Oblate bloid with p and q attempt 3 PhL 2.14.10

Note added 2.28.10. This attempt was going well, but I made an error (see blue and red colored text below), and then at once jumped off to the attempt 4 version where I fixed the error and went on the get a good solution.

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1. The p and q functions. The goal is to use these "real atom" functions in the ζ dimension,

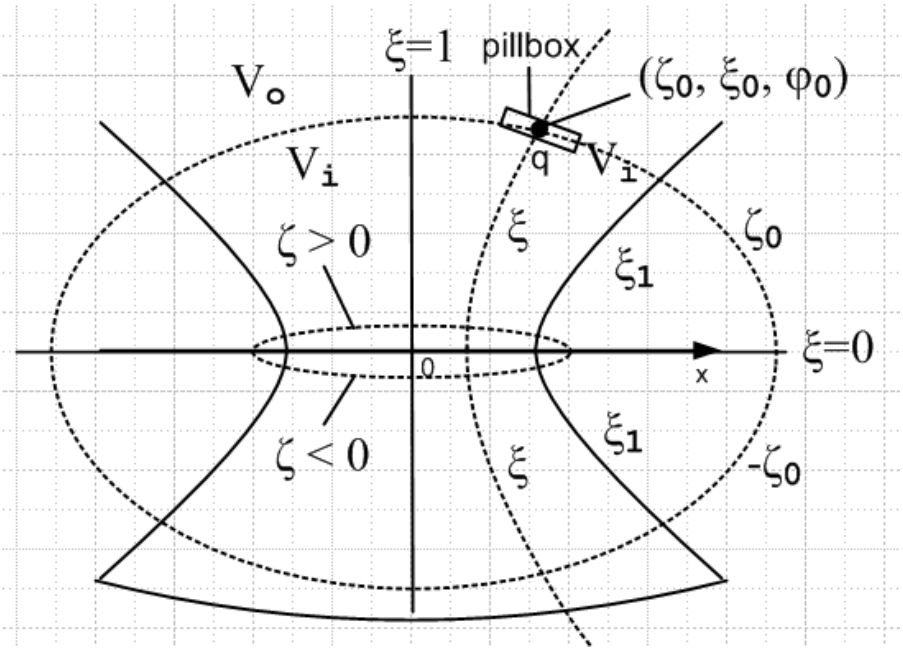
$$q_n^m(\zeta) = (-i)^{2m} (\pm i)^{n+1} Q_n^m(i\zeta)$$

$$p_n^m(\zeta) = [(\mp i)^{-m} P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$$

$$= (\pm i)^{-n} 2\cos[(n+m)\pi/2] P_n^m(i\zeta) - (\pm i)^{-m/2} (-i)^{2m} (2/\pi) \sin\pi(n+m) Q_n^m(i\zeta)$$

These functions are real when n and m are and ζ are real. You must use the upper sign when Re(ζ) > 0, otherwise the lower sign.

2. The Smythian form, picture, and matching in the Neck . First, the picture



We shall try a more general form than we used in Attempt #2. Let's first concentrate just on the upper bloid region and try this:

Upper half bloid:

$V_o(\zeta,\xi) = \Sigma_{nm} [\quad A_{nm}\, q_n^m(\zeta)]\ P_n^m(\xi) \cos(m\varphi)$

$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ P_n^m(\xi) \cos(m\varphi)$ //upper $\zeta \geq 0$

We must have the q alone for the outside part due to large ζ decay, so this is about as general as we can get.

