Exterior Green's Function for an Oblate Spheroid PhL 1.25.10

Note: after doing this problem, I want to do the Green's function for an Oblate Hyperboloid, where we will have the condition $V_o = 0$ on the bloid ξ_1 . Then I want to take the limit $\xi_1 = 0$ to get the Green's Function for a charge near an iris. Then I will put the point charge into the hole in the iris and try to obtain the very simple result for the surface charge claimed in Smythe Problem 38.

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Overview (10.4.10, 3 pages)

In **Section 1** I propose a Smythian inside/outside form which matches on the spheroidal boundary containing the Green's point charge, which has the proper large ζ behavior outside this boundary (Q), and which vanishes on the metal spheroid $\zeta=\zeta_1$. This form uses oblate atoms where the ξ (like $z=\cos\theta$) and φ are oscillatory, and ζ is radial/expo.

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(j\zeta_1)Q_n^m(j\zeta_0) - Q_n^m(j\zeta_1)P_n^m(j\zeta_0)] Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(j\zeta_1) Q_n^m(j\zeta) - Q_n^m(j\zeta_1)P_n^m(j\zeta)] Q_n^m(j\zeta_0) P_n^m(\xi) e^{im\varphi}$$

In **Section 2** I compute the pillbox condition's LHS to be the following,

$$\partial_\zeta V_i(\zeta,\xi) - \partial_\zeta V_o(\zeta,\xi) = [j /(1+\zeta_0^2)] \Sigma_{nm} (-1)^m [(n+m)! / (n-m)!] E_{nm} P_n^m(\xi) Q_n^m(j\zeta_1)e^{im\varphi}$$

In **Section 3** I set the pillbox condition RHS to $(q/\varepsilon) (h_2/h_3h_1) \delta(\varphi-\varphi_0) \delta(\xi - \xi_0)$ with $q_1, q_2, q_3 = \xi, \zeta, \varphi$, and I state expressions for the h_i scale factors. Then,

In **Section 4** I apply orthogonality twice to solve for the coefficients, with this result,

$$E_{nm} = (q/4\pi j c_1\varepsilon)(-1)^m(2n+1) [(n-m)!/(n+m)!]^2 P_n^m(\xi_0) e^{-im\varphi 0}$$

In **Section 5** I insert this coefficient to obtain the final result for the exterior Green's function for the oblate spheroid labeled by $\zeta = \zeta_1$,

$$g(\zeta,\xi,\varphi \mid \zeta_0,\xi_0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm} (-1)^m (2n+1) [(n-m)!/(n+m)!]^2 *$$

$$[\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] Q_n^m(j\zeta_>) P_n^m(\xi_0) P_n^m(\xi) e^{im(\varphi-\varphi 0)}$$

$\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$ ζ_1 = surface on which g = 0

I verify that g is in fact real, making use of $P_n^m(-z) = P_\nu^m(z) (-1)^n$ and $Q_n^m(-z) = Q_\nu^m(z) (-1)^{n+1}$. The result is manifestly symmetric under $\zeta,\xi,\varphi \leftrightarrow \zeta_0,\xi_0,\varphi_0$. Dimensions are correct, Q/L.

In **Section 6** I slide the Green's point charge to the on-axis point so $\xi_0 \rightarrow 1$ (charge at north pole), and I recover the on-axis result I obtained in "On-axis Green's Function for an Oblate Spheroid.doc" which was necessary to do Smythe's problem 85. This is the only "verification" I have of the general result above.

In **Section 7** I instead move the Green's point charge into the equatorial plane of the spheroid $\xi_0 \rightarrow 0$, to get the following result which arises since $P_n^m(x=0) = 0$ when n+m = odd (spheroid = ζ_1 still)

$$g(\zeta,\xi,\varphi \mid \zeta_0,0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm,n+m\ even} (-1)^{(n-m)/2} (2n+1) [(n-m)!/(n+m)!]^2 [(n+m-1)!!/ (n-m)!!]$$

$$[\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] Q_n^m(j\zeta_>) P_n^m(\xi) e^{im(\varphi-\varphi 0)}$$

I make an attempt to simplify the gamma function stuff, but this effort fails. Still a double sum!

In **Section 8** I start again with the general Section 5 result and I squash the spheroid to a disk $\zeta_1 = 0$,

$$g(\zeta,\xi,\varphi \mid \zeta_0,\xi_0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm,n+m\ even} (-1)^m (2n+1) [(n-m)!/(n+m)!]^2 *$$

$$[(2/\pi) (+i)^{2n+1} Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] Q_n^m(j\zeta_>) P_n^m(\xi_0) P_n^m(\xi) e^{im(\varphi-\varphi 0)}$$

$\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$ ζ_1 = (surface on which g = 0) = 0

Here ζ_0. ξ_0, φ_0 determines where the Green's charge is located, as usual.

