now look back at the Gauss business. Using $-j\,F_{nm} = H_{nm}$ we had

$$(q/[c_1\varepsilon])\,\delta(\varphi-\varphi_0)\,\delta(\xi-\xi_0) = \Sigma_n \Sigma_{m=0}^{\infty} H_{nm}\,(-1)^m f(n,m)\, P_n^m(\xi)\cos(m\varphi)$$

Everything here is REAL. The charge density of the point charge is real, and we do our orthogs and find the real numbers H_{nm} . This came from:

$$\partial_\zeta V_i(\zeta,\xi,\varphi) - \partial_\zeta V_o(\zeta,\xi,\varphi) = (1+\zeta_0^2)^{-1} \Sigma_n \Sigma_{m=0}^{\infty} H_{nm}\,(-1)^m\,[\Gamma(1+m+n)/\Gamma(1-m+n)]\, P_n^m(\xi)\cos(m\varphi)$$

Now suppose we take the Re and Im parts of both sides. We now have a PDE question. Suppose we have

$$f(x,y) = \mathrm{Re}(f(x,y)) + i\,\mathrm{Im}(f(x,y))$$

$$\partial_x f(x,y) = \partial_x \mathrm{Re}(f(x,y)) + i\,\partial_x \mathrm{Im}(f(x,y))$$

Since the function g(x) = Re(f(x,y)) is real for all values of x, for any fixed y, $\partial_x g(x)$ must be real. And in fact this derivative is real for all y. Therefore, we can write $V_i = \mathrm{Re}(V_i) + i\,\mathrm{Im}(V_i)$ and for V_o. Then

$$\partial_\zeta V_i(\zeta,\xi,\varphi) - \partial_\zeta V_o(\zeta,\xi,\varphi) =$$

$$(\partial_\zeta \mathrm{ReV}_i(\zeta,\xi,\varphi) - \partial_\zeta \mathrm{ReV}_o(\zeta,\xi,\varphi)) + i\ (\partial_\zeta \mathrm{ImV}_i(\zeta,\xi,\varphi) - \partial_\zeta \mathrm{ImV}_o(\zeta,\xi,\varphi))$$

Then our equation above becomes:

$$(\partial_\zeta \mathrm{ReV}_i(\zeta,\xi,\varphi) - \partial_\zeta \mathrm{ReV}_o(\zeta,\xi,\varphi)) =$$
$$(1+\zeta_0^2)^{-1} \Sigma_n \Sigma_{m=0}^{\infty} H_{nm} (-1)^m [\Gamma(1+m+n)/\Gamma(1-m+n)]\ P_n^m(\xi)\cos(m\varphi)$$

$$(\partial_\zeta \mathrm{ImV}_i(\zeta,\xi,\varphi) - \partial_\zeta \mathrm{ImV}_o(\zeta,\xi,\varphi)) = 0$$

This tells me that $\mathrm{ReV}_i(\zeta,\xi,\varphi)$ is a *candidate* real solution to this problem because it "sees" the point charge via the pillbox, whereas $\mathrm{ImV}_i(\zeta,\xi,\varphi)$ is NOT a candidate solution since it sees no charge at all, It would *seem* then that $u = \mathrm{ImV}_i(\zeta,\xi,\varphi)$ would be a solution of Laplace which has u = 0 on the bloid. But consider:

$$\mathrm{Re}\ V_i(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} H_{nm}\ \mathrm{Re}\,[\,j\,Q_n^m(j\zeta_0)\,P_n^m(j\zeta)]\,P_n^m(\xi)\cos(m\varphi)$$

Is this a solution to the Laplace equation? I think that is extremely unlikely, because we can write each partial wave here as A Re $[P_n^m(j\zeta)]$ + B Im $[P_n^m(j\zeta)]$ and we just showed above that things like Re $[P_n^m(j\zeta)]$ do *not* solve the ζ separated equation, and so Re $[P_n^m(j\zeta)]\,P_n^m(\xi)\cos(m\varphi)$ does *not* solve the Laplace equation. So this whole avenue leads nowhere I am afraid.

The complex nature of j $Q_n^m(j\zeta_0)$ $P_n^m(j\zeta)$] for general real n.