(a) get matching at the boundary between upper Vi and Vo . Continuity at the boundary then tells us that

$V_o(\zeta_0,\xi) = V_i(\zeta_0,\xi)$

$\Sigma_{nm} [A_{nm}\, q_n^m(\zeta_0)]\ P_n^m(\xi) \cos(m\varphi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ P_n^m(\xi) \cos(m\varphi)$

Matching terms tells us that

$A_{nm}\, q_n^m(\zeta_0) = B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)$

So multiply and divide the upper form like this:

$V_o(\zeta,\xi) = \Sigma_{nm} [\quad A_{nm}\, q_n^m(\zeta_0) q_n^m(\zeta)]\ P_n^m(\xi) \cos(m\varphi) / q_n^m(\zeta_0)$

$= \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi) / q_n^m(\zeta_0)$

Then we have:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi) / q_n^m(\zeta_0)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)\ ]\ P_n^m(\xi) \cos(m\varphi)$$

Now define $B_{nm}' = B_{nm}/ q_n^m(\zeta_0)$ and the same for C, then we have

$$V_o(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta_0) + C'_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi)$$

Now removed the primes to reduce clutter, and we arrive at this Symthian form:

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad //\text{upper} \quad \zeta \geq 0$$

This is what it has to *look like* to match at the boundary between V_o and V_i .

(b) the lower bloid form. Now, by the same argument, the lower bloid must have this form (new meaning for primes):

Lower half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(-\zeta_0) + C'_{nm}\ q_n^m(-\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad //\text{lower} \quad \zeta \leq 0$$

The matching surface here has $\zeta = -\zeta_0$, so have installed that value.

(c) Smythian form for the upper and lower bloids:

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad //\text{upper} \quad \zeta \geq 0$$

Lower half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(-\zeta_0) + C'_{nm}\ q_n^m(-\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad //\text{lower} \quad \zeta \leq 0$$

(d) Match the potential at the neck which is $\zeta = 0$. We have at the neck:

$$V_i(0+,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad //\text{upper}$$
$$V_i(0-,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(0-) + C'_{nm}\ q_n^m(0-)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad //\text{lower}$$

But our p and q functions are all perfectly symmetric, so rewrite the lower line

$$V_i(0+,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+)]\ \ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper}$$
$$V_i(0-,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(0+) + C'_{nm}\ q_n^m(0+)]\ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower}$$

The condition is then:

$$B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+) = B'_{nm}\ p_n^m(0+) + C'_{nm}\ q_n^m(0+)$$

$$(B_{nm} - B'_{nm})\ p_n^m(0+) = -(C_{nm} - C'_{nm})\ q_n^m(0+)$$

The good news is that we are not forced right here to have B = B' and C = C'. If we *were*, then due to the perfect symmetry of our P and Q functions, we would have the exact same potential on both upper and lower and we know this cannot be true since the Green's Charge is only in the upper. So, to coin a phrase, once again "we are still alive".

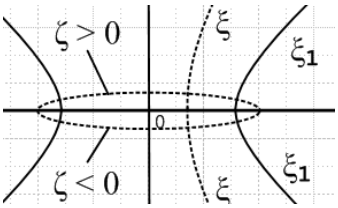
(e) Match the ζ derivative of the potential at the neck which is ζ = 0.

This is going to be the tricky one I think. We start with

$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper}\ \ \zeta \geq 0$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)]\ q_n^m(-\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower}\ \ \zeta \leq 0$$

As usual, we pause to review the scale factors to make sure we don't trip up:

$$h_1 = c_1 \sqrt{\xi^2 + \zeta^2} / \sqrt{1-\xi^2} \qquad q_1, q_2, q_3 = \xi, \zeta, \varphi$$
$$h_2 = c_1 \sqrt{\xi^2 + \zeta^2} / \sqrt{1+\zeta^2}$$
$$h_3 = c_1 \sqrt{1+\zeta^2}\ \sqrt{1-\xi^2}$$



When we talk about the derivative normal to the neck disk plane, just above the plane we have our $\hat{\mathbf{n}}$ vector pointing up and we say $\partial_n = (1/h_2)\,\partial_\zeta$. Both n and ζ increase "going up". From above we know that $h_2 = c_1\xi$ at the neck, where ξ is a positive number in the range $(\xi_0, 1)$. So $\partial_n = (c_1\xi)^{-1}\,\partial_\zeta$.

