Oblate bloid with p and q attempt 4: success! PhL 2.14-21.10

The problem here is to find the Green's Function for an oblate hyperboloid.

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1. The p and q functions. The goal is to use these "real atom" functions in the ζ dimension,

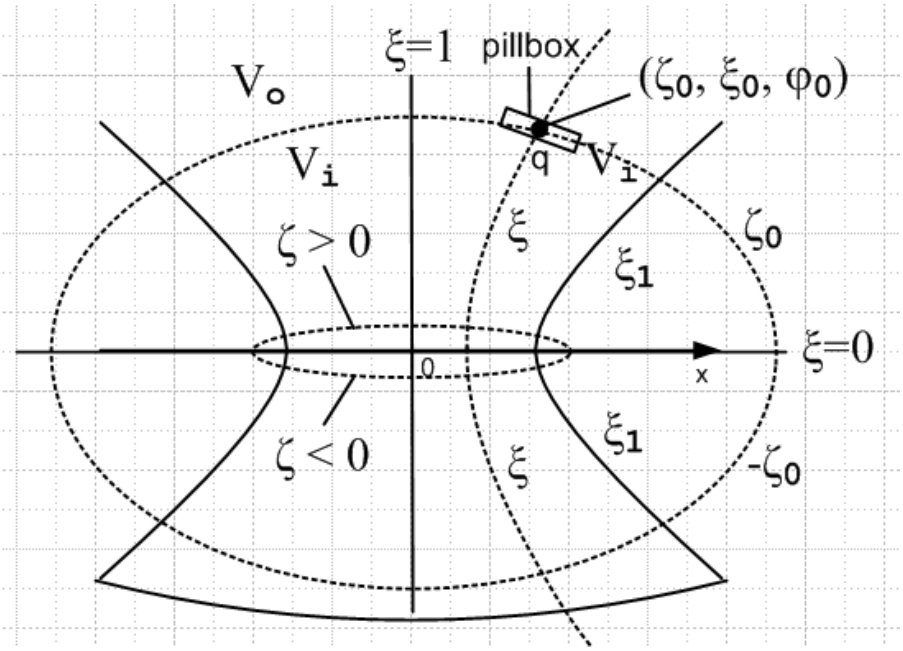
$$q_n^m(\zeta) = (-i)^{2m} (\pm i)^{n+1} Q_n^m(i\zeta)$$

$$p_n^m(\zeta) = [(\mp i)^{-m} P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$$

$$= (\pm i)^{-n} 2\cos[(n+m)\pi/2] P_n^m(i\zeta) - (\pm i)^{-m/2} (-i)^{2m} (2/\pi) \sin\pi(n+m) Q_n^m(i\zeta)$$

These functions are real when n and m are and ζ are real. You must use the upper sign when Re(ζ) > 0, otherwise the lower sign.

2. The Smythian form, picture, and matching in the Neck . First, the picture



We shall try a more general form than we used in Attempt #2. Let's first concentrate just on the upper bloid region and try this:

Upper half bloid:

$V_o(\zeta,\xi) = \Sigma_{nm} [\quad A_{nm} q_n^m(\zeta)] P_n^m(\xi) \cos(m\varphi)$

$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm} p_n^m(\zeta) + C_{nm} q_n^m(\zeta)] P_n^m(\xi) \cos(m\varphi)$ //upper $\zeta \geq 0$

We must have the q alone for the outside part due to large ζ decay, so this is about as general as we can get.

(a) get matching at the boundary between upper Vi and Vo . Continuity at the boundary then tells us that

$$V_o(\zeta_0,\xi) = V_i(\zeta_0,\xi)$$

$$\Sigma_{nm} [A_{nm} q_n^m(\zeta_0)] P_n^m(\xi) \cos(m\varphi) = \Sigma_{nm} [B_{nm} p_n^m(\zeta_0) + C_{nm} q_n^m(\zeta_0)] P_n^m(\xi) \cos(m\varphi)$$

Matching terms tells us that

$$A_{nm}\, q_n^m(\zeta_0) = B_{nm}\, p_n^m(\zeta_0) + C_{nm}\, q_n^m(\zeta_0)$$

So multiply and divide the upper form like this:

$$V_o(\zeta,\xi) = \Sigma_{nm} [\quad A_{nm}\, q_n^m(\zeta_0) q_n^m(\zeta)]\; P_n^m(\xi) \cos(m\varphi) / q_n^m(\zeta_0)$$

$$= \Sigma_{nm} [B_{nm}\, p_n^m(\zeta_0) + C_{nm}\, q_n^m(\zeta_0)]\; q_n^m(\zeta)\; P_n^m(\xi) \cos(m\varphi) / q_n^m(\zeta_0)$$

Then we have:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\, p_n^m(\zeta_0) + C_{nm}\, q_n^m(\zeta_0)]\; q_n^m(\zeta)\; P_n^m(\xi) \cos(m\varphi) / q_n^m(\zeta_0)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\, p_n^m(\zeta) + C_{nm}\, q_n^m(\zeta)\,]\; P_n^m(\xi) \cos(m\varphi)$$

Now define $B_{nm}' = B_{nm}/ q_n^m(\zeta_0)$ and the same for C, then we have

$$V_o(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\, p_n^m(\zeta_0) + C'_{nm}\, q_n^m(\zeta_0)]\; q_n^m(\zeta)\; P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\, p_n^m(\zeta) + C'_{nm}\, q_n^m(\zeta)\,]\; q_n^m(\zeta_0)\, P_n^m(\xi) \cos(m\varphi)$$

Now removed the primes to reduce clutter, and we arrive at this Symthian form:

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\, p_n^m(\zeta_0) + C_{nm}\, q_n^m(\zeta_0)]\; q_n^m(\zeta)\; P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\, p_n^m(\zeta) + C_{nm}\, q_n^m(\zeta)\,]\; q_n^m(\zeta_0)\, P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

This is what it has to *look like* to match at the boundary between V_o and V_i .

(b) the lower bloid form. Now, by the same argument, the lower bloid must have this form (new meaning for primes):

Lower half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\, p_n^m(-\zeta_0) + C'_{nm}\, q_n^m(-\zeta_0)]\; q_n^m(\zeta)\; P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\, p_n^m(\zeta) + C'_{nm}\, q_n^m(\zeta)\,]\; q_n^m(-\zeta_0)\, P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

The matching surface here has $\zeta = -\zeta_0$, so have installed that value.

(c) Smythian form for the upper and lower bloids:

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\, p_n^m(\zeta_0) + C_{nm}\, q_n^m(\zeta_0)]\; q_n^m(\zeta)\; P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\, p_n^m(\zeta) + C_{nm}\, q_n^m(\zeta)\,]\; q_n^m(\zeta_0)\, P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

Lower half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(-\zeta_0) + C'_{nm}\ q_n^m(-\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \le 0$$

(d) Match the potential at the neck which is $\zeta = 0$. We have at the neck:

$$V_i(0+,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper}$$
$$V_i(0-,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(0-) + C'_{nm}\ q_n^m(0-)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower}$$

But our p and q functions are all perfectly symmetric, so rewrite the lower line

$$V_i(0+,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper}$$
$$V_i(0-,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(0+) + C'_{nm}\ q_n^m(0+)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower}$$

The condition is then:

$$B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+) = B'_{nm}\ p_n^m(0+) + C'_{nm}\ q_n^m(0+)$$

$$(B_{nm} - B'_{nm})\ p_n^m(0+) = -(C_{nm} - C'_{nm})\ q_n^m(0+)$$

The good news is that we are not forced right here to have B = B' and C = C'. If we *were*, then due to the perfect symmetry of our P and Q functions, we would have the exact same potential on both upper and lower and we know this cannot be true since the Green's Charge is only in the upper. So, to coin a phrase, once again "we are still alive".

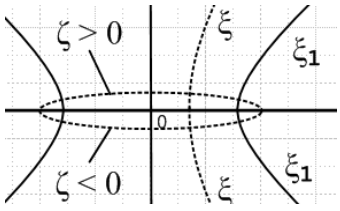
(e) Match the ζ derivative of the potential at the neck which is $\zeta = 0$.

This is going to be the tricky one I think. We start with

$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \ge 0$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \le 0$$

As usual, we pause to review the scale factors to make sure we don't trip up:

$$h_1 = c_1 \sqrt{\xi^2 + \zeta^2} / \sqrt{1-\xi^2} \qquad q_1, q_2, q_3 = \xi, \zeta, \varphi$$
$$h_2 = c_1 \sqrt{\xi^2 + \zeta^2} / \sqrt{1+\zeta^2}$$
$$h_3 = c_1 \sqrt{1+\zeta^2}\ \sqrt{1-\xi^2}$$



When we talk about the derivative normal to the neck disk plane, just above the plane we have our $\hat{\mathbf{n}}$ vector pointing up and we say $\partial_n = (1/h_2)\,\partial_\zeta$. Both n and ζ increase "going up". From above we know that $h_2 = c_1\xi$ at the neck, where ξ is a positive number in the range (ξ0, 1). So $\partial_n = (c_1\xi)^{-1}\,\partial_\zeta$.

Now what about just below the neck? As we move up, ζ gets less negative so dζ is positive, and dn is also positive, so we have the exact same rule: $\partial_n = (c_1\xi)^{-1}\,\partial_\zeta$. And of course ξ is the same on both sides of the neck. So we can ignore the factor $(c_1\xi)^{-1}$ in our match since it is the same on both sides, and then our matching condition is to make $\partial_\zeta V_i$ be the same on both sides of the neck! We have:

$$V_i(\zeta,\xi) = \Sigma_{nm}\,[B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\,P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper}\quad \zeta \geq 0$$
$$V_i(\zeta,\xi) = \Sigma_{nm}\,[B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)]\ q_n^m(-\zeta_0)\,P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower}\quad \zeta \leq 0$$

$$\partial_\zeta V_i(\zeta,\xi) = \Sigma_{nm}\,[B_{nm}\ p_n^{m\,\prime}(\zeta) + C_{nm}\ q_n^{m\,\prime}(\zeta)]\ \ q_n^m(\zeta_0)\,P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper}\quad \zeta \geq 0$$
$$\partial_\zeta V_i(\zeta,\xi) = \Sigma_{nm}\,[B'_{nm}\ p_n^{m\,\prime}(\zeta) + C'_{nm}\ q_n^{m\,\prime}(\zeta)]\ q_n^m(-\zeta_0)\,P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower}\quad \zeta \leq 0$$

But we want this match to occur at ζ = 0. To allow for the possibility that derivatives are different above and below the plane of the neck, we write [we also replace $q_n^m(-\zeta_0) = q_n^m(\zeta_0)$ in the second line]

$$\partial_\zeta V_i(0+,\xi) = \Sigma_{nm}\,[B_{nm}\ p_n^{m\,\prime}(0+) + C_{nm}\ q_n^{m\,\prime}(0+)]\ \ q_n^m(\zeta_0)\,P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper}$$
$$\partial_\zeta V_i(0-,\xi) = \Sigma_{nm}\,[B'_{nm}\ p_n^{m\,\prime}(0-) + C'_{nm}\ q_n^{m\,\prime}(0-)]\ \ q_n^m(\zeta_0)\,P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower}\quad \zeta \leq 0$$

Since there is no surface charge layer in the neck, we require these to be the same. Therefore our matching condition is then

$$B_{nm}\ p_n^{m\,\prime}(0+) + C_{nm}\ q_n^{m\,\prime}(0+) = B'_{nm}\ p_n^{m\,\prime}(0-) + C'_{nm}\ q_n^{m\,\prime}(0-)$$

Feb 21, 2001: I now realize that, although $p_n^{m\,\prime}(0\pm) = 0$, this is NOT TRUE for $q_n^{m\,\prime}(0\pm)$ as I originally thought. So starting here, things will be different. I will call this attempt 4!