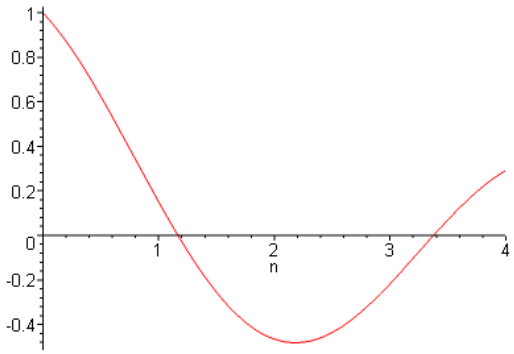
Well, I am now wondering about the above some more. Suppose we have a complete set of real functions on some interval $\varphi_n(x)$. Then suppose $f(x) = \Sigma_n a_n \varphi_n(x)$ where f(x) is real. Can the a_n be complex in this situation? We would have Im f = 0 = $\Sigma_n \mathrm{Im}(a_n)\,\varphi_n(x)$, then orthogonality would tell us $\mathrm{Im}(a_n) = 0$. So this same argument holds for a complete set $\varphi_{n,m}(\xi,\varphi)$. Therefore, if $V_o(\zeta,\xi)$ is in fact real, then it must be true that every single {j $Q_n^m(j\zeta)$ $P_n^m(j\zeta_0)$} must be separately real.

Is this true for m = 0? Well, I can find a value of n for which it is not true:

```
I*LegendreQ(1.2,0,I*2)*LegendreP(1.2,0,I*1):evalf(%);
                    .05939268641 + .001646033692 I
```

Now any n you pick will be the zero of $P_n(\xi_1)$ for some ζ_1 . For example, if $\xi_1 = .15$ we find

```
plot(Re(LegendreP(n,0,.15)),n=0..4);
```



so we could tune ξ_1 as needed to get a zero exactly at 1.2. But then j $Q_n(j2)$ $P_n(j1) \neq$ real as shown above. So I am changing my tune now. I think when I write

$$V_o(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} H_{nm} \{j Q_n^m(j\zeta) P_n^m(j\zeta_0)\} P_n^m(\xi) \cos(m\varphi) \qquad (***)$$

that V_o really is complex. After all, I could do double orthogonality to find

$$H_{nm} \{j Q_n^m(j\zeta) P_n^m(j\zeta_0)\} \sim \int d\varphi \int d\xi P_n^m(\xi) \cos(m\varphi) V_o(\zeta,\xi)$$

If V_0 really were real, then the RHS is real, but I know the LHS is not real! So our Smythian form seems to be generating non-real potentials. This is a problem. The potential is caused by the point charge and the induced charge and it must be real.

Again, consider

$$\text{Im } V_i(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} H_{nm} \text{ Im } [j Q_n^m(j\zeta_0) P_n^m(j\zeta)] P_n^m(\xi) \cos(m\varphi)$$

If we apply orthog go this we are going to get

$$\int \int \text{Im } V_i(\zeta,\xi) P_n^m(\xi) \cos(m\varphi) \sim H_{nm} \text{ Im } [j Q_n^m(j\zeta_0) P_n^m(j\zeta)]$$

and since we KNOW that Im $[j Q_n^m(j\zeta_0) P_n^m(j\zeta)] \neq 0$ (we just did an example above), then we know that we cannot possibly have Im $V_i(\zeta,\xi) = 0$.

Big Problem: Our Smythian form generates complex potentials, not real ones.

Idea #1: Maybe I need to imitate the $(r/a)^{-n-1}$ idea of the cone in spherical solution, something like this for a Smythian form:

$$V_o(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} L_{nm} [Q_n^m(j\zeta)/ Q_n^m(j\zeta_0)] [P_n^m(j\zeta_0)/ P_n^m(j0+)] P_n^m(\xi) \cos(m\varphi) \qquad (***)$$
$$V_i(\zeta,\xi) = \Sigma_n \Sigma_{m=0}^{\infty} L_{nm} [Q_n^m(j\zeta_0)/ Q_n^m(j0+)] [P_n^m(j\zeta)/ P_n^m(j\zeta_0)] P_n^m(\xi) \cos(m\varphi)$$

I know this does not work, but something that has ratios everywhere so maybe we can meet our new "hidden boundary condition" that the potential has to be real! Maybe the entire Smythe method has to be thrown out. I don't see him doing any problems like this in his book (let's check one more time tomorrow).

A Paradox to Worry About.