Now what about just below the neck? As we move up, ζ gets less negative so dζ is positive, and dn is also positive, so we have the exact same rule: $\partial_n = (c_1\xi)^{-1}\,\partial_\zeta$. And of course ξ is the same on both sides of the neck. So we can ignore the factor $(c_1\xi)^{-1}$ in our match since it is the same on both sides, and then our matching condition is to make $\partial_\zeta V_i$ be the same on both sides of the neck! We have:

$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper}\ \ \zeta \geq 0$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)]\ q_n^m(-\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower}\ \ \zeta \leq 0$$

$$\partial_\zeta V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m{}'(\zeta) + C_{nm}\ q_n^m{}'(\zeta)]\ \ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \quad \text{//upper} \quad \zeta \geq 0$$
$$\partial_\zeta V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m{}'(\zeta) + C'_{nm}\ q_n^m{}'(\zeta)]\ \ q_n^m(-\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \quad \text{//lower} \quad \zeta \leq 0$$

But we want this match to occur at $\zeta = 0$. To allow for the possibility that derivatives are different above and below the plane of the neck, we write [we also replace $q_n^m(-\zeta_0) = q_n^m(\zeta_0)$ in the second line]

$$\partial_\zeta V_i(0+,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m{}'(0+) + C_{nm}\ q_n^m{}'(0+)]\ \ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \quad \text{//upper}$$
$$\partial_\zeta V_i(0-,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m{}'(0-) + C'_{nm}\ q_n^m{}'(0-)]\ \ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \quad \text{//lower} \quad \zeta \leq 0$$

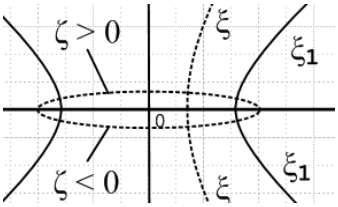
Since there is no surface charge layer in the neck, we require these to be the same. Therefore our matching condition is then

$$B_{nm}\ p_n^m{}'(0+) + C_{nm}\ q_n^m{}'(0+) = B'_{nm}\ p_n^m{}'(0-) + C'_{nm}\ q_n^m{}'(0-)$$

Now an unusual thing happens. Because p and q are symmetric, all four derivatives are 0!! Therefore, we obtain no condition at all here!! The condition is satisfied for any B, B', C, C' . So once again, we are "still alive" in our Attempt #3. So we move on to the next matching requirement:

The above blue is wrong. Switching now to Attempt 4. -PhL 2.21.10

(f) Match the ξ derivative of the potential at the neck which is ζ = 0. Here we have $\partial_n = (1/h_1)\,\partial_\xi$ and $h_1 = c_1\xi / \sqrt{1-\xi^2}$ which is a positive real number.



This is a "radial" ρ-like derivative on the disk. In our picture on the right side of the center axis we have, as we move radially inward, we have $d\xi > 0$, so we can think of $\hat{\mathbf{n}}$ as pointing inward. We just want to make sure this inward radial derivative is the same just above and just below the disk. The positive factor h_1 can then be ignored. We then have,

$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \quad \text{//upper} \quad \zeta \geq 0$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)]\ \ q_n^m(-\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \quad \text{//lower} \quad \zeta \leq 0$$

$$\partial_\xi V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\ P_n^m{}'(\xi)\cos(m\varphi) \quad \text{//upper} \quad \zeta \geq 0$$
$$\partial_\xi V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)]\ \ q_n^m(-\zeta_0)\ P_n^m{}'(\xi)\cos(m\varphi) \quad \text{//lower} \quad \zeta \leq 0$$

$$\partial_\xi V_i(0+,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+)]\ \ q_n^m(\zeta_0)\ P_n^m{}'(\xi)\cos(m\varphi) \quad \text{//upper}$$
$$\partial_\xi V_i(0-,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(0-) + C'_{nm}\ q_n^m(0-)]\ \ q_n^m(-\zeta_0)\ P_n^m{}'(\xi)\cos(m\varphi) \quad \text{//lower}$$

These will be equal if we set

$$B_{nm}\ p_n^m(0+)\ + C_{nm}\ q_n^m(0+)\ = B'_{nm}\ p_n^m(0-)\ + C'_{nm}\ q_n^m(0-)$$

But this is exactly the same condition we get for the potential match, so we get no new condition in this way, and we are still alive!