In **Section 9** I take both the above limits at the same time, so we then have the Green's function for a disk where the Green's point charge is in the plane of the disk. The result is this:

$$g(\zeta,\xi,\varphi \mid \zeta_0,0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm,n+m\ even} (-1)^{(n-m)/2} (2n+1) [(n-m)!/(n+m)!]^2 [(n+m-1)!!/ (n-m)!!]$$

$$[\{(2/\pi) (+i)^{2n+1}\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] Q_n^m(j\zeta_>) P_n^m(\xi) e^{im(\varphi-\varphi 0)}$$

$\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$ ζ_1 = surface on which g = 0

where now ζ_0=0. ξ_0, φ_0 determines where the Green's charge is located in the disk plane. I recall being rather disappointed at the complexity of this result, still a double sum since there is azi dependence. I then compute the derivative that is proportional to the charge density on the disk and find that

$$\partial_\xi g(\zeta,\xi,\varphi \mid \zeta_0,0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm,\,n+m\ \text{even}} (-1)^{(n-m)/2} (2n+1)\,[(n-m)!/(n+m)!]^2\,[(n+m-1)!!/(n-m)!!]$$

$$[\{(2/\pi)(+i)^{2n+1}\}Q_n^m(j\zeta) - P_n^m(j\zeta)]\,Q_n^m(j\zeta_0)\;P_n^{m\,\prime}(\xi)\,e^{im(\varphi-\varphi_0)}$$

Somehow these two results must be equivalent to get the famous results I compute in my "doing the disk by inversion.doc" where the Green's function and charge density come out being these expressions,

$$\Phi_{disk}(\rho,\varphi,z) = q_1 (2/\pi r_1) \cos^{-1}\{\sqrt{2}\,R\,r_1 \,/\sqrt{m + (c^2-R^2)\sqrt{[(R-\rho)^2+z^2]\,[(R+\rho)^2+z^2]}}\}$$

$$m = (\rho^2+z^2)(c^2-R^2) + R^2(R^2-c^2+2r_1^2)$$

$$r_1^2 = \rho^2 + c^2 + z^2 - 2c\rho\cos(\varphi-\varphi_0)$$

$$\sigma_{disk}(\mathbf{r}) = -\,q_1(1/(2\pi^2 r_1^2))\,\sqrt{c^2-R^2}\,/\sqrt{R^2-\rho^2} \qquad \text{// each side}$$

$$r_1^2 = (\rho^2 + c^2 - 2c\rho\cos(\varphi-\varphi_0))$$

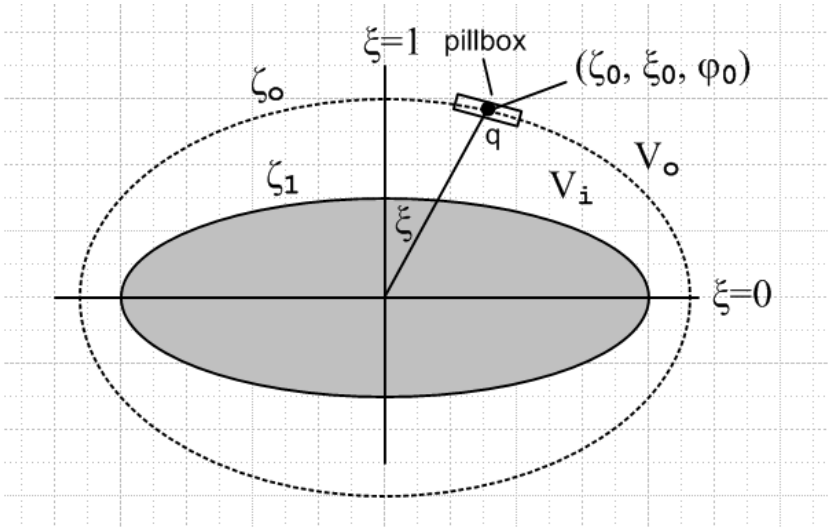
total charge induced on the disk = $-\,q_1\,(2/\pi)\tan^{-1}(R/\sqrt{c^2-R^2})$

Here the results are given in cylindrical coordinates ρ,z,φ. The radius of the disc is R (which would be c_1 in the oblate solution), and c (= ρ_0) is the distance from disc center to the point charge which is located in the z_0= 0 plane of the disk.