We have these results from "reality of P and Q" doc:

$$p_n^{m\,\prime}(0\pm) = 0$$
$$q_n^{m\,\prime}(0\pm) = \mp\sqrt{\pi}\ 2^m\ [\,\Gamma(1+n/2+m/2) \,/\, \Gamma(1/2+n/2-m/2)]$$

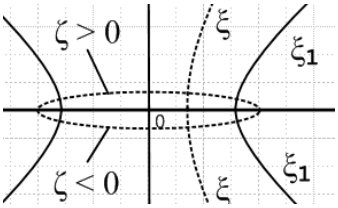
Therefore our above derivative matching condition is this:

$$+ C_{nm}\ q_n^{m\,\prime}(0+) = + C'_{nm}\ q_n^{m\,\prime}(0-)$$

$$C'_{nm} = -\,C_{nm}$$

Wee move on to the next matching requirement:

(f) Match the ξ derivative of the potential at the neck which is ζ = 0. Here we have $\partial_n = (1/h_1)\,\partial_\xi$ and $h_1 = c_1\xi \,/\, \sqrt{1-\xi^2}$ which is a positive real number.



This is a "radial" ρ-like derivative on the disk. In our picture on the right side of the center axis we have, as we move radially inward, we have dξ > 0, so we can think of $\hat{\mathbf{n}}$ as pointing inward. We just want to make sure this inward radial derivative is the same just above and just below the disk. The positive factor h_1 can then be ignored. We then have,

$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper}\ \ \zeta \geq 0$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)]\ q_n^m(-\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower}\ \ \zeta \leq 0$$

$$\partial_\xi V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\ P_n^{m\,\prime}(\xi)\cos(m\varphi) \qquad \text{//upper}\ \ \zeta \geq 0$$
$$\partial_\xi V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)]\ q_n^m(-\zeta_0)\ P_n^{m\prime}(\xi)\cos(m\varphi) \qquad \text{//lower}\ \ \zeta \leq 0$$

$$\partial_\xi V_i(0+,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+)]\ \ q_n^m(\zeta_0)\ P_n^{m\,\prime}(\xi)\cos(m\varphi) \qquad \text{//upper}$$
$$\partial_\xi V_i(0-,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(0-) + C'_{nm}\ q_n^m(0-)]\ q_n^m(-\zeta_0)\ P_n^{m\prime}(\xi)\cos(m\varphi) \qquad \text{//lower}$$

These will be equal if we set

$$B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+) = B'_{nm}\ p_n^m(0-) + C'_{nm}\ q_n^m(0-)$$

But this is exactly the same condition we get for the potential match, so we get no new condition in this way, and we are still alive!

(g) Match the φ derivative of the potential at the neck which is ζ = 0.

This will work the same way as ξ, but instead of having $P_n^{m\,\prime}(\xi)$ replacing $P_n^m(\xi)$ we will have $-m\sin(m\varphi)$ replacing $\cos(m\varphi)$. Our matching condition will then be:

$$m(B_{nm}\ p_n^m(0+) + C_{nm}\ q_n^m(0+)) = m(B'_{nm}\ p_n^m(0-) + C'_{nm}\ q_n^m(0-))$$

which again gives us nothing new, whether or not m = 0. So we are STILL alive after doing all these matches at the neck.

(h) Status after doing all the neck matches:

We install the fact that $C'_{nm} = -C_{nm}$ to get

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ \ q_n^m(\zeta)\ P_n^m(\xi)\cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper}\ \ \zeta \geq 0$$

Lower half bloid:

$V_o(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(-\zeta_0) - C_{nm}\ q_n^m(-\zeta_0)]\ q_n^m(\zeta)\ \ P_n^m(\xi) \cos(m\varphi)$

$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta)\ -\ C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi) \cos(m\varphi)$ //lower $\zeta \le 0$

with conditions:

$$(B_{nm} - B'_{nm})\ p_n^m(0+) = -(C_{nm} - C'_{nm})\ q_n^m(0+)$$
$$C'_{nm} = -C_{nm}$$

Use the second to eliminate C' from the first

$$(B_{nm} - B'_{nm})\ p_n^m(0+) = -(2C_{nm})\ q_n^m(0+)$$

We still seem to have two sets of coefficients that are completely free.

(i) What are the advantages of our Smythian Form:

- inside the upper bloid we are continuous at the boundary $\zeta = \zeta_0$
- inside the lower bloid we are continuous at the boundary $\zeta = -\zeta_0$
- $\partial_\zeta V$ is also continuous at $\zeta = \zeta_0$ in the upper bloid so there is no surface charge out on that spheroid, except at the Green's charge. And the same thing will happen in the lower bloid.
- Going off to infinity in any direction is properly damped by the nature of $q_n^m(\zeta)$.
- The condition $P_n^m(\xi_0) = 0$ determines the n spectrum and makes V = 0 for upper *and* lower bloid.
- Since ξ is only in the limited range $(\xi_0, 1)$, we do not hit $\xi = -1$, and we then have no conflicting condition there requiring n to be integral.

(j) Folding of the m sum. One imagines that the initial sums are m = $(-\infty,\infty)$ all integers since this is the full spectrum of the φ ODE. However, each factor in the integrand has a symmetry property, to wit

$q_n^{-m}(\zeta) = f(n,-m)\ q_n^m(\zeta)$ m = integer, n = general, ζ = anything
$p_n^{-m}(\zeta) = f(n,-m)\ p_n^m(\zeta)$

$P_n^{-m}(\xi) = (-1)^m f(n,-m) P_n^m(\xi)$ // this (only!) is the on-the-cut version of P
$\cos(-m\varphi) = \cos(m\varphi)$

Therefore, we can take the sum $-\infty,-1$ and "fold it over" into the positive sum by redefining the coefficients. We assume this was done right at the very start before we defined B,C,B',C' We use the short hand notation Σ_{nm} to stand for $\Sigma_{m=0}^{\infty}\Sigma_n$.

(k) The spectrum of n. This is obtained from $P_n^m(\xi_1) = 0$ since ξ_1 defines the surface of the entire metallic hyperboloid (=bloid). As shown elsewhere, (1) the eigenvalues of n are real and in general non integral; (2) the eigenvalues of n depend on m and ξ_1 so we could write $n_i = n_i(m, \xi_1)$, i = 1,2,3... as a

way to enumerate these zeros. The n sum therefore goes inside the m sum. We shall just write Σ_n with this spectrum understood.

3. Apply Gauss's Law to pillbox around point charge q.

(a) The Boiler Plate Warmup. I show in "Smythe Greens oblate spheroid.doc" that, for a Green's point charge q located at point $(\zeta_0, \xi_0, \varphi_0)$, Gauss's law says

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi - \xi_0)$$

where $q_1, q_2, q_3 = \xi, \zeta, \varphi$ which is the key to these results,

$$h_1 = c_1 \sqrt{\xi^2+ \zeta^2} / \sqrt{1-\xi^2}$$
$$h_2 = c_1 \sqrt{\xi^2+ \zeta^2} / \sqrt{1+\zeta^2}$$
$$h_3 = c_1 \sqrt{1+\zeta^2}\ \sqrt{1-\xi^2}$$

$$h_1h_3 = c_1 \sqrt{\xi^2+ \zeta^2} / \sqrt{1-\xi^2}\ \ c_1 \sqrt{1+\zeta^2}\ \sqrt{1-\xi^2}\ = c_1^2 \sqrt{1+\zeta^2}\ \sqrt{\xi^2+ \zeta^2}$$
$$h_2 = c_1 \sqrt{\xi^2+ \zeta^2} / \sqrt{1+\zeta^2}$$

$$(1+\zeta^2)\, c_1\, h_2 = h_1h_3 \quad \Rightarrow \quad (h_2/h_3h_1) = [(1+\zeta^2)\, c_1]^{-1}$$

The charge density on the spheroid surface is given by [this is Gauss's law for the pillbox]

$$\sigma/\varepsilon = \partial_n V_i - \partial_n V_o \qquad \partial_n = (1/h_2)\,\partial_\zeta \quad \Rightarrow \quad \sigma/\varepsilon = (1/h_2)[\,\partial_\zeta V_i - \partial_\zeta V_o\,]$$

Therefore, the h_2 cancels when we write σ. Here are our two main results:

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi - \xi_0) \qquad (**)$$

$$\sigma/\varepsilon = (1/h_2)[\,\partial_\zeta V_i - \partial_\zeta V_o\,] = (q/\varepsilon)\,(1/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi - \xi_0)$$

$$dA_2 = h_3h_1 dq_3 dq_1 = h_3h_1 d\varphi\, d\xi \qquad \text{// area of pillbox side}$$

This then is the effective surface charge distribution "of the point charge q". In the doc last referenced, we took this double delta limit to the north pole which was an extra complication. Here we work as is.

We can simplify

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi - \xi_0) = (q/\varepsilon)\,[(1+\zeta^2)\, c_1]^{-1}\,\delta(\varphi-\varphi_0)\,\delta(\xi - \xi_0)$$

$$= (q/c_1\varepsilon)\,\delta(\varphi-\varphi_0)\,\delta(\xi - \xi_0)/\,(1+\zeta_0^2)$$

(b) Now we compute [∂ζVi - ∂ζVo] using our Smythian upper form above, which is this:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

$$\partial_\zeta V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m{}'(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$\partial_\zeta V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m{}'(\zeta) + C_{nm}\ q_n^m{}'(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

$$\partial_\zeta V_i - \partial_\zeta V_o = \Sigma_{nm} P_n^m(\xi) \cos(m\varphi) *$$
$$[B_{nm}\ p_n^m{}'(\zeta) + C_{nm}\ q_n^m{}'(\zeta)\ ]\ q_n^m(\zeta_0) - [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m{}'(\zeta)$$

But we only care at $\zeta = \zeta_0$ so we have

$$\partial_\zeta V_i - \partial_\zeta V_o = \Sigma_{nm} P_n^m(\xi) \cos(m\varphi) *$$
$$[B_{nm}\ p_n^m{}'(\zeta_0) + C_{nm}\ q_n^m{}'(\zeta_0)\ ]\ q_n^m(\zeta_0) - [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m{}'(\zeta_0)$$

The two C terms exactly cancel (meaning we will learn nothing about the C coefficients from this pillbox exercise) leaving us with:

$$\partial_\zeta V_i - \partial_\zeta V_o = \Sigma_{nm} P_n^m(\xi) \cos(m\varphi) *$$
$$B_{nm}\ p_n^m{}'(\zeta_0)\ q_n^m(\zeta_0) - B_{nm}\ p_n^m(\zeta_0)\ q_n^m{}'(\zeta_0)$$

$$= -\Sigma_{nm} P_n^m(\xi) \cos(m\varphi)\ B_{nm}\ W[p_n^m(\zeta_0), q_n^m(\zeta_0)]$$

For $\zeta_0 > 0$ we look this up in P and Q to find

$$W[p_n^m(\zeta), q_n^m(\zeta)] = +\ 2\cos[(n+m)\pi/2] f(n,m) *\ 1/(1+\zeta^2)$$

so we have our result that

$$\partial_\zeta V_i - \partial_\zeta V_o = -\Sigma_{nm} P_n^m(\xi) \cos(m\varphi)\ B_{nm}\ 2\cos[(n+m)\pi/2] f(n,m) *\ 1/(1+\zeta_0^2)$$

$$= -\ 2/(1+\zeta_0^2)\ \Sigma_{nm}\ B_{nm} \cos[(n+m)\pi/2] f(n,m)\ P_n^m(\xi) \cos(m\varphi)$$