(g) Match the φ derivative of the potential at the neck which is ζ = 0.

This will work the same way as ξ, but instead of having $P_n^m{}'(\xi)$ replacing $P_n^m(\xi)$ we will have -msin(mφ) replacing cos(mφ). Our matching condition will then be:

$$m(B_{nm}\ p_n^m(0+)\ + C_{nm}\ q_n^m(0+))\ = m(B'_{nm}\ p_n^m(0-)\ + C'_{nm}\ q_n^m(0-))$$

which again gives us nothing new, whether or not m = 0. So we are STILL alive after doing all these matches at the neck.

(h) Status after doing all the neck matches:

Upper half bloid:

$$V_o(\zeta,\xi)\ = \Sigma_{nm}\,[B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi)\cos(m\varphi)$$
$$V_i(\zeta,\xi)\ = \Sigma_{nm}\,[B_{nm}\ p_n^m(\zeta)\ + C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper}\quad \zeta \geq 0$$

Lower half bloid:

$$V_o(\zeta,\xi)\ = \Sigma_{nm}\,[B'_{nm}\ p_n^m(-\zeta_0) + C'_{nm}\ q_n^m(-\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi)\cos(m\varphi)$$
$$V_i(\zeta,\xi)\ = \Sigma_{nm}\,[B'_{nm}\ p_n^m(\zeta)\ + C'_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower}\quad \zeta \leq 0$$

with condition:

$$(B_{nm} - B'_{nm})\ p_n^m(0+)\ = -\ (\ C_{nm} - C'_{nm})\ q_n^m(0+)$$

This condition causes some kind of connection between the upper and lower coefficients, but we still seem to have three sets of coefficients that are completely free.

(i) What are the advantages of our Smythian Form:

- inside the upper bloid we are continuous at the boundary $\zeta = \zeta_0$
- inside the lower bloid we are continuous at the boundary $\zeta = -\zeta_0$
- $\partial_\zeta V$ is also continuous at $\zeta = \zeta_0$ in the upper bloid so there is no surface charge out on that spheroid, except at the Green's charge. And the same thing will happen in the lower bloid.
- Going off to infinity in any direction is properly damped by the nature of $q_n^m(\zeta)$.
- The condition $P_n^m(\xi_0) = 0$ determines the n spectrum and makes V = 0 for upper *and* lower bloid.
- Since ξ is only in the limited range $(\xi_0, 1)$, we do not hit ξ = -1, and we then have no conflicting condition there requiring n to be integral.

(j) Folding of the m sum. One imagines that the initial sums are m = (-∞,∞) all integers since this is the full spectrum of the φ ODE. However, each factor in the integrand has a symmetry property, to wit

$q_n^{-m}(\zeta) = f(n,-m)\, q_n^m(\zeta)$ m = integer, n = general, ζ = anything
$p_n^{-m}(\zeta) = f(n,-m)\, p_n^m(\zeta)$

$P_n^{-m}(\xi) = (-1)^m f(n,-m) P_n^m(\xi)$ // this (only!) is the on-the-cut version of P
$\cos(-m\varphi) = \cos(m\varphi)$

Therefore, we can take the sum -∞,-1 and "fold it over" into the positive sum by redefining the coefficients. We assume this was done right at the very start before we defined B,C,B',C' We use the short hand notation Σ_{nm} to stand for $\Sigma_{m=0}^{\infty}\Sigma_n$.