Before I knew about the inversion method, I was hoping this oblate spheroid path would give the simple charge density shown above. I think it would be a very challenging math problem to show the equivalence of these two sets of results! No doubt a certain "oblate spheroidal addition theorem" would be involved, but I have not yet developed my 1/R doc for oblates, so I don't have this theorem.

My last act is to show that the messy expression for ∂_ξg above (~ σ supposedly) does in fact vanish outside the disk, just the way σ should behave.

In "Smythe Greens oblate spheroid META.doc" I outline the steps taken to do this problem for the point charge being on-axis. Here we shall attempt the problem for a **general position of the Green's charge**.



1. The Smythian Form. In analogy with the above-mentioned document, consider this form

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(j\zeta_1)Q_n^m(j\zeta_0) - Q_n^m(j\zeta_1)P_n^m(j\zeta_0)] Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(j\zeta_1) Q_n^m(j\zeta) - Q_n^m(j\zeta_1)P_n^m(j\zeta)] Q_n^m(j\zeta_0) P_n^m(\xi) e^{im\varphi}$$

where we have added the azimuthal dependence. This form has these characteristics:

- $V_i = V_o$ on the surface $\zeta=\zeta_0$ which is the boundary between our two regions, away from charge q
- $V_i = 0$ when $\zeta = \zeta_1$. $\xi=0$
- Each form is composed of oblate atoms and thus satisfies Laplace

Note added 2.14.10. For the on-axis problem solved as noted above, we had only Σ_n and we used the ξ equation SL problem to determine the spectrum of n, being n = 0,1,2... Here we add a sum Σ_m which is over *all* integer values of m. It is probably possible to fold the negative side into the positive, but I did not do that here.

2. Compute Radial Derivatives and take difference, use Wronskian.

So already a huge amount of work has been done and we have just a single set of coefficients E_{nm} to determine. The next step is to compute the "radial" derivatives: (look for prime marks!!)

$$\partial_\zeta V_o(\zeta,\xi,\varphi) = j \Sigma_{nm} E_{nm} [P_n^m(j\zeta_1)Q_n^m(j\zeta_0) - Q_n^m(j\zeta_1)P_n^m(j\zeta_0)] \quad Q_n^m{}'(j\zeta) P_n^m(\xi) e^{im\varphi}$$
$$\partial_\zeta V_i(\zeta,\xi,\varphi) = j \Sigma_{nm} E_{nm} [P_n^m(j\zeta_1) Q_n^m{}'(j\zeta) - Q_n^m(j\zeta_1)P_n^m{}'(j\zeta)] Q_n^m(j\zeta_0) P_n^m(\xi) e^{im\varphi}$$

Then we shall need the difference:

$$\partial_\zeta V_i(\zeta,\xi) - \partial_\zeta V_o(\zeta,\xi) = j \Sigma_{nm} E_{nm} P_n^m(\xi) e^{im\varphi} *$$

$$[P_n^m(j\zeta_1) Q_n^m{}'(j\zeta) - Q_n^m(j\zeta_1)P_n^m{}'(j\zeta)] Q_n^m(j\zeta_0) - [P_n^m(j\zeta_1)Q_n^m(j\zeta_0) - Q_n^m(j\zeta_1)P_n^m(j\zeta_0)] Q_n^m{}'(j\zeta)$$

But we are only going to want this quantity at $\zeta = \zeta_0$ which marks the spheroid holding our point charge, so set $\zeta = \zeta_0$

$$\partial_\zeta V_i(\zeta,\xi) - \partial_\zeta V_o(\zeta,\xi) = j \Sigma_{nm} E_{nm} P_n^m(\xi) e^{im\varphi} *$$

$$[P_n^m(j\zeta_1) Q_n^m{}'(j\zeta_0) - Q_n^m(j\zeta_1) P_n^m{}'(j\zeta_0)] Q_n^m(j\zeta_0) - [P_n^m(j\zeta_1) Q_n^m(j\zeta_0) - Q_n^m(j\zeta_1) P_n^m(j\zeta_0)] Q_n^m{}'(j\zeta_0)$$