What happens if we let our Green's point charge slide down the dotted ξ line until it finds itself in the hole disk? So it ends up at $\zeta_0 = 0$. We start with

$$V(\xi,\zeta,\varphi) = (jq/[2\pi c_1\varepsilon]) \Sigma_{m=0}^{\infty}(2-\delta_{m,0}) (-1)^m \cos(m[\varphi-\varphi_0]) \; x \qquad \text{// upper region}$$
$$\Sigma_n f(n,-m) Q_n^m(j\zeta_>) P_n^m(j\zeta_<) P_n^m(\xi_0) P_n^m(\xi)/K_n(\xi_1)$$

But of course we now have $\zeta_> = \max(|\zeta|,\zeta_0) = |\zeta|$ and $\zeta_< = \min(\zeta,\zeta_0) = 0$ so this becomes

$$V(\xi,\zeta,\varphi) = (jq/[2\pi c_1\varepsilon]) \Sigma_{m=0}^{\infty}(2-\delta_{m,0}) (-1)^m \cos(m[\varphi-\varphi_0]) \; x \qquad \text{// upper region}$$
$$\Sigma_n f(n,-m) Q_n^m(j\,|\zeta|) P_n^m(j0+) P_n^m(\xi_0) P_n^m(\xi)/K_n(\xi_1)$$

Now, we know that

$$P_n^m(z=0\pm) = 2^m\sqrt{\pi}\; e^{\mp i n m/2} / [\, \Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2)\,] \qquad \text{// p 126 (22)}$$

Since n takes on weird values in its spectrum, these are not likely to cause poles in the denominator here, so in general all these $P_n^m(z=0\pm)$ will be non-zero (OK, some might be zero, but certainly not all). So specifically we have this result:

$$V(\xi,\zeta,\varphi) = (jq/[2\pi c_1\varepsilon]) \Sigma_{m=0}^{\infty}(2-\delta_{m,0}) (-1)^m \cos(m[\varphi-\varphi_0]) \; x \qquad \text{// upper region}$$
$$\Sigma_n f(n,-m) Q_n^m(j\,|\zeta|) \{\, 2^m\sqrt{\pi}\; e^{-i n m/2} / [\, \Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2)\,] \,\} P_n^m(\xi_0) P_n^m(\xi)/K_n(\xi_1)$$

For the lower region, we have this same formula but with the $(-1)^m$ missing. The paradox is this: for our Green's charge right in the hole disk, we would expect V in the lower region to be the *same* as V in the upper region, with $\zeta \rightarrow -\zeta$ in the formula, but we have this $(-1)^m$ factor sitting there. This paradox would be resolved if somehow the sum contained only even m terms. Physically you would think this would be the case. But what could make that happen? The only factor in our V formula that can detect that the Green's charge has gone down into this disk is the factor $P_n^m(j\zeta_<)$. So there is our real paradox. Is our formula just not valid in this limit? Are the n_i values magically integers?

There is no use worrying about the iris with a charge in the hole, if the general case is illogical.

Recall this: when I tried to compute the potential of a charged iris, I could not start with the asymmetric form Smythe used for his E field on one side problem. I had to start with a more general form that allowed for a symmetric solution. So maybe in this problem, we have to start from scratch to handle a Green's function exactly in the hole plane and accept that the limit just does not work.

The Hole in Plate Limit

The bloid folds down and becomes an iris whose hole has radius c_1. The limiting bloid has $\xi_1 = 0$. This means the ξ SL problem is now on the interval (0,1) which is NOT the same as (-1,1).

For me, the big question is this SL problem. Is there some simplification of it?

(a) The spectrum of n. Our eigenvalues are now the n which make $P_n^m(\xi_1 = 0) = 0$. But we know a lot about these values. From our Leg prop doc,

$$P_\nu^\mu(x=0) = 2^\mu\sqrt{\pi} \; / \; [\, \Gamma(1/2 - \nu/2 - \mu/2)\, \Gamma(1 + \nu/2 - \mu/2) \,]$$

$$P_n^m(\xi_1 = 0) = 2^m\sqrt{\pi} \; / \; [\, \Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2) \,]$$

So we will get $P_n^m(\xi_1 = 0) = 0$ wherever either of these gamma functions has a pole! Consider first the leftmost gamma function. It has poles here

$1/2 - n/2 - m/2 = 0,-1,-2....$

$1 - n - m = 0,-2,-4....$

$- n - m = -1,-3,-5....$

$n + m = 1,3,5....$ or $n+m =$ positive odd integer

$n = -m +$ odd $\quad$ odd = 1,3,5...