(k) The spectrum of n. This is obtained from $P_n^m(\xi_1) = 0$ since ξ_1 defines the surface of the entire metallic hyperboloid (=bloid). As shown elsewhere, (1) the eigenvalues of n are real and in general non integral; (2) the eigenvalues of n depend on m and ξ_1 so we could write $n_i = n_i(m, \xi_1)$, i = 1,2,3... as a way to enumerate these zeros. The n sum therefore goes inside the m sum. We shall just write Σ_n with this spectrum understood.

3. Apply Gauss's Law to pillbox around point charge q.

(a) The Boiler Plate Warmup. I show in "Smythe Greens oblate spheroid.doc" that, for a Green's point charge q located at point $(\zeta_0, \xi_0, \varphi_0)$, Gauss's law says

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi - \xi_0)$$

where $q_1, q_2, q_3 = \xi, \zeta, \varphi$ which is the key to these results,

$$h_1 = c_1\sqrt{\xi^2+\zeta^2}\,/\sqrt{1-\xi^2}$$
$$h_2 = c_1\sqrt{\xi^2+\zeta^2}\,/\sqrt{1+\zeta^2}$$
$$h_3 = c_1\sqrt{1+\zeta^2}\,\sqrt{1-\xi^2}$$

$$h_1h_3 = c_1\sqrt{\xi^2+\zeta^2}\,/\sqrt{1-\xi^2}\;\; c_1\sqrt{1+\zeta^2}\;\sqrt{1-\xi^2} = c_1^2\sqrt{1+\zeta^2}\;\sqrt{\xi^2+\zeta^2}$$
$$h_2 = c_1\sqrt{\xi^2+\zeta^2}\,/\sqrt{1+\zeta^2}$$

$$(1+\zeta^2)\,c_1\,h_2 = h_1h_3 \quad\Rightarrow\quad (h_2/h_3h_1) = [(1+\zeta^2)\,c_1]^{-1}$$

The charge density on the spheroid surface is given by [this is Gauss's law for the pillbox]

$$\sigma/\varepsilon = \partial_n V_i - \partial_n V_o \qquad \partial_n = (1/h_2)\,\partial_\zeta \quad\Rightarrow\quad \sigma/\varepsilon = (1/h_2)[\,\partial_\zeta V_i - \partial_\zeta V_o\,]$$

Therefore, the h_2 cancels when we write σ. Here are our two main results:

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi - \xi_0) \qquad (**)$$

$$\sigma/\varepsilon = (1/h_2)[\,\partial_\zeta V_i - \partial_\zeta V_o\,] = (q/\varepsilon)\,(1/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi-\xi_0)$$

$$dA_2 = h_3h_1 dq_3 dq_1 = h_3h_1 d\varphi\, d\xi \qquad \text{// area of pillbox side}$$

This then is the effective surface charge distribution "of the point charge q". In the doc last referenced, we took this double delta limit to the north pole which was an extra complication. Here we work as is.

We can simplify

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi-\xi_0) = (q/\varepsilon)\,[(1+\zeta^2)\,c_1]^{-1}\,\delta(\varphi-\varphi_0)\,\delta(\xi-\xi_0)$$

$$= (q/c_1\varepsilon)\,\delta(\varphi-\varphi_0)\,\delta(\xi-\xi_0)/\,(1+\zeta_0^2)$$

(b) Now we compute [∂_ζVi - ∂_ζVo] using our Smythian upper form above, which is this:

$$V_o(\zeta,\xi) = \Sigma_{nm}\,[B_{nm}\,p_n^m(\zeta_0) + C_{nm}\,q_n^m(\zeta_0)]\;q_n^m(\zeta)\;P_n^m(\xi)\cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm}\,[B_{nm}\,p_n^m(\zeta) + C_{nm}\,q_n^m(\zeta)\,]\;q_n^m(\zeta_0)\,P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