Restate the final result:

$$\partial_\zeta V_i - \partial_\zeta V_o = -\ 2/(1+\zeta_0^2)\ \Sigma_{nm}\ B_{nm} \cos[(n+m)\pi/2] f(n,m)\ P_n^m(\xi) \cos(m\varphi)$$

(c) compute the B coefficients:

So combining part a and b above, we have

$$(q/c_1\varepsilon)\ \delta(\varphi-\varphi_0)\ \delta(\xi - \xi_0)/\ (1+\zeta_0^2) = -\ 2/(1+\zeta_0^2)\ \Sigma_{nm}\ B_{nm} \cos[(n+m)\pi/2] f(n,m)\ P_n^m(\xi) \cos(m\varphi)$$

$$(q/c_1\varepsilon)\ \delta(\varphi-\varphi_0)\ \delta(\xi - \xi_0) = -\ 2\Sigma_{nm}\ B_{nm} \cos[(n+m)\pi/2] f(n,m)\ P_n^m(\xi) \cos(m\varphi)$$

First, apply $\int_0^{2\pi} d\varphi \cos(m'\varphi)$ to both sides and use Schaum p 96 15.27 on the RHS,

$$\int_0^{2\pi} d\varphi \cos(m'\varphi) \cos(m\varphi) = \delta_{m,m'} 2\pi/(2-\delta_{m,0})$$

to get (first go to primes, then replaced m' by m, two steps),

$$\cos(m\varphi_0) (q/c_1\varepsilon) \delta(\xi - \xi_0) = - 4\pi\Sigma_n (2-\delta_{m,0})^{-1} B_{nm} \cos[(n+m)\pi/2] f(n,m) P_n^m(\xi)$$

But we assumed at the start in our cos(mφ) form that $\varphi_0 = 0$ so we have

$$(q/c_1\varepsilon) \delta(\xi - \xi_0) = - 4\pi (2-\delta_{m,0})^{-1} \Sigma_n B_{nm} \cos[(n+m)\pi/2] f(n,m) P_n^m(\xi)$$ schk

Now our next orthogonality is this (taken from "the Smyth method for Green's...",

$$\int_0^{\alpha} d\theta \sin\theta P_n^m(\cos\theta) P_{n'}^m(\cos\theta) = K_n^m(\alpha) \delta_{n,n'}$$

$$K_n^m(\alpha) = - \sin^2\alpha /(2n+1) * \partial_z P_n^m(\cos\alpha) \partial_n P_n^m(\cos\alpha)$$ schk from memory

or

$$\int_{\xi_1}^1 d\xi P_n^m(\xi) P_{n'}^m(\xi) = k_n^m(\xi_1) \delta_{n,n'}$$

$$K_n^m(\xi_1) = - (1-\xi_1^2) /(2n+1) * \partial_\xi P_n^m(\xi_1) \partial_n P_n^m(\xi_0)$$ // ignore chg in fnctnl form of K

So apply $\int_{\xi_1}^1 P_{n'}^m(\xi)$ to both sides of (*) to get (again, use n', then replace n' by n)

$$(q/c_1\varepsilon) P_n^m(\xi_0) = + 4\pi (2-\delta_{m,0})^{-1} B_{nm} \cos[(n+m)\pi/2] f(n,m) K_n^m(\xi_1)$$

$$(2-\delta_{m,0}) (q/[4\pi c_1\varepsilon]) P_n^m(\xi_0) = + B_{nm} \cos[(n+m)\pi/2] f(n,m) K_n^m(\xi_1)$$

$$B_{nm} = (2-\delta_{m,0}) (q/[4\pi c_1\varepsilon]) P_n^m(\xi_0) \sec[(n+m)\pi/2] f(n,-m)/ K_n^m(\xi_1)$$

So this tells us our B coefficients and of course they are REAL.

(d) "pillbox" in the lower bloid. Suppose we just repeat the above steps in the lower bloid. Since $q_n^m(-\zeta_0) = q_n^m(\zeta_0)$, the *only* differences we get would be these:

1. Replace B with B' and C with C'.
2. $\zeta < 0$ in the lower bloid.
3. Since there is no point charge there, we get $\partial_\zeta V_i - \partial_\zeta V_o = 0$ at the image point of the Green charge.

Then as before, C' cancels away and we learn nothing about it, but we would then find that

$$0 = +2\Sigma_{nm} B'_{nm} \cos[(n+m)\pi/2] f(n,m) P_n^m(\xi) \cos(m\varphi)$$

where I show a sign change because the Wronskian would have the other sign. But no matter, this tells us a very important fact:

$$B'_{nm} = 0$$

This is the condition such that the radial derivative $\partial_\zeta V$ be continuous at the boundary between V_i and V_0 down in the lower bloid.

(e) Tangential derivatives at the bloid surface ?

This is a little new to me, first time I have considered it. We know that we get V = 0 on the bloid surface from the $P_n^m(\xi_1)$ spectrum. We can get the charge σ from the ∂_ξ derivatives. But are we sure that both the tangential derivatives at the bloid surface will be 0 (as they must be). Once again,

$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0) P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

$$\partial_\zeta V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^{m\,\prime}(\zeta) + C_{nm}\ q_n^{m\,\prime}(\zeta)]\ \ q_n^m(\zeta_0) P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

$$\partial_\zeta V_i(\zeta,\xi_1) = \Sigma_{nm} [B_{nm}\ p_n^{m\,\prime}(\zeta) + C_{nm}\ q_n^{m\,\prime}(\zeta)]\ \ q_n^m(\zeta_0) P_n^m(\xi_1) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

But $P_n^m(\xi_1) = 0$ so we are OK. Similarly, if we look at the φ tangential derivative, we get this same $P_n^m(\xi_1)$ factor, so I think it will be OK as well.

(f) Matching tangential derivatives on the spheroid surface?

This is another aspect I have not considered. Again as have on the top

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ \ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0) P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

$$\partial_\xi V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ \ q_n^m(\zeta)\ P_n^{m\,\prime}(\xi) \cos(m\varphi)$$
$$\partial_\xi V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ \ q_n^m(\zeta_0) P_n^{m\,\prime}(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

If we then set $\zeta = \zeta_0$ for our spheroid, they match just fine. Same will happen for lower bloid. And the same argument shows that we have a match for the φ derivative which is the other tangent.

(g) So here is how we now stand:

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ \ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$

$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

Lower half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [-C_{nm}\ q_n^m(-\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [-C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower } \zeta \leq 0$$

with condition

$$(B_{nm})\ p_n^m(0+) = -(2C_{nm})\ q_n^m(0+) \qquad C_{nm} = (-1/2)\ (p_n^m(0+)/\ q_n^m(0+)\ B_{nm}$$

and with pillbox evaluation

$$B_{nm} = (2-\delta_{m,0})\ (q/[4\pi c_1 \varepsilon])\ P_n^m(\xi_0)\ \sec[(n+m)\pi/2]\ f(n,-m)/\ K_n^m(\xi_1)$$

This almost looks like a solution to me!!! We know from reality of PQ doc that

$$p_n^m(0\pm) = 2^{m+1}\sqrt{\pi}\ /\ [\ \Gamma(1/2 - n/2 - m/2)\ \Gamma(1 + n/2 - m/2)\ ]$$
$$q_n^m(0\pm) = (-1)^m \sqrt{\pi}\ 2^{m-1}\ [\ \Gamma(1/2 + n/2 + m/2)\ /\ \Gamma(1 + n/2 - m/2)]$$

So we find then that

$$(p_n^m(0+)/\ q_n^m(0+)$$

$$= 2^{m+1}\sqrt{\pi}\ \Gamma(1 + n/2 - m/2)\ /\ [(-1)^m \sqrt{\pi}\ 2^{m-1}\ \Gamma(1/2 + n/2 + m/2)\ \Gamma(1/2 - n/2 - m/2)\ \Gamma(1 + n/2 - m/2)\ ]$$

$$= 2^{+1}\ /\ [(-1)^m\ 2^{-1}\ \Gamma(1/2 + n/2 + m/2)\ \Gamma(1/2 - n/2 - m/2)\ ]$$

$$= 4\ (-1)^m\ /\ [\ \Gamma(1/2 + n/2 + m/2)\ \Gamma(1/2 - n/2 - m/2)\ ]$$

Therefore we have

$$C_{nm} = (-1/2)\ (p_n^m(0+)/\ q_n^m(0+)\ B_{nm} =$$

$$= (-1/2)\ B_{nm}\ 4\ (-1)^m\ /\ [\ \Gamma(1/2 + n/2 + m/2)\ \Gamma(1/2 - n/2 - m/2)\ ]$$

$$= 2(-1)^{m+1}\ B_{nm}\ /\ [\ \Gamma(1/2 + n/2 + m/2)\ \Gamma(1/2 - n/2 - m/2)\ ]$$

Restate the tentative solution:

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta) + C_{nm}\ q_n^m(\zeta)]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$

Lower half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm}\,[-C_{nm}\ q_n^m(-\zeta_0)]\ q_n^m(\zeta)\ \ P_n^m(\xi)\cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm}\,[-C_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower} \quad \zeta \le 0$$

$$B_{nm} = (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon])\ P_n^m(\xi_0)\sec[(n+m)\pi/2]\ f(n,-m)/\ K_n^m(\xi_1)$$

$$C_{nm} = 2(-1)^{m+1}\,B_{nm}\ /\ [\ \Gamma(1/2 + n/2 + m/2)\,\Gamma(1/2 - n/2 - m/2)\ ]$$

Let's define

$$1/[\ \Gamma(1/2 + n/2 + m/2)\,\Gamma(1/2 - n/2 - m/2)\ ] = g_3(n,m)$$

Then

$$C_{nm} = 2(-1)^{m+1}\,g_3(n,m)\,B_{nm}$$

and we write

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm}\,B_{nm}\,[p_n^m(\zeta_0) - 2(-1)^m\ g_3(n,m)q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi)\cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm}\,B_{nm}\,[p_n^m(\zeta) - 2(-1)^m\ g_3(n,m)q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper} \quad \zeta \ge 0$$

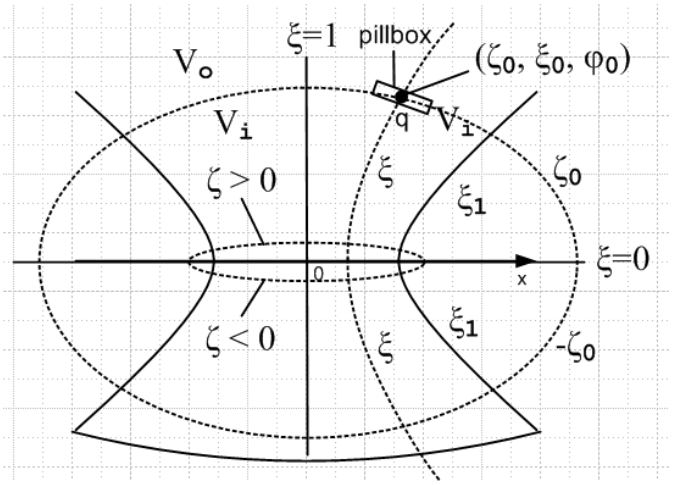
Lower half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm}\,B_{nm}\,[2(-1)^m\,g_3(n,m)\ q_n^m(-\zeta_0)]\ q_n^m(\zeta)\ \ P_n^m(\xi)\cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm}\,B_{nm}\,[2(-1)^m\,g_3(n,m)\ q_n^m(\zeta)\ ]\ q_n^m(-\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower} \quad \zeta \le 0$$

$$B_{nm} = -(2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon])\ P_n^m(\xi_0)\sec[(n+m)\pi/2]\ f(n,-m)/\ K_n^m(\xi_1)$$