The other gamma function is the $n \rightarrow -n-1$ of the first, so it has poles here

$-n-1 = -m +$ odd

$-n = -m +$ even $\quad$ even = 2,4,6.. (but not 0)

$n = m -$ even

So we can <u>summarize as follows</u>. $P_n^m(\xi_1 = 0) = 0$ for the following spectrum of n values:

n1 = -m + odd $\quad$ odd = 1,3,5...
n2 = m - even $\quad$ even = 2,4,6.. (but not 0)

Let's "plot" this spectrum for some positive values of m :

Start with m = 4:

```
      *   *  -m   *   *   *   0   *   *   *   m   *   *   *   *
n1    .   .   .   x   .   x   .   x   .   x   .   x   .   x
n2    x   .   x   .   x   .   x   .   x   .   .   .   .   .   .
n     x   .   x   x   x   x | x   x   x   x   .   x   .   x   .
```

Then do m = 3:

```
      *   *   *  -m   *   *   0   *   *   m   *   *   *   *   *
n1    .   .   .   .   x   .   x   .   x   .   x   .   x   .   x
n2    .   x   .   x   .   x   .   x   .   .   .   .   .   .   .
n     .   x   .   x   x   x | x   x   x   .   x   .   x   .   x
```

Then do m = 2:

```
      *   *   *   *  -m   *   0   *   m   *   *   *   *   *   *
n1        .   .   .   .   x   .   x   .   x   .   x   .   x   .
n2    x   .   x   .   x   .   x   .   .   .   .   .   .   .
n     x   .   x   .   x   x | x   x   .   x   .   x   .   x
```

Then do m = 1:

```
      *   *   *   *   *  -m   0   m   *   *   *   *   *   *   *
n1        .   .   .   .   .   x   .   x   .   x   .   x   .   x
n2    .   x   .   x   .   x   .   .   .   .   .   .   .   .   .
n     .   x   .   x   .   x | x   .   x   .   x   .   x   .   x
```

Then do m =0:

```
      *   *   *   *   *   *  0=m  *   *   *   *   *   *   *   *
n1            .   .   .   .   .   x   .   x   .   x   .   x   .
n2    x   .   x   .   x   .   .   .   .   .   .   .   .   .   .
n     x   .   x   .   x   . | .   x   .   x   .   x   .   x   .
```

For example, this says that $P_0^0(0) \neq 0$ which is correct, since $P_0 = 1$. As expected, in all cases we have the spectrum mirroring around the non-spectral value n = -1/2,

Now, let's keep only the n values to the right of n = -1/2 because we know the ones to the left are redundant. How do we express the spectrum for general m?

<u>The non-redundant spectrum of n for a given m ≥ 0 value:</u>

m ≥0, even	n = all positive odds + [evens in range (0,m-1)]	**n spectrum**
m= 0	n = all positive odds	
m > 0, odd	n = all positive evens + [odds in range (0,m-1)]	

This certainly seems unusual, but I think it is correct.

Now what can we say about our orthogonality?

$$\int_{\xi 1}^{1} d\xi\, P_n^m(\xi)\, P_{n'}^m(\xi) = \delta_{n,n'}\, K_n(\xi_1) \qquad \text{// for n,n' in the spectrum}$$

where $K_n(\xi_1) = -[(1-\xi_1^2)/(2n+1)]\, \partial_\xi P_n^m(\xi_1)\, \partial_n P_n^m(\xi_1)$

$$\int_{0}^{1} d\xi\, P_n^m(\xi)\, P_{n'}^m(\xi) = \delta_{n,n'}\, K_n(0)$$

where $K_n(0) = -[1/(2n+1)]\, \partial_\xi P_n^m(0)\, \partial_n P_n^m(0)$

I am hoping there is some simplification here which allows a simple form for the K's. The integral does not appear in GR in any obvious place. We want to know this:

$$\int_{0}^{1} d\xi\, [P_n^m(\xi)]^2 = K_n(0) = ???$$

We know that

$P_\nu^\mu(-x) = P_\nu^\mu(x)(-1)^{\nu+\mu}$ // $\nu+\mu$ = integer

$P_n^m(-x) = P_n^m(x)(-1)^{n+m}$ // $n+\mu$ = integer

Thus we can say that

$$\int_{-1}^{0} d\xi\, [P_n^m(\xi)]^2 = \int_{0}^{1} d\xi'\, [P_n^m(-\xi')]^2 = \int_{0}^{1} d\xi'\, [P_n^m(\xi')]^2 = \int_{0}^{1} d\xi\, [P_n^m(\xi)]^2$$