$$\partial_\zeta V_o(\zeta,\xi) = \Sigma_{nm}\,[B_{nm}\,p_n^m(\zeta_0) + C_{nm}\,q_n^m(\zeta_0)]\;q_n^{m\,\prime}(\zeta)\;P_n^m(\xi)\cos(m\varphi)$$
$$\partial_\zeta V_i(\zeta,\xi) = \Sigma_{nm}\,[B_{nm}\,p_n^{m\,\prime}(\zeta) + C_{nm}\,q_n^{m\,\prime}(\zeta)\,]\;q_n^m(\zeta_0)\,P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

$$\partial_\zeta V_i - \partial_\zeta V_o = \Sigma_{nm}\,P_n^m(\xi)\cos(m\varphi) \;*$$
$$[B_{nm}\,p_n^{m\,\prime}(\zeta) + C_{nm}\,q_n^{m\,\prime}(\zeta)\,]\;q_n^m(\zeta_0) - [B_{nm}\,p_n^m(\zeta_0) + C_{nm}\,q_n^m(\zeta_0)]\;q_n^{m\,\prime}(\zeta)$$

But we only care at $\zeta = \zeta_0$ so we have

$$\partial_\zeta V_i - \partial_\zeta V_o = \Sigma_{nm}\,P_n^m(\xi)\cos(m\varphi) \;*$$
$$[B_{nm}\,p_n^{m\,\prime}(\zeta_0) + C_{nm}\,q_n^{m\,\prime}(\zeta_0)\,]\;q_n^m(\zeta_0) - [B_{nm}\,p_n^m(\zeta_0) + C_{nm}\,q_n^m(\zeta_0)]\;q_n^{m\,\prime}(\zeta_0)$$

The two C terms exactly cancel (meaning we will learn nothing about the C coefficients from this pillbox exercise) leaving us with:

$$\partial_\zeta V_i - \partial_\zeta V_o = \Sigma_{nm}\,P_n^m(\xi)\cos(m\varphi) \;*$$
$$B_{nm}\,p_n^{m\,\prime}(\zeta_0)\;q_n^m(\zeta_0) - B_{nm}\,p_n^m(\zeta_0)\,q_n^{m\,\prime}(\zeta_0)$$

$$= -\,\Sigma_{nm}\,P_n^m(\xi)\cos(m\varphi)\,B_{nm}\,W[p_n^m(\zeta_0),\,q_n^m(\zeta_0)]$$

For $\zeta_0 > 0$ we look this up in P and Q to find

$$W[p_n^m(\zeta),\,q_n^m(\zeta)] = +\,2\cos[(n+m)\pi/2]f(n,m) \;*\; 1/(1+\zeta^2)$$

so we have our result that

$$\partial_\zeta V_i - \partial_\zeta V_o = -\Sigma_{nm} P_n^m(\xi) \cos(m\varphi) B_{nm} 2\cos[(n+m)\pi/2] f(n,m) * 1/(1+\zeta_0^2)$$

$$= -2/(1+\zeta_0^2) \Sigma_{nm} B_{nm} \cos[(n+m)\pi/2] f(n,m) P_n^m(\xi) \cos(m\varphi)$$

Restate the final result:

$$\partial_\zeta V_i - \partial_\zeta V_o = -2/(1+\zeta_0^2) \Sigma_{nm} B_{nm} \cos[(n+m)\pi/2] f(n,m) P_n^m(\xi) \cos(m\varphi)$$

(c) compute the B coefficients:

So combining part a and b above, we have

$$(q/c_1\varepsilon)\, \delta(\varphi-\varphi_0)\, \delta(\xi - \xi_0)/ (1+\zeta_0^2) = -2/(1+\zeta_0^2) \Sigma_{nm} B_{nm} \cos[(n+m)\pi/2] f(n,m) P_n^m(\xi) \cos(m\varphi)$$

$$(q/c_1\varepsilon)\, \delta(\varphi-\varphi_0)\, \delta(\xi - \xi_0) = -2\Sigma_{nm} B_{nm} \cos[(n+m)\pi/2] f(n,m) P_n^m(\xi) \cos(m\varphi)$$