Notice now that the two lines for the lower bloid are the same! This is reasonable since there is nothing on the dividing line to divide the regions, so we can now give a

Final Result (ha ha)



Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} B_{nm} [p_n^m(\zeta_0) - 2(-1)^m \; g_3(n,m) q_n^m(\zeta_0)] \; q_n^m(\zeta) \; P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} B_{nm} [p_n^m(\zeta) - 2(-1)^m \; g_3(n,m) q_n^m(\zeta)] \; q_n^m(\zeta_0) \; P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

Lower half bloid:

$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} \; 2(-1)^m \; g_3(n,m) \; q_n^m(\zeta_0) \; q_n^m(\zeta) \; P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

We get the q function "all the way" in the lower bloid! Here is all supporting data:

$$B_{nm} = (2-\delta_{m,0}) \; (q/[4\pi c_1 \varepsilon]) \; P_n^m(\xi_0) \; \sec[(n+m)\pi/2] \; f(n,-m)/ \; K_n^m(\xi_1)$$
$$f(n,-m) = \Gamma(1+n-m)/ \; \Gamma(1+n+m)$$
$$K_n^m(\xi_1) = - \; (1-\xi_1^2) \; /(2n+1) \; * \; \partial_\xi P_n^m(\xi_1) \; \partial_n P_n^m(\xi_1)$$
$$g_3(n,m) = 1/ \; [\; \Gamma(1/2 + n/2 + m/2) \; \Gamma(1/2 - n/2 - m/2) \;]$$

Below I find this result

$$\pi \sec((n+m)\pi/2) = \Gamma(1/2 + n/2+m/2) \; \Gamma(1/2 - n/2-m/2) \; = [1/ \; g_3(n,m)]$$

which allows us to write an alternate form for B_{nm}

$$B_{nm} = (2-\delta_{m,0}) \; (q/[4\pi^2 c_1 \varepsilon]) \; P_n^m(\xi_0) \; [1/g_3(n,m)] \; f(n,-m)/ \; K_n^m(\xi_1)$$

We could write the solution even more densely as follows with min/max of (ζ,ζ_0) :

$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [p_n^m(\zeta_<) - 2(-1)^m \; g_3(n,m) \; q_n^m(\zeta_<)] \; q_n^m(\zeta_>) \; P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$
$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [\qquad 2(-1)^m \; g_3(n,m) \; q_n^m(\zeta_0)] \; q_n^m(\zeta) \; P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

(h) Discussion.

I cannot think of any more boundary conditions. Everything seems right at the neck, at the bloid surface, at the Green's charge, and at infinity.

Here is a comment from my attempt 3 after it failed:

"**Here is a very easy way to see the problem here:** for the bloid, we know we are trying to come up with a solution which is *different* in the upper and lower regions. But our atoms p and q are the same in these two regions, and so can only generate symmetric solutions. The form is dead on arrival. "

Why is this statement wrong? Here is our original Smythian form:

Upper half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm} \; p_n^m(\zeta_0) + C_{nm} \; q_n^m(\zeta_0)] \; q_n^m(\zeta) \; P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B_{nm} \; p_n^m(\zeta) + C_{nm} \; q_n^m(\zeta)] \; q_n^m(\zeta_0) \; P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

Lower half bloid:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta_0) + C'_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$$
$$V_i(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(\zeta) + C'_{nm}\ q_n^m(\zeta)\ ]\ q_n^m(\zeta_0)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \le 0$$

It is true that both $p_n^m(\zeta)$ and $q_n^m(\zeta)$ are symmetric. It is true that the forms shown here are exactly the same for upper and lower, apart from primes on the coefficients. But it is NOT true that the coefficients are the same, so upper and lower are in fact not the same.

Here is another statement:

"So I need to go off and ponder this idea of my p and q functions supposedly being independent solutions of our ζ ODE, and yet both are even. But they are supposed to be a complete set like cos(mφ) and sin(mφ) to allow you to construct a general function f(φ), but in ζ I can only construct EVEN functions of ζ. "

I guess this limitation of our real functions did not prevent us from solving the oblate bloid, as shown above. I am still not sure WHY these independent real solutions to the ζ flipped Legendre equation are symmetric, but I will ponder that.

Feb 26. I have pondered it and I have all the answers ("Why are p and q symmetric.doc"). The two independent solutions of the flipped Legendre ζ equation can be and should be taken as the symmetric p and q functions. There exists another p' function, call it, which is odd, but of course it is a lincomb of p and q. Here is how you can make a non-even function out of p and q: $f(\zeta) = 3p(\zeta) \pm 2q(\zeta)$ where the sign is sign(ζ), so that is not really the problem I thought it was. Even though p and q are even, f is not even. And of course $g(\zeta) = \pm q(\zeta)$ is an odd function.

I am now ready to continue on taking limits of the above "Final Result".

4. The Cone Limit

I took this limit in "Green's Function for an Oblate Bloid using cone method v3.doc" where I had my solution being unfortunately complex not real, but I think some of the data there will be useful here. Bateman's cone solution was this:

r<a $\quad V_i = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r/a)^n \quad (-1)^m P_n^m(z) \cos(m\varphi)$

r>a $\quad V_e = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r/a)^{-n-1} (-1)^m P_n^m(z) \cos(m\varphi)$ ok

a = his distance of the point charge from the cone origin
q = 1 = his point charge value

which we can write as a one-liner where $r_>$ and $r_<$ are max and min of r and a.

$$V_{cone} = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r_</a)^n (r_>/a)^{-n-1} \quad (-1)^m P_n^m(z) \cos(m\varphi)$$

I have added $(-1)^m$ above because these are on the cut P functions and Smythe has that phase difference. The A_{mn} are given by (Bateman P here as well) [again, from above quoted doc]

$$A_{mn} = (2-\delta_{m,0})\,(2\pi\varepsilon a)^{-1}\,(-1)^m\,P_n^m(\xi_0) \,/\, [(2n+1)K_n(\xi_1)]$$

In the ξ coordinate, I am using entirely Bateman on-the-cut P functions.

So much for the Smythe cone solution which I want to use as a check. My bloid result was this, for the distant region in the upper bloid

$$V(\zeta,\xi) = \Sigma_{nm}\, B_{nm}\,[p_n^m(\zeta_<) - 2(-1)^m\, g_3(n,m) q_n^m(\zeta_<)]\, q_n^m(\zeta_>)\, P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

I am allowed now to think of both $\zeta_<$ and $\zeta_>$ as large, so will need large ζ limits of the p and q functions. And of course we have

$$B_{nm} = (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon])\; P_n^m(\xi_0)\,\sec[(n+m)\pi/2]\, f(n,-m)/\, K_n^m(\xi_1)$$

which needs no limits taken. Now I can see that for large $\zeta_<$, the p term will dominate the q term due to the usual large-z behavior rules, so I can replace my V above with this simpler one in the cone limit:

$$V(\zeta,\xi) = \Sigma_{nm}\, B_{nm}\, p_n^m(\zeta_<)\; q_n^m(\zeta_>)\; P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

which looks promising. The limits I need are from P and Q doc,

$$q_n^m(\zeta) \;\rightarrow\; \sqrt{\pi}\;\; [\,\Gamma(1+n+m)/\,\Gamma(n+3/2)]\; 2^{-n-1}|\zeta|^{-n-1} \qquad \text{large } |\zeta|$$

$$p_n^m(\zeta) \rightarrow\; \cos[(n+m)\pi/2]\; (2^{n+1}/\sqrt{\pi}\,)\; [\Gamma(n+1/2)/\,\Gamma(1+n-m)]\; |\zeta|^n \qquad \text{large } |\zeta|$$

We also know that

$$\zeta = r/c_1 \qquad \text{and} \qquad \zeta_0 = a/c_1 \qquad \Rightarrow \qquad \zeta_> = r_>/c_1 \quad \text{and } \zeta_< = r_</c_1$$

Then we have

$$q_n^m(\zeta_>) \;\rightarrow\; \sqrt{\pi}\;\; [\,\Gamma(1+n+m)/\,\Gamma(n+3/2)]\; 2^{-n-1}(r_>/c_1)^{-n-1}$$

$$p_n^m(\zeta_<) \rightarrow\; \cos[(n+m)\pi/2]\; (2^{n+1}/\sqrt{\pi}\,)\; [\Gamma(n+1/2)/\,\Gamma(1+n-m)]\; (r_</c_1)^n$$

so that

$$p_n^m(\zeta_<)\, q_n^m(\zeta_>) = \sqrt{\pi}\;\; [\,\Gamma(1+n+m)/\,\Gamma(n+3/2)]\; 2^{-n-1}(r_>/c_1)^{-n-1}$$
$$* \;\cos[(n+m)\pi/2]\; (2^{n+1}/\sqrt{\pi}\,)\; [\Gamma(n+1/2)/\,\Gamma(1+n-m)]\; (r_</c_1)^n$$

$$= \;\; [\,\Gamma(1+n+m)/\,\Gamma(n+3/2)]\; (r_>/c_1)^{-n-1} \;*\; \cos[(n+m)\pi/2]\; [\Gamma(n+1/2)/\,\Gamma(1+n-m)]\; (r_</c_1)^n$$

Now write

$$(r_>/c_1)^{-n-1}\,(r_</c_1)^{n} \;=\; (r_>)^{-n-1}\,(r_<)^{n}\;c_1$$
$$(r_>/a)^{-n-1}\,(r_</a)^{n} \;=\; (r_>)^{-n-1}\,(r_<)^{n}\;a$$

$$\Rightarrow \qquad (r_>/c_1)^{-n-1}\,(r_</c_1)^{n} \;=(r_>/a)^{-n-1}\,(r_</a)^{n} \;*\,(c_1/a)$$

Then we have

$$p_n^m(\zeta_<)\,q_n^m(\zeta_>) \;=\; [\,f(n,m)/\,\Gamma(n+3/2)]\,(r_>/a)^{-n-1} \;*\; \cos[(n+m)\pi/2]\,[\Gamma(n+1/2)]\,(r_</a)^{n} \;\; (c_1/a)$$

$$= (c_1/a)\,f(n,m)\,\cos[(n+m)\pi/2]\,[\Gamma(n+1/2)/\,\Gamma(n+3/2)] \;\; (r_</a)^{n}\,(r_>/a)^{-n-1}$$

Now recall that

$$B_{nm} \;=\; (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon]) \;\; P_n^m(\xi_0)\,\sec[(n+m)\pi/2]\,f(n,-m)/\,K_n^m(\xi_1)$$

so we can form this combination:

$$B_{nm}\,p_n^m(\zeta_<)\,q_n^m(\zeta_>) = (c_1/a)\,f(n,m)\,\cos[(n+m)\pi/2]\,[\Gamma(n+1/2)/\,\Gamma(n+3/2)] \;\; (r_</a)^{n}\,(r_>/a)^{-n-1}$$

$$*\;\; (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon]) \;\; P_n^m(\xi_0)\,\sec[(n+m)\pi/2]\,f(n,-m)/\,K_n^m(\xi_1)$$

$$= \;(c_1/a) \;\; [\Gamma(n+1/2)/\,\Gamma(n+3/2)] \;\; (r_</a)^{n}\,(r_>/a)^{-n-1} \;\; * \;\; (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon]) \;\; P_n^m(\xi_0) \;\; /\,K_n^m(\xi_1)$$