Then

$$\int_{-1}^{1} d\xi\, [P_n^m(\xi)]^2 = 2 \int_{0}^{1} d\xi\, [P_n^m(\xi)]^2$$

All we are saying is that the function $f(\xi) = [P_n^m(\xi)]^2$ is an even function of ξ for all n and m both integers. Therefore we use this fact

$$\int_{-1}^{1} dz\, P_n^m(z) P_k^m(z) = \delta_{n,k}\, (n+1/2)^{-1}\, [(n+m)!/(n-m)!] \qquad n,k = m, m+1, m+2 \ldots\ldots \infty$$

$$= \delta_{n,k}\, k_n^m \qquad k_n^m = (n+1/2)^{-1}\, (n+m)!/(n-m)!$$

which says

$$\int_{-1}^{1} dz\, [P_n^m(z)]^2 = k_n^m = (n+1/2)^{-1}\, (n+m)!\,/\,(n-m)! = 2\, f(n,m)\,/(2n+1)$$

and this we learn that

$$\int_{0}^{1} d\xi\, [P_n^m(\xi)]^2 = (1/2) \int_{-1}^{1} dz\, [P_n^m(z)]^2 = f(n,m)\,/(2n+1) = K_n(0)$$

so we can write $1/K_n(0) = (2n+1)\, f(n,-m)$. This is very good news indeed.

So, let's take a look at our big solution in this limit:

$$V(\xi,\zeta,\varphi) = (jq/[2\pi c_1 \varepsilon]) \sum_{m=0}^{\infty}(2-\delta_{m,0})\,(-1)^m \cos(m[\varphi-\varphi_0])\ x \qquad \text{// upper region}$$
$$\sum_n f(n,-m)\, Q_n^m(j\zeta_>)\, P_n^m(j\zeta_<)\, P_n^m(\xi_0)\, P_n^m(\xi)/K_n(\xi_1 = 0)$$

$$= (jq/[2\pi c_1 \varepsilon]) \sum_{m=0}^{\infty}(2-\delta_{m,0})\,(-1)^m \cos(m[\varphi-\varphi_0])\ x \text{ // upper region}$$
$$\sum_n (2n+1)[f(n,-m)]^2\, Q_n^m(j\zeta_>)\, P_n^m(j\zeta_<)\, P_n^m(\xi_0)\, P_n^m(\xi)$$

So this is simpler only inasmuch as the sum Σ_n is over certain integer values (see spectrum above), and the hard-to-compute $1/K_n(\xi_1)$ factor has been replaced by something very easy to compute. So this is the result for the iris with the Green's point charge at an arbitrary location. [except it is complex]

Now there are several specific iris locations of interest for the point charge. A big location of interest for me is when the point charge sits right inside the hole, which means $\zeta_0 = 0$. We just let the charge drop down slowly onto the positive side of the center disk. In this case, we have

$$\zeta_> = \max(\zeta,0) = \zeta \qquad \text{and} \quad \zeta_< = \min(\zeta,0) = 0$$

so we get

$$V(\xi,\zeta,\varphi) = (jq/[2\pi c_1 \varepsilon]) \sum_{m=0}^{\infty}(2-\delta_{m,0})\,(-1)^m \cos(m[\varphi-\varphi_0])\ x \qquad \text{// upper region}$$
$$\sum_n (2n+1)[f(n,-m)]^2\, Q_n^m(j\zeta)\, P_n^m(j0+)\, P_n^m(\xi_0)\, P_n^m(\xi)$$

We then call upon our Leg prop doc which says:

$$P_\nu^\mu(z=0\pm) = 2^\mu\sqrt{\pi}\ e^{\mp i\pi\mu/2} / [\ \Gamma(1/2 - \nu/2 - \mu/2)\, \Gamma(1 + \nu/2 - \mu/2)\ ] \qquad \text{// p 126 (22)}$$

$$P_n^m(j0+) = 2^m\sqrt{\pi}\ e^{-i\pi m/2} / [\ \Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2)\ ] \qquad \text{// p 126 (22)}$$

But recall our result above where we figured out the spectrum for n,

$$P_n^m(\xi_1 = 0) = 2^m\sqrt{\pi}\ / [\ \Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2)\ ]$$

This tells us that for every point in the n spectrum, we will have $P_n^m(j0+) = 0$. Therefore, we learn the interesting fact that $V \equiv 0$ everywhere! So something is wrong with our limiting process!