First, apply $\int_0^{2\pi} d\varphi \cos(m'\varphi)$ to both sides and use Schaum p 96 15.27 on the RHS,

$$\int_0^{2\pi} d\varphi \cos(m'\varphi) \cos(m\varphi) = \delta_{m,m'} 2\pi/(2-\delta_{m,0})$$

to get (first go to primes, then replaced m' by m, two steps),

$$\cos(m\varphi_0)\, (q/c_1\varepsilon)\, \delta(\xi - \xi_0) = -4\pi\Sigma_n (2-\delta_{m,0})^{-1} B_{nm} \cos[(n+m)\pi/2] f(n,m) P_n^m(\xi)$$

But we assumed at the start in our cos(mφ) form that $\varphi_0 = 0$ so we have

$$(q/c_1\varepsilon)\, \delta(\xi - \xi_0) = -4\pi\, (2-\delta_{m,0})^{-1} \Sigma_n B_{nm} \cos[(n+m)\pi/2] f(n,m) P_n^m(\xi)$$

Now our next orthogonality is this (taken from "the Smyth method for Green's...",

$$\int_0^{\alpha} d\theta \sin\theta\, P_n^m(\cos\theta)\, P_{n'}^m(\cos\theta) = K_n^m(\alpha)\ \delta_{n,n'}$$

$$K_n^m(\alpha) = -\sin^2\alpha /(2n+1) * \partial_z P_n^m(\cos\alpha)\, \partial_n P_n^m(\cos\alpha)$$

or

$$\int_{\xi_1}^{1} d\xi\, P_n^m(\xi)\, P_{n'}^m(\xi) = k_n^m(\xi_0)\ \delta_{n,n'}$$

$$K_n^m(\xi_0) = -(1-\xi_0^2) /(2n+1) * \partial_\xi P_n^m(\xi_0)\, \partial_n P_n^m(\xi_0) \qquad \text{// ignore chg in fnctnl form}$$

So apply $\int_{\xi_1}^{1} P_{n'}^m(\xi)$ to both sides of (*) to get (again, use n', then replace n' by n)

$$(q/c_1\varepsilon)\, P_n^m(\xi_0) = +4\pi\,(2-\delta_{m,0})^{-1} B_{nm} \cos[(n+m)\pi/2] f(n,m)\, K_n^m(\xi_0)$$

$$(2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon])\;\; P_n^m(\xi_0) = + B_{nm} \cos[(n+m)\pi/2] f(n,m)\, K_n^m(\xi_0)$$

$$B_{nm} = + (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon])\;\; P_n^m(\xi_0)\, \sec[(n+m)\pi/2]\, f(n,-m)/\, K_n^m(\xi_0)$$

So this tells us our B coefficients and of course they are REAL.

(d) "pillbox" in the lower bloid. Suppose we just repeat the above steps in the lower bloid. Since $q_n^m(-\zeta_0) = q_n^m(\zeta_0)$, the *only* differences we get would be these:

1. Replace B with B' and C with C'.
2. $\zeta < 0$ in the lower bloid.
3. Since there is no point charge there, we get $\partial_\zeta V_i - \partial_\zeta V_o = 0$ at the image point of the Green charge.

Then as before, C' cancels away and we learn nothing about it, but we would then find that

$$0 = + 2\Sigma_{nm} B'_{nm} \cos[(n+m)\pi/2] f(n,m)\, P_n^m(\xi) \cos(m\varphi)$$

where I show a sign change because the Wronskian would have the other sign. But no matter, this tells us a very important fact:

$$B'_{nm} = 0$$

This is the condition such that the radial derivative $\partial_\zeta V$ be continuous at the boundary between V_i and V_0 down in the lower bloid.

(e) Tangential derivatives at the bloid surface ?