But $\Gamma(n+3/2) = (n+1/2)\,\Gamma(n+1/2) \;= \Gamma(n+1/2)\,(2n+1)/2$ so we continue with the above

$$= \;(c_1/a) \;\; [2/(2n+1)] \;\; (r_</a)^{n}\,(r_>/a)^{-n-1} \;\; * \;\; (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon]) \;\; P_n^m(\xi_0) \;\; /\,K_n^m(\xi_1)$$

$$= \;(2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon])\,(c_1/a)\;2\;(r_</a)^{n}\,(r_>/a)^{-n-1} \;\; * \;\; P_n^m(\xi_0) \;\; /\,[(2n+1)\,K_n^m(\xi_1)$$

$$= \;(2-\delta_{m,0})\,(q/[2\pi a\varepsilon])\,(r_</a)^{n}\,(r_>/a)^{-n-1} \;\; P_n^m(\xi_0) \;\; /\,[(2n+1)\,K_n^m(\xi_1)$$

$$= \{\,q(-1)^m\,\} \;\; * \;\{\,(2-\delta_{m,0})\,(1/[2\pi a\varepsilon])\,(r_</a)^{n}\,(r_>/a)^{-n-1} \;\; (-1)^m\,P_n^m(\xi_0) \;\; /\,[(2n+1)\,K_n^m(\xi_1)$$

$$= \;\{\,q(-1)^m\,\}\;A_{mn} \;\; (r_</a)^{n}\,(r_>/a)^{-n-1}$$

Therefore, my bloid limit is this:

$$V(\zeta,\xi) \;= \Sigma_{nm}\,B_{nm}\,p_n^m(\zeta_<) \;\; q_n^m(\zeta_>) \;\; P_n^m(\xi)\,\cos(m\varphi) \qquad\qquad //\text{upper} \quad \zeta \geq 0$$

$$= \Sigma_{nm}\,\{q\,(-1)^m\,\}\;A_{mn} \;\; (r_</a)^{n}\,(r_>/a)^{-n-1} \;\; P_n^m(\xi)\,\cos(m\varphi)$$

and we compare this to the Smythe cone result above which was (he has q = 1)

$$V_{cone} = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r_</a)^n (r_>/a)^{-n-1} \quad (-1)^m P_n^m(z) \cos(m\varphi)$$

and we agree exactly! By the way, somehow this confirms the idea that I have only $m \geq 0$ in my Σ_{nm} sum. If it were a full range sum, we would not match the cone limit. This has something to do with completeness of the m-spectrum, let's try to avoid worrying about that right now. I think $m \geq$ provides the complete spectrum and so that is the sum you need to use.

Now what happens in the lower bloid? Recall our "final result" for the entire bloid:

$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [p_n^m(\zeta_<) - 2(-1)^m g_3(n,m) q_n^m(\zeta_<)] q_n^m(\zeta_>) P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$
$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [\quad 2(-1)^m g_3(n,m) q_n^m(\zeta_<)] q_n^m(\zeta_>) P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower } \zeta \leq 0$$

In the upper region we threw out the q term relative to the p term for large arg. But in the lower bloid, all we have is the $q(\zeta)$ term. We in the lower bloid we want to evaluate this quantity:

$$B_{nm} q_n^m(\zeta_<) q_n^m(\zeta_>)$$

$$q_n^m(\zeta_>) \rightarrow \sqrt{\pi} \; [\Gamma(1+n+m)/\Gamma(n+3/2)] \, 2^{-n-1} (r_>/c_1)^{-n-1}$$
$$q_n^m(\zeta_<) \rightarrow \sqrt{\pi} \; [\Gamma(1+n+m)/\Gamma(n+3/2)] \, 2^{-n-1} (r_</c_1)^{-n-1}$$

so

$$V_{lower} \sim B_{nm} q_n^m(\zeta_<) q_n^m(\zeta_>) \sim (r_>/c_1)^{-n-1} (r_</c_1)^{-n-1}$$

which we can compare to what happens in the upper bloid

$$V_{upper} \sim B_{nm} p_n^m(\zeta_<) q_n^m(\zeta_>) \sim (r_>/c_1)^{-n-1} (r_</c_1)^{+n}$$

So here is the ratio of potential strength within a partial wave n,m:

$$V_{lower}/V_{upper} \sim (r_</c_1)^{-2n-1}$$

So you get the idea that things are very weak in the lower bloid. As we truly go to the cone limit, the neck completely closes down, this means $c_1 \rightarrow 0$ and the above ratio $\rightarrow 0$ for any $n > -1/2$ which range includes all n in our n spectrum. In this limit, the potential in the lower bloid is 0 everywhere. But in the upper bloid it is this, where Green's charge is at radial distance a from the cone apex at the origin:

$$V_{cone} = \Sigma_{m=0}^{\infty} \Sigma_n A_{mn} (r_</a)^n (r_>/a)^{-n-1} \quad (-1)^m P_n^m(z) \cos(m\varphi)$$

$$A_{mn} = (2-\delta_{m,0}) (2\pi\varepsilon a)^{-1} (-1)^m P_n^m(\xi_0) / [(2n+1)K_n(\xi_1)]$$

$$K_n^m(\xi_1) = - (1-\xi_1^2) /(2n+1) \; * \; \partial_\xi P_n^m(\xi_1) \, \partial_n P_n^m(\xi_1)$$

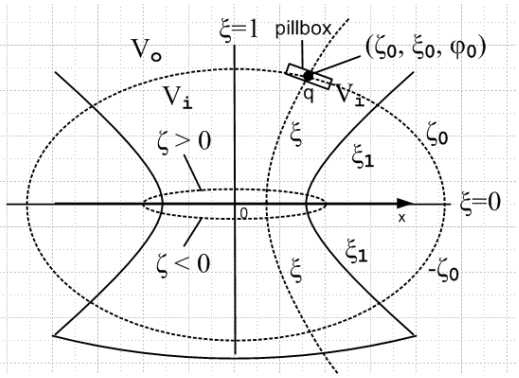
5. The On-Axis Limit

Our starting position is this,

$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [p_n^m(\zeta_<) - 2(-1)^m g_3(n,m) q_n^m(\zeta_<)] q_n^m(\zeta_>) P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$
$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [\quad 2(-1)^m g_3(n,m) q_n^m(\zeta_<)] q_n^m(\zeta_>) P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower } \zeta \leq 0$$

$B_{nm} = (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon])\;P_n^m(\xi_0)\,\sec[(n+m)\pi/2]\,f(n,-m)/\,K_n^m(\xi_1)$
$f(n,-m) = \Gamma(1+n-m)/\,\Gamma(1+n+m)$
$K_n^m(\xi_1) = -\,(1-\xi_1^2)\,/(2n+1)\;*\;\partial_\xi P_n^m(\xi_1)\,\partial_n P_n^m(\xi_1)$
$g_3(n,m) = 1/\,[\,\Gamma(1/2 + n/2 + m/2)\,\Gamma(1/2 - n/2 - m/2)\,]$

Putting the Green's charge on axis means $\xi_0 = 1$.



The only effect this has is on the B_{nm} . Consulting our leg prop doc we find that

$P_\nu^\mu(1) = \delta^\mu_0$ // μ = integers and general ν

Therefore we have

$B_{nm} = (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon])\;P_n^m(1)\,\sec[(n+m)\pi/2]\,f(n,-m)/\,K_n^m(\xi_1)$
$= (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon])\;\delta^m_0\,\sec[(n+m)\pi/2]\,f(n,-m)/\,K_n^m(\xi_1)$
$= (1)\,(q/[4\pi c_1\varepsilon])\;\delta^m_0\,\sec[(n)\pi/2]\;1/\,K_n^0(\xi_1)$
$= \delta^m_0\,(q/[4\pi c_1\varepsilon])\;\sec[n\pi/2]\,/\,K_n^0(\xi_1)$

$B_{n0} = (q/[4\pi c_1\varepsilon])\;\sec[n\pi/2]\,/\,K_n^0(\xi_1)$

We have now the n-spectrum $P_n(\xi_1) = 0$, and $K_n^0(\xi_1)$ is still nothing simple I know of, so leave. Our solutions is then this for on-axis:

$V(\zeta,\xi) = \Sigma_n\,B_{n0}\,[p_n^0(\zeta_<) - 2\;g_3(n,0)q_n^0(\zeta_<)]\;q_n^0(\zeta_>)\;P_n(\xi)$ //upper $\zeta \geq 0$
$V(\zeta,\xi) = \Sigma_n\,B_{n0}\,[\qquad 2\;g_3(n,0)q_n^0(\zeta_<)]\;q_n^0(\zeta_>)\;P_n(\xi)$ //lower $\zeta \leq 0$

$p_n^0(\zeta_<) = [P_n(i\zeta) + P_n(-i\zeta)]$
$q_n^0(\zeta) = (\pm i)^{n+1}\,Q_n(i\zeta)$
$B_{n0} = (q/[4\pi c_1\varepsilon])\;\sec[n\pi/2]\,/\,K_n^0(\xi_1)$
$K_n^0(\xi_1) = -\,(1-\xi_1^2)\,/(2n+1)\;*\;\partial_\xi P_n(\xi_1)\,\partial_n P_n(\xi_1)$
$g_3(n,0) = 1/\,[\,\Gamma(1/2 + n/2)\,\Gamma(1/2 - n/2\,)\,]$

On scratch I show that

$\Gamma(1/2 + n/2)\,\Gamma(1/2 - n/2) = \Gamma(1+x)\Gamma(-x)$ $\qquad x = n/2-1/2$

$= -\pi \csc[\,\pi(n/2-1/2)] \;=\; +\pi \sec(n\pi/2)$

so that

$$B_{n0}\, g_3(n,0) = +[\pi \sec(n\pi/2)]^{-1}\,(q/[4\pi c_1\varepsilon])\;\sec[n\pi/2] / K_n^0(\xi_1)$$
$$= (q/[4\pi^2 c_1\varepsilon]) / K_n^0(\xi_1)$$

So the on-axis result is pretty similar to the general result, except the m sum is gone and m = 0.

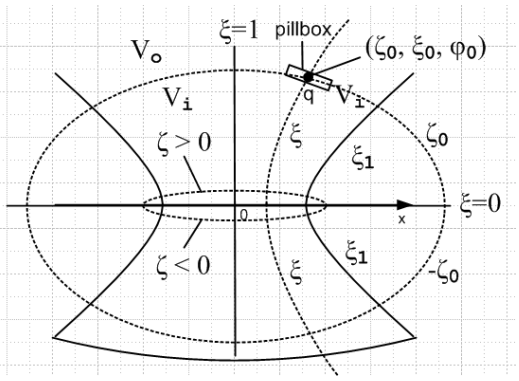
6. In-the-neck Limit

Our starting position is this,

$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [p_n^m(\zeta_<) - 2(-1)^m\; g_3(n,m) q_n^m(\zeta_<)]\; q_n^m(\zeta_>)\; P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$
$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [\qquad 2(-1)^m\; g_3(n,m) q_n^m(\zeta_<)]\; q_n^m(\zeta_>)\; P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

$$B_{nm} = (2-\delta_{m,0})\,(q/[4\pi c_1\varepsilon])\; P_n^m(\xi_0)\,\sec[(n+m)\pi/2]\; f(n,-m)/\, K_n^m(\xi_1)$$
$$f(n,-m) = \Gamma(1+n-m)/\,\Gamma(1+n+m)$$
$$K_n^m(\xi_1) = -\,(1-\xi_1^2)\,/(2n+1) \;*\; \partial_\xi P_n^m(\xi_1)\,\partial_n P_n^m(\xi_1)$$
$$g_3(n,m) = 1/\,[\,\Gamma(1/2 + n/2 + m/2)\,\Gamma(1/2 - n/2 - m/2)\,]$$

Putting the Green's charge in the neck means $\zeta_0 = 0$.



Thus, $\zeta_0 = 0$ will always be $\zeta_<$ and then $\zeta_<$ will always be ζ, so we may write

$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [p_n^m(0) - 2(-1)^m\; g_3(n,m)\; q_n^m(0)]\; q_n^m(\zeta)\; P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$
$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [\qquad 2(-1)^m\; g_3(n,m)\; q_n^m(0)]\; q_n^m(\zeta)\; P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

Our "reality of P and Q doc" tells us that

$$p_n^m(0) = 2^{m+1}\sqrt{\pi}\; / [\,\Gamma(1/2 - n/2 - m/2)\,\Gamma(1 + n/2 - m/2)\,]$$
$$q_n^m(0) = (-1)^m \sqrt{\pi}\; 2^{m-1}\,[\,\Gamma(1/2 + n/2 + m/2) / \Gamma(1 + n/2 - m/2)]$$

and now

$$g_3(n,m)\, q_n^m(0) = 1/\left[\,\Gamma(1/2 + n/2 + m/2)\,\Gamma(1/2 - n/2 - m/2)\,\right] * (-1)^m \sqrt{\pi}\, 2^{m-1}\left[\,\Gamma(1/2 + n/2 + m/2) / \Gamma(1 + n/2 - m/2)\right]$$

$$= 1/\left[\Gamma(1/2 - n/2 - m/2)\,\right] * (-1)^m \sqrt{\pi}\, 2^{m-1}\left[\,1 / \Gamma(1 + n/2 - m/2)\right]$$

$$= (-1)^m \sqrt{\pi}\, 2^{m-1}\; 1/\left[\Gamma(1/2 - n/2 - m/2)\;\Gamma(1 + n/2 - m/2)\right]$$

And then

$$\left[p_n^m(0) - 2(-1)^m\; g_3(n,m)\, q_n^m(0)\right]$$

$$= \left(2^{m+1}\sqrt{\pi}\; - 2(-1)^m\;(-1)^m \sqrt{\pi}\, 2^{m-1}\right) / \left[\Gamma(1/2 - n/2 - m/2)\;\Gamma(1 + n/2 - m/2)\right]$$

$$= \sqrt{\pi}\;\left(2^{m+1}\; -\; 2^m\right) / \left[\Gamma(1/2 - n/2 - m/2)\;\Gamma(1 + n/2 - m/2)\right]$$

$$= \sqrt{\pi}\, 2^m\,(2\; - 1\,) / \left[\Gamma(1/2 - n/2 - m/2)\;\Gamma(1 + n/2 - m/2)\right]$$

$$= \sqrt{\pi}\, 2^m\; / \left[\Gamma(1/2 - n/2 - m/2)\;\Gamma(1 + n/2 - m/2)\right]$$

Also, we can say

$$2(-1)^m\, g_3(n,m)\, q_n^m(0) = 2(-1)^m(-1)^m \sqrt{\pi}\, 2^{m-1}\; 1/\left[\Gamma(1/2 - n/2 - m/2)\;\Gamma(1 + n/2 - m/2)\right]$$

$$= \sqrt{\pi}\, 2^m\; 1/\left[\Gamma(1/2 - n/2 - m/2)\;\Gamma(1 + n/2 - m/2)\right]$$

If we now define

$$h(n,m) = 1/\left[\Gamma(1/2 - n/2 - m/2)\;\Gamma(1 + n/2 - m/2)\right]$$

we have then showed that

$$2(-1)^m\, g_3(n,m)\, q_n^m(0) = \sqrt{\pi}\, 2^m\, h(n,m)$$

$$\left[p_n^m(0) - 2(-1)^m\; g_3(n,m)\, q_n^m(0)\right] = \sqrt{\pi}\, 2^m\, h(n,m)$$

so we can rewrite our result this way

$$V(\zeta,\xi) = \Sigma_{nm}\, B_{nm}\left[\sqrt{\pi}\, 2^m\, h(n,m)\right] q_n^m(\zeta)\; P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

$$V(\zeta,\xi) = \Sigma_{nm}\, B_{nm}\left[\sqrt{\pi}\, 2^m\, h(n,m)\right] q_n^m(\zeta)\; P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

$$B_{nm} = (2-\delta_{m,0})\,(q/[4\pi c_1 \varepsilon])\; P_n^m(\xi_0)\,\sec[(n+m)\pi/2]\; f(n,-m)/\; K_n^m(\xi_1)$$

$$f(n,-m) = \Gamma(1+n-m)/\;\Gamma(1+n+m)$$

$$K_n^m(\xi_1) = -\,(1-\xi_1^2)\,/(2n+1)\; *\; \partial_\xi P_n^m(\xi_1)\,\partial_n P_n^m(\xi_1)$$

$h(n,m) = 1/[\Gamma(1/2 - n/2 - m/2)\ \Gamma(1 + n/2 - m/2)]$
$q_n^m(\zeta) \equiv (-i)^{2m}\ (\pm i)^{n+1}\ Q_n^m(i\zeta)$

and we get the up/down symmetry that we expect in this situation !! So if Green's in the neck, you get the Q decay in both directions.

There might be some further simplification of the constants but I don't think it adds much.

I could now take the on-axis limit of the above neck result and I would then find that

$$B_{nm} = \delta^m{}_0\ (q/[4\pi c_1 \varepsilon])\ \sec[n\pi/2] / K_n^0(\xi_1)$$

and our result would then be

$$V(\zeta,\xi) = \Sigma_n\ B_{n0}\ [\sqrt{\pi}\ h(n,0)]\ q_n^0(\zeta)\ P_n(\xi) \qquad \text{//upper and lower, in neck, on axis}$$

$h(n,0) = 1/[\Gamma(1/2 - n/2)\ \Gamma(1 + n/2)]$
$B_{n0} = (q/[4\pi c_1 \varepsilon])\ \sec[n\pi/2] / K_n^0(\xi_1)$
$q_n^0(\zeta) \equiv (\pm i)^{n+1}\ Q_n(i\zeta)$

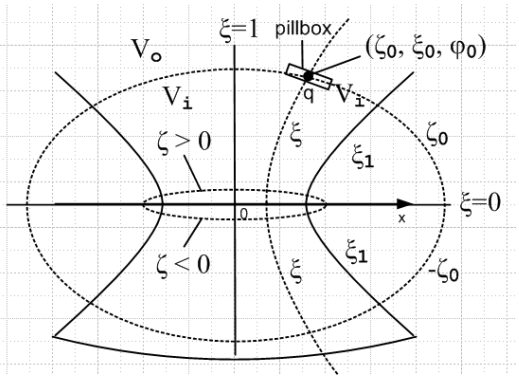
7. The Iris Limit

This is the limit I have long been seeking! Our starting position is this,

$$V(\zeta,\xi) = \Sigma_{nm}\ B_{nm}\ [p_n^m(\zeta_<) - 2(-1)^m\ g_3(n,m) q_n^m(\zeta_<)]\ q_n^m(\zeta_>)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//upper } \zeta \geq 0$$
$$V(\zeta,\xi) = \Sigma_{nm}\ B_{nm}\ [\qquad 2(-1)^m\ g_3(n,m) q_n^m(\zeta_<)]\ q_n^m(\zeta_>)\ P_n^m(\xi)\cos(m\varphi) \qquad \text{//lower } \zeta \leq 0$$

$B_{nm} = (2-\delta_{m,0})\ (q/[4\pi c_1 \varepsilon])\ P_n^m(\xi_0)\ \sec[(n+m)\pi/2]\ f(n,-m)/ K_n^m(\xi_1)$
$f(n,-m) = \Gamma(1+n-m)/\ \Gamma(1+n+m)$
$K_n^m(\xi_1) = -\ (1-\xi_1^2)\ /(2n+1)\ *\ \partial_\xi P_n^m(\xi_1)\ \partial_n P_n^m(\xi_1)$
$g_3(n,m) = 1/\ [\ \Gamma(1/2 + n/2 + m/2)\ \Gamma(1/2 - n/2 - m/2)\ ]$

The Iris Limit means $\zeta_1 = 0$.



I am not sure I have fully dealt with this limit properly in another doc, but I will look now. Found stuff in "Green's Function for an Oblate Bloid using cone method v3.doc". If we regard the n-spectrum as being complete for n values to the right of n = -1/2, we find this slightly strange n-spectrum,

The non-redundant spectrum of n for a given $m \geq 0$ value:

$m \geq 0$, even	n = all positive odds + [evens in range (0,m-1)]	**n spectrum**
m= 0	n = all positive odds	
m > 0, odd	n = all positive evens + [odds in range (0,m-1)]	

This spectrum comes from this fact:

$$0 = P_n^m(0) = 2^\mu\sqrt{\pi} \,/\, [\, \Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2)\,] = 2^\mu\sqrt{\pi}\, h(n,m)$$

Thus, for a given m, every n value in the n-spectrum causes h(n,m) = 0.

I go on to show that

$$1/ K_n(0) = (2n+1)\, f(n,-m)$$

So our solution now becomes

$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [p_n^m(\zeta_<) - 2(-1)^m\, g_3(n,m) q_n^m(\zeta_<)]\, q_n^m(\zeta_>)\, P_n^m(\xi) \cos(m\varphi) \quad \text{//upper} \quad \zeta \geq 0$$
$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [\quad 2(-1)^m\, g_3(n,m) q_n^m(\zeta_<)]\, q_n^m(\zeta_>)\, P_n^m(\xi) \cos(m\varphi) \quad \text{//lower} \quad \zeta \leq 0$$

$$B_{nm} = (2-\delta_{m,0})\, (q/[4\pi c_1 \varepsilon])\, P_n^m(\xi_0) \sec[(n+m)\pi/2]\, f(n,-m)/ K_n^m(0)$$
$$f(n,-m) = \Gamma(1+n-m)/ \Gamma(1+n+m)$$
$$1/ K_n(0) = (2n+1)\, f(n,-m)$$
$$g_3(n,m) = 1/ [\, \Gamma(1/2 + n/2+ m/2)\, \Gamma(1/2 - n/2 - m/2)\,]$$

So nothing really simplifies much except the n-spectrum and the K expression.

8. First put Green's charge in the neck, then go to the Iris Limit

The in neck limit ($\zeta_0 = 0$) gives this from above

$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [\sqrt{\pi}\, 2^m\, h(n,m)]\, q_n^m(\zeta)\, P_n^m(\xi) \cos(m\varphi) \quad \text{// upper and lower}$$

$$B_{nm} = (2-\delta_{m,0})\, (q/[4\pi c_1 \varepsilon])\, P_n^m(\xi_0) \sec[(n+m)\pi/2]\, f(n,-m)/ K_n^m(\xi_1)$$
$$f(n,-m) = \Gamma(1+n-m)/ \Gamma(1+n+m)$$
$$K_n^m(\xi_1) = - (1-\xi_1^2) /(2n +1) \; * \; \partial_\xi P_n^m(\xi_1)\, \partial_n P_n^m(\xi_1)$$
$$h(n,m) = 1/[\Gamma(1/2 - n/2 - m/2)\; \Gamma(1 + n/2- m/2)]$$
$$q_n^m(\zeta) \equiv (-i)^{2m}\, (\pm i)^{n+1}\, Q_n^m(i\zeta)$$

We then add our iris limit ($\xi_1 = 0$) to this and we get

$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [\sqrt{\pi}\, 2^m h(n,m)]\, q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi)$ // upper and lower

$B_{nm} = (2-\delta_{m,0})\, (q/[4\pi c_1 \varepsilon])\ P_n^m(\xi_0) \sec[(n+m)\pi/2]\, f(n,-m)/ K_n^m(0)$
$f(n,-m) = \Gamma(1+n-m)/ \Gamma(1+n+m)$
$1/ K_n(0) = (2n+1)\, f(n,-m)$
$h(n,m) = 1/[\Gamma(1/2 - n/2 - m/2)\ \Gamma(1 + n/2- m/2)]$
$q_n^m(\zeta) \equiv (-i)^{2m}\, (\pm i)^{n+1}\, Q_n^m(i\zeta)$

The spectrum of n was those n which made

$P_n^m(x=0) = 2^\mu \sqrt{\pi}\ /\ [\ \Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2)\ ] = 2^\mu \sqrt{\pi}\ \ h(n,m)$

But this *seems* to say that at every point in the n spectrum, h(n,m) = 0, and therefore our entire potential collapses to 0. Again, this is for iris and Green's point in the neck somewhere. This time and last time I am just doing something wrong in my taking of this iris limit!