This is a little new to me, first time I have considered it. We know that we get V = 0 on the bloid surface from the $P_n^m(\xi_1)$ spectrum. We can get the charge σ from the ∂_ξ derivatives. But are we sure that both the tangential derivatives at the bloid surface will be 0 (as they must be). Once again,

$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\; p_n^m(\zeta) + C_{nm}\; q_n^m(\zeta)\,]\quad q_n^m(\zeta_0)\, P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper}\quad \zeta \geq 0$$

$$\partial_\zeta V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\; p_n^{m\,\prime}(\zeta) + C_{nm}\; q_n^{m\,\prime}(\zeta)\,]\quad q_n^m(\zeta_0)\, P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper}\quad \zeta \geq 0$$

$$\partial_\zeta V_i(\zeta,\xi_1) = \Sigma_{nm} [B_{nm}\; p_n^{m\,\prime}(\zeta) + C_{nm}\; q_n^{m\,\prime}(\zeta)\,]\quad q_n^m(\zeta_0)\, P_n^m(\xi_1) \cos(m\varphi) \quad \text{//upper}\quad \zeta \geq 0$$

But $P_n^m(\xi_1) = 0$ so we are OK. Similarly, if we look at the φ tangential derivative, we get this same $P_n^m(\xi_1)$ factor, so I think it will be OK as well.

(f) Matching tangential derivatives on the spheroid surface?

This is another aspect I have not considered. Again as have on the top

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

$$\partial_\xi V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^{m\,\prime}(\xi) \cos(m\varphi)$$
$$\partial_\xi V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^{m\,\prime}(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

If we then set $\zeta = \zeta_0$ for our spheroid, they match just fine. Same will happen for lower bloid. And the same argument shows that we have a match for the φ derivative which is the other tangent.

(g) So here is how we now stand:

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

Lower half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [C'_{nm}\ q_n^m(-\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [C'_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

with condition due to matching at the neck in all ways,

$$(B_{nm})\ p_n^m(0+) = -(C_{nm} - C'_{nm})\ q_n^m(0+)$$

or

$$B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+) = C'_{nm}\ q_n^m(0+)$$

and with pillbox evaluation

$$B_{nm} = +(2-\delta_{m,0})\ (q/[4\pi c_1 \varepsilon])\ P_n^m(\xi_0)\ \sec[(n+m)\pi/2]\ f(n,-m)/\ K_n^m(\xi_0)$$

(h) Discussion.

I cannot think of any more boundary conditions. Everything seems right at the neck, at the bloid surface, at the Green's charge, and at infinity. But there must be something else for the following reason. We have only one condition on three coefficients, so I ought to be able to set $C'_{nm} = 0$ and have zero potential in the lower bloid! But we know that cannot be right.

OK, I think I have found the kiss of death for Attempt #3. Consider just

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$

$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

This tells us that

$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

$$\partial_\zeta V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m{}'(\zeta) + C_{nm}\ q_n^m{}'(\zeta)]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

$$\partial_\zeta V_i(\zeta=0+,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m{}'(0+) + C_{nm}\ q_n^m{}'(0+)]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi)$$

$$= 0$$

Our Smythian form is not general enough because it does not allow for a neck-normal derivative to exist on the neck plane! This is due to the fact that the p and q functions are symmetric, so their derivatives at $\zeta = 0$ are 0. I *knew* that symmetry was going to get me sooner or later!

Here is a very easy way to see the problem here: for the bloid, we know we are trying to come up with a solution which is *different* in the upper and lower regions. But our atoms p and q are the same in these two regions, and so can only generate symmetric solutions. The form is dead on arrival.

So I need to go off and ponder this idea of my p and q functions supposedly being independent solutions of our ζ ODE, and yet both are even. But they are supposed to be a complete set like $\cos(m\varphi)$ and $\sin(m\varphi)$ to allow you to construct a general function $f(\varphi)$, but in ζ I can only construct EVEN functions of ζ.