Let's write out all the gammas and see if this is really true. First, copy down the following from above

$\Gamma(1/2 + n/2)\, \Gamma(1/2 - n/2) = \Gamma(1+x)\Gamma(-x)$ $x = n/2-1/2$

$= -\pi \csc[\, \pi(n/2-1/2)] = +\pi \sec(n\pi/2)$

$\Rightarrow$ $\pi \sec(n\pi/2) = \Gamma(1/2 + n/2)\, \Gamma(1/2 - n/2)$

In this last equation, try replacing $n \rightarrow n+m$:

$\pi \sec((n+m)\pi/2) = \Gamma(1/2 + n/2+m/2)\, \Gamma(1/2 - n/2-m/2)$

Then consider

$\sec[(n+m)\pi/2]\ h(n,m) = \pi^{-1}\, \Gamma(1/2 + n/2+m/2)\, \Gamma(1/2 - n/2-m/2)\, /[\Gamma(1/2 - n/2 - m/2)\ \Gamma(1 + n/2- m/2)]$

$= \pi^{-1}\ \Gamma(1/2 + n/2+m/2) / \Gamma(1 + n/2- m/2)$

With this little preliminary, we now write:

$B_{nm}\, h(n,m) = (2-\delta_{m,0})\, (q/[4\pi c_1 \varepsilon])\ P_n^m(\xi_0) \sec[(n+m)\pi/2]\, f(n,-m)/ K_n^m(0) * h(n,m)$

$= (2-\delta_{m,0})\, (q/[4\pi c_1 \varepsilon])\, (2n+1)\, P_n^m(\xi_0)\ \{\ [f(n,-m)]^2 \sec[(n+m)\pi/2]\, h(n,m)\ \}$

$= (2-\delta_{m,0})\, (q/[4\pi^2 c_1 \varepsilon])\, (2n+1)\, P_n^m(\xi_0)\ [f(n,-m)]^2\, \Gamma(1/2 + n/2+m/2) / \Gamma(1 + n/2- m/2)$

Now we have killed off one of the gamma in h(n,m). I know from the prev men doc that the denom gamma you see above kills off $B_{nm}\, h(n,m)$ when

n = m - even = 2,4,6.. (but not 0)

so these terms will not survive in the n-sum. But the rest of the spectrum will. Those terms are these:

n = -m + odd odd = 1,3,5... n + m = odd

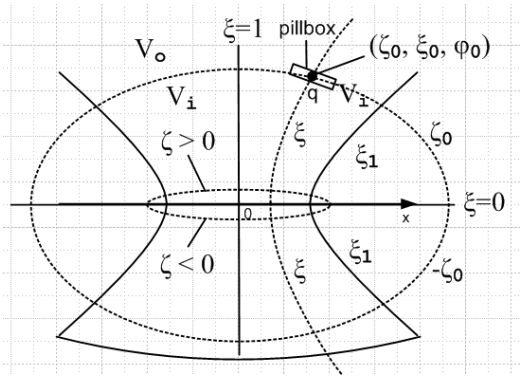
So here is our result written out:

$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [\sqrt{\pi}\, 2^m h(n,m)] q_n^m(\zeta) P_n^m(\xi) \cos(m\varphi) =$$

or

$$V(\zeta,\xi) = \Sigma_{m=0}^{\infty} \Sigma_{n=-m+odd} \sqrt{\pi}\, 2^m (2-\delta_{m,0}) (q/[4\pi^2 c_1 \varepsilon]) [\, \Gamma(1+n-m)/ \Gamma(1+n+m)]^2 * (2n+1) [\Gamma(1/2+n/2+m/2) / \Gamma(1+n/2- m/2)] P_n^m(\xi_0) q_n^m(\zeta) P_n^m(\xi) \cos(m\varphi)$$

This is supposedly the potential everywhere for the bloid bent down to an iris, and the Green's charge in the hole! Green's is at (ξ_0, $\zeta_0 = 0$, $\varphi_o = 0$), observation point is (ξ,ζ,φ).



Let's try to simplify some of the gamma functions. Consider that

$$2^{2x-1} \Gamma(x)\Gamma(x+1/2) = \sqrt{\pi}\ \Gamma(2x)$$

=>

$$\Gamma(x)/ \Gamma(2x) = \sqrt{\pi}\, 2^{1-2x} / \Gamma(x+1/2)$$

Try this with x = 1/2+n/2+m/2 so 2x = 1+n+m and x+1/2 = 1+n/2+m/2 and 1-2x = -n-m. Then we have

$$\Gamma(1/2+n/2+m/2)/ \Gamma(1+n+m) = \sqrt{\pi}\, 2^{-n-m} / \Gamma(1+n/2+m/2)$$

and we negate m to get this further result:

$$\Gamma(1/2+n/2-m/2)/ \Gamma(1+n-m) = \sqrt{\pi}\, 2^{-n+m} / \Gamma(1+n/2-m/2)$$

Therefore,

$$f(n,-m) * [\Gamma(1/2+n/2+m/2) / \Gamma(1+n/2- m/2)]$$

$$[\, \Gamma(1+n-m)/ \Gamma(1+n+m)] * [\Gamma(1/2+n/2+m/2) / \Gamma(1+n/2- m/2)]$$

$= \Gamma(1/2+n/2+m/2) / \Gamma(1+n+m) \ * \ [\Gamma(1+n/2- m/2)/ \Gamma(1+n-m)]^{-1}$

$= \sqrt{\pi}\ 2^{-n-m} / \Gamma(1+n/2+m/2) \ * \ [\sqrt{\pi}\ 2^{-n+m} / \Gamma(1+n/2-m/2)]^{-1}$

$= 2^{-2m} / \Gamma(1+n/2+m/2) \ * \ [1 / \Gamma(1+n/2-m/2)]^{-1}$

$= 2^{-2m}\ \Gamma(1+n/2-m/2)/ \Gamma(1+n/2+m/2) \ = 2^{-2m}\ f(n/2,-m/2)$

so this gives at least some reduction of our result above:

$$V(\zeta,\xi) \ = \Sigma_{m=0}^{\infty}\ \Sigma_{n=-m+odd}\ \sqrt{\pi}\ 2^{m}\ (2-\delta_{m,0})\ (q/[4\pi^2 c_1 \varepsilon])\ [\ \Gamma(1+n-m)/\ \Gamma(1+n+m)]^2$$
$$* \ (2n+1)\ [\Gamma(1/2+n/2+m/2) / \Gamma(1+n/2- m/2)]\ P_n^m(\xi_0)\ q_n^m(\zeta)\ \ P_n^m(\xi)\ \cos(m\varphi)$$

$$= \Sigma_{m=0}^{\infty}\ \Sigma_{n=-m+odd}\ \ 2^{m}\ (2-\delta_{m,0})\ (q/[4\pi^2 c_1 \varepsilon])\ [\ \Gamma(1+n-m)/\ \Gamma(1+n+m)]\ 2^{-2m}$$

$$* \ (2n+1)\ [\Gamma(1+n/2-m/2)/\ \Gamma(1+n/2+m/2)]\ P_n^m(\xi_0)\ q_n^m(\zeta)\ \ P_n^m(\xi)\ \cos(m\varphi)$$

or

$$V(\zeta,\xi) \ = \Sigma_{m=0}^{\infty}\ \Sigma_{n=-m+odd}\ \sqrt{\pi}\ 2^{-m}(2-\delta_{m,0})\ (q/[4\pi^2 c_1 \varepsilon])\ \ f(n,-m)\ f(n/2,-m/2)\ (2n+1)$$
$$* \ P_n^m(\xi_0)\ P_n^m(\xi)\ q_n^m(\zeta)\ \cos(m\varphi)$$

<u>This result is then for Green's charge in the hole, and bloid folded down to an iris.</u>

(a) Potential on the iris and in the hole.

Suppose now we try to set $\xi = 0$ so we are then <u>right on the metal iris.</u> Then factor $P_n^m(\xi) = 0$ and we obtain the desired fact that V = 0 on the entire iris.

Suppose we want to know the <u>potential just in the hole itself</u>. Then we set $\zeta = 0$ and use

$$q_n^m(0) \ = (-1)^m \sqrt{\pi}\ 2^{m-1}\ [\ \Gamma(1/2 + n/2+ m/2) / \Gamma(1 + n/2- m/2)]$$

Using our same reduction above, we will get

$$f(n,-m)\ q_n^m(0) = \ (-1)^m \sqrt{\pi}\ 2^{m-1}\ \Gamma(1+n-m)/\ \Gamma(1+n+m) \ * \ [\ \Gamma(1/2 + n/2+ m/2) / \Gamma(1 + n/2- m/2)]$$

$$= (-1)^m \sqrt{\pi}\ 2^{m-1} * 2^{-2m}\ \Gamma(1+n/2-m/2)/\ \Gamma(1+n/2+m/2)$$

$$= (-1)^m \sqrt{\pi}\ 2^{-m-1}\ [\Gamma(1+n/2-m/2)/\ \Gamma(1+n/2+m/2)] \ = \ (-1)^m \sqrt{\pi}\ 2^{-m-1}\ f(n/2,-m/2)$$

We then get

$$V(\zeta,\xi) \ = \Sigma_{m=0}^{\infty}\ \Sigma_{n=-m+odd}\ \sqrt{\pi}\ 2^{-m}(2-\delta_{m,0})\ (q/[4\pi^2 c_1 \varepsilon])\ \ f(n,-m)\ f(n/2,-m/2)\ (2n+1)$$

$$* P_n^m(\xi_0)\, P_n^m(\xi)\, q_n^m(\zeta) \cos(m\varphi)$$

$$= \Sigma_{m=0}^{\infty} \Sigma_{n=-m+odd} \sqrt{\pi}\, 2^{-m}(2-\delta_{m,0})\, (q/[4\pi^2 c_1 \varepsilon])\ \ f(n,-m)\, [f(n/2,-m/2)]^2\, (2n+1)\, (-1)^m \sqrt{\pi}\, 2^{-m-1}$$
$$* P_n^m(\xi_0)\, P_n^m(\xi) \cos(m\varphi)$$

$$= (q/[4\pi c_1 \varepsilon])\, \Sigma_{m=0}^{\infty} (-1)^m\ \ 2^{-2m-1}(2-\delta_{m,0})\, \Sigma_{n=-m+odd}\ \ (2n+1)\, f(n,-m)\, [f(n/2,-m/2)]^2$$
$$* P_n^m(\xi_0)\, P_n^m(\xi) \cos(m\varphi)$$

I suppose this could possibly be right. It has to blow up as $\xi \to \xi_0$ since then we are moving our observation point right to the Green's charge. Maybe this is reflected in that fact that, in this case, everything in the summand is positive (I think).

(b) Charge density on the iris // small correction made 3.8.10

Now, suppose we are after the <u>charge density on the iris</u>. This will involve a ∂_ξ derivative of the potential then evaluated at $\xi = 0$. So start with the general potential for the iris with charge in the hole at ξ_0:

$$V(\zeta,\xi) \ = \Sigma_{m=0}^{\infty} \Sigma_{n=-m+odd} \sqrt{\pi}\, 2^{-m}(2-\delta_{m,0})\, (q/[4\pi^2 c_1 \varepsilon])\ \ f(n,-m)\, f(n/2,-m/2)\, (2n+1)$$
$$* P_n^m(\xi_0)\, P_n^m(\xi)\, q_n^m(\zeta) \cos(m\varphi)$$

$$\partial_\xi V(\zeta,\xi) \ = \ \Sigma_{m=0}^{\infty} \Sigma_{n=-m+odd} \sqrt{\pi}\, 2^{-m}(2-\delta_{m,0})\, (q/[4\pi^2 c_1 \varepsilon])\ \ f(n,-m)\, f(n/2,-m/2)\, (2n+1)$$
$$* P_n^m(\xi_0)\, P_n^{m\,\prime}(\xi)\, q_n^m(\zeta) \cos(m\varphi)$$

and leg doc says [recall that all ξ related P functions are on the cut functions]

$$P_\nu^{m\,\prime}(x=0) \ = \ -\, P_\nu^{m+1}(x=0) \qquad\qquad m \geq 0 \text{ and integer}$$

and we know that

$$P_\nu^\mu(x=0) \ = 2^\mu \sqrt{\pi}\ / [\, \Gamma(1/2 - \nu/2 - \mu/2)\, \Gamma(1 + \nu/2 - \mu/2)\,]$$

which we adjust to be

$$P_\nu^{m\,\prime}(x=0) \ = \ -\, P_\nu^{m+1}(x=0) \qquad\qquad m \geq 0 \text{ and integer}$$
$$P_n^{m+1}(x=0) \ = 2^{m+1} \sqrt{\pi}\ / [\, \Gamma(1/2 - n/2 - (m+1)/2)\, \Gamma(1 + n/2 - (m+1)/2)\,]$$
$$= 2^{m+1} \sqrt{\pi}\ / [\, \Gamma(- n/2 - m/2)\, \Gamma(1/2 + n/2 - m/2)\,]$$

=>

$$P_n^{m\,\prime}(x=0) \ = \ -\ 2^{m+1} \sqrt{\pi}\ / [\, \Gamma(- n/2 - m/2)\, \Gamma(1/2 + n/2 - m/2)\,]$$

so we then have

$$\partial_\xi V(\zeta,0) \ = \ \Sigma_{m=0}^{\infty} \Sigma_{n=-m+odd} \sqrt{\pi}\, 2^{-m}(2-\delta_{m,0})\, (q/[4\pi^2 c_1 \varepsilon])\ \ f(n,-m)\, f(n/2,-m/2)\, (2n+1)$$
$$* P_n^m(\xi_0)\, \{\, -2^{m+1} \sqrt{\pi}\ / [\, \Gamma(- n/2 - m/2)\, \Gamma(1/2 + n/2 - m/2)\,]\} \, q_n^m(\zeta) \cos(m\varphi)$$

Now we have

$$f(n/2,-m/2) / \Gamma(1/2 + n/2 - m/2) = [\ \Gamma(1/2 + n/2 - m/2)/ \Gamma(1/2 + n/2 + m/2] / \Gamma(1/2 + n/2 - m/2)$$

$$= 1/ \Gamma(1/2 + n/2 + m/2)$$

so then

$$\partial_\xi V(\zeta,0) = \Sigma_{m=0}^{\infty} \Sigma_{n=-m+odd} \sqrt{\pi}\ 2^{-m}(2-\delta_{m,0})\ (q/[4\pi^2 c_1 \varepsilon])\ f(n,-m)\ (2n+1)$$
$$* P_n^m(\xi_0)\ \{ -2^{m+1}\sqrt{\pi}\ / [\ \Gamma(- n/2 - m/2)\ \Gamma(1/2 + n/2 + m/2)\ ]\}\ q_n^m(\zeta)\ \cos(m\varphi)$$

$$= -\ 2\ (q/[4\pi c_1 \varepsilon])\Sigma_{m=0}^{\infty}(2-\delta_{m,0})\ \Sigma_{n=-m+odd}\ f(n,-m)\ (2n+1)$$
$$* P_n^m(\xi_0)\ \{ 1 / [\ \Gamma(- n/2 - m/2)\ \Gamma(1/2 + n/2 + m/2)\ ]\}\ q_n^m(\zeta)\ \cos(m\varphi)$$

In this expression, we have:

$(\zeta_0, \xi_0, \varphi_0) = (0,\xi_0,0)$ = location of the Green's point charge in the hole
$(\zeta, \xi, \varphi) = (\zeta, 0, \varphi)$ = location on the iris where we are looking at σ

$\xi_0,\zeta_0=0,\varphi_0=0$ is the location of the charge in the hole, ζ labels the spheroid which marks your radial distance out on the iris from the hole center, and φ is your azimuth on the iris. The charge density on the iris will be this result combined with the h_ξ scale factor. With some work we could convert this thing to 2D polar coordinates on the iris. It is hard to imagine much simplification! At least the result is real! There is no obvious way to remove the P and q functions from this result! You could draw a nice iris picture and write the distance of the charge from the origin ρ_0 and the distance of the observation point from the origin ρ (and angle φ), but those P and q functions will always be there. So we are far away from the Problem 38 result quoted below. The expansion above could in fact be correct, but it is not the form of the answer we want.

(c) Comment: This brings a long voyage to an end. I got interested in the Green's for the bloid problem in the first place, because I somehow hoped that this final charge density formula would come out simple. But it is not simple, and it took about a month or more to get to this point! Let's look back now at a certain sequence of Smythe problems that got me motivated on this (Smythe page 203)

> **38.** A charge q is at a distance b from the center of a circular hole of radius a in an infinite earthed flat conducting sheet and lies in the same plane. Show that the charge density induced on the sheet at a distance r from the charge and c from the center of the hole is $-q(a^2 - b^2)^{\frac{1}{2}}/[2\pi^2 r^2(c^2 - a^2)^{\frac{1}{2}}]$.

So my path via the oblate bloid to get to this charge density gives the answer you see above, more or less, but obviously there is some much simpler method to do it. This problem in Smythe's problem list appears well before any problems which relate the oblate P and Q functions! Once you get the above solved, you can then do inversion and learn something about spherical bowls in the following problems.

9. A comparison between the real and complex bloid solutions.

While I was writing this section just now, I realized why the Smythian form which led to the complex solution was invalid, resolving that mystery. I added a paragraph at the top of that complex-solution document explaining this detail. (see "Green's Function for an Oblate Bloid using cone method v3.doc".)

10. Restatement of the General Bloid Solution:

$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [p_n^m(\zeta_<) - 2(-1)^m\ g_3(n,m)\ q_n^m(\zeta_<)]\ q_n^m(\zeta_>)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$
$$V(\zeta,\xi) = \Sigma_{nm} B_{nm} [\qquad 2(-1)^m\ g_3(n,m)\ q_n^m(\zeta_<)]\ q_n^m(\zeta_>)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

$$B_{nm} = (2-\delta_{m,0})\ (q/[4\pi c_1 \varepsilon])\ P_n^m(\xi_0)\ \sec[(n+m)\pi/2]\ f(n,-m)/\ K_n^m(\xi_1)$$
$$f(n,-m) = \Gamma(1+n-m)/\ \Gamma(1+n+m)$$
$$K_n^m(\xi_1) = -\ (1-\xi_1^2)\ /(2n+1)\ *\ \partial_\xi P_n^m(\xi_1)\ \partial_n P_n^m(\xi_1)$$
$$g_3(n,m) = 1/\ [\ \Gamma(1/2 + n/2 + m/2)\ \Gamma(1/2 - n/2 - m/2)\ ]$$

$$\pi \sec((n+m)\pi/2) = \Gamma(1/2 + n/2+m/2)\ \Gamma(1/2 - n/2-m/2) = [1/\ g_3(n,m)]$$
$$B_{nm} = (2-\delta_{m,0})\ (q/[4\pi^2 c_1 \varepsilon])\ P_n^m(\xi_0)\ [1/g_3(n,m)]\ f(n,-m)/\ K_n^m(\xi_1)$$

The summation is really this: $\Sigma_{nm} = \Sigma_{m=0}^{\infty}\ \Sigma_{n=n(i,m)}$ where n(i,m) is the spectrum of n with n > -1/2

Suppose we make this new definitions

$$g_4(n,m) = [\ \Gamma(1/2 + n/2+ m/2)\ \Gamma(1/2 - n/2 - m/2)\ ] \qquad / 1/g_3$$

$$C_{nm} = B_{nm}\ g_3(n,m) = (2-\delta_{m,0})\ (q/[4\pi^2 c_1 \varepsilon])\ P_n^m(\xi_0)\ f(n,-m)/\ K_n^m(\xi_1)$$

$$B_{nm} = C_{nm}\ g_4(n,m)$$

Then our solution is this, where I now supply all supporting data:

Final Solution:

$$V(\zeta,\xi,\varphi) = \Sigma_{nm} C_{nm} [g_4(n,m)\ p_n^m(\zeta_<) - 2(-1)^m\ q_n^m(\zeta_<)]\ q_n^m(\zeta_>)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$
$$V(\zeta,\xi,\varphi) = \Sigma_{nm} C_{nm} [\qquad 2(-1)^m\ q_n^m(\zeta_<)]\ q_n^m(\zeta_>)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

$$C_{nm} = (2-\delta_{m,0})\ (q/[4\pi^2 c_1 \varepsilon])\ P_n^m(\xi_0)\ f(n,-m)/\ K_n^m(\xi_1)$$
$$g_4(n,m) = \Gamma(1/2 + n/2+ m/2)\ \Gamma(1/2 - n/2 - m/2)$$
$$f(n,-m) = \Gamma(1+n-m)/\ \Gamma(1+n+m)$$
$$K_n^m(\xi_1) = -\ (1-\xi_1^2)\ /(2n+1)\ *\ \partial_\xi P_n^m(\xi_1)\ \partial_n P_n^m(\xi_1)$$
$$\Sigma_{nm} = \Sigma_{m=0}^{\infty}\ \Sigma_{n=n(i,m)} \text{ where } P_{n(i,m)}(\xi_1) = 0,\ i = 1,2,3...$$
$$q_n^m(\zeta) \equiv (-i)^{2m}\ (\pm i)^{n+1}\ Q_n^m(i\zeta) \qquad \text{Re}(\zeta) > 0 \text{ upper sign, } \text{Re}(\zeta) < 0 \text{ lower sign}$$
$$p_n^m(\zeta) \equiv [\ (\mp i)^{-m}\ P_n^m(i\zeta)\ + (\pm i)^{-m} P_n^m(-i\zeta)]$$
$P_n^m(\xi)$ is Bateman on the cut, other P and Q are Bateman off the cut

c_1 = focal distance of the oblate hyperboloid
potential of point charge is $q/(4\pi\varepsilon)$
location of Green's charge as shown below but with $\varphi_0 = 0$; observation point (ζ,ξ,φ),