The first and third terms are equal and cancel, so we get

$$\partial_\zeta V_i(\zeta,\xi) - \partial_\zeta V_o(\zeta,\xi) = j \Sigma_{nm} E_{nm} P_n^m(\xi) e^{im\varphi} *$$

$$[- Q_n^m(j\zeta_1) P_n^m{}'(j\zeta_0) Q_n^m(j\zeta_0) + Q_n^m(j\zeta_1) P_n^m(j\zeta_0) Q_n^m{}'(j\zeta_0)]$$

or

$$\partial_\zeta V_i(\zeta,\xi) - \partial_\zeta V_o(\zeta,\xi) = j \Sigma_{nm} E_{nm} P_n^m(\xi) e^{im\varphi} * Q_n^m(j\zeta_1) W[P_n^m(j\zeta_0), Q_n^m(j\zeta_0)]$$

and we are very happy to see that Wronskian. We are doing "off axis Legendre functions" here, so make sure you use the right form. This is discussed in my "the Smythe method for Green's...doc" where I process Bateman's page 123 result to get

$$W[P_n^m(z), Q_n^m(z)] = (-1)^m (n+m)! / (n-m)! * 1/(1-z^2)$$

which in our case becomes

$$W[P_n^m(j\zeta_0), Q_n^m(j\zeta_0)] = (-1)^m (n+m)! / (n-m)! * 1/(1+\zeta_0^2)$$

So, our derivative difference is now

$$\partial_\zeta V_i(\zeta,\xi) - \partial_\zeta V_o(\zeta,\xi) = [j /(1+\zeta_0^2)] \Sigma_{nm} (-1)^m [(n+m)! / (n-m)!] E_{nm} P_n^m(\xi) Q_n^m(j\zeta_1) e^{im\varphi} \quad (*)$$

which I would say is "not too bad".

3. Apply Gauss's Law to pillbox around point charge q.

I show in "Smythe Greens oblate spheroid.doc" that, for a Green's point charge q located at point (ζ_0, ξ_0, φ_0), Gauss's law says [this is not the right doc! This is the doc where I did it roughly for the first time and not in the following notation. The correct doc is

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon) (h_2/h_3h_1) \delta(\varphi-\varphi_0) \delta(\xi - \xi_0)$$

where $q_1, q_2, q_3 = \xi, \zeta, \varphi$ which is the key to these results,

$$h_1 = c_1^2\xi\left[\frac{x^2}{(-c_1^2\xi^2)^2} + \frac{\rho^2}{c_1^4(1-\xi^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1-\xi^2}\right)^{\frac{1}{2}} \qquad (6)$$

$$h_2 = c_1^2\zeta\left[\frac{x^2}{c_1^4\zeta^4} + \frac{\rho^2}{c_1^4(1+\zeta^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1+\zeta^2}\right)^{\frac{1}{2}} \qquad (7)$$

$$h_3 = \frac{\rho d\phi}{d\phi} = \rho = c_1[(1+\zeta^2)(1-\xi^2)]^{\frac{1}{2}} \qquad (8)$$

or in my writing (copied from "the Smythe method for Greens...doc)

$h_1 = c_1\sqrt{\xi^2+\zeta^2}\,/\sqrt{1-\xi^2}$
$h_2 = c_1\sqrt{\xi^2+\zeta^2}\,/\sqrt{1+\zeta^2}$
$h_3 = c_1\sqrt{1+\zeta^2}\ \sqrt{1-\xi^2}$

$h_1h_3 = c_1\sqrt{\xi^2+\zeta^2}\,/\sqrt{1-\xi^2}\ \ c_1\sqrt{1+\zeta^2}\ \sqrt{1-\xi^2}\ = c_1^2\sqrt{1+\zeta^2}\ \sqrt{\xi^2+\zeta^2}$
$h_2 = c_1\sqrt{\xi^2+\zeta^2}\,/\sqrt{1+\zeta^2}$

$(1+\zeta^2)\,c_1\,h_2 = h_1h_3 \qquad \Rightarrow \qquad (h_2/h_3h_1) = [(1+\zeta^2)\,c_1]^{-1}$

The charge density on the spheroid surface is given by [this is Gauss's law for the pillbox]

$\sigma/\varepsilon = \partial_n V_i - \partial_n V_o \qquad \partial_n = (1/h_2)\,\partial_\zeta \quad \Rightarrow \qquad \sigma/\varepsilon = (1/h_2)[\,\partial_\zeta V_i - \partial_\zeta V_o\,]$

Therefore, the h_2 cancels when we write σ. Here are our two main results:

$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi-\xi_0) = q/[\varepsilon c_1(1+\zeta^2)]\ \delta(\varphi-\varphi_0)\,\delta(\xi-\xi_0) \qquad (**)$

$\sigma/\varepsilon = (1/h_2)[\,\partial_\zeta V_i - \partial_\zeta V_o\,] = (q/\varepsilon)\,(1/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi-\xi_0)$

$dA_2 = h_3h_1dq_3dq_1 = h_3h_1d\varphi\,d\xi \qquad$ // area of pillbox side

This then is the effective surface charge distribution "of the point charge q". In the doc last referenced, we took this double delta limit to the north pole which was an extra complication. Here we work as is.

So we now know from (*) and (**) that

$(q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi_0)\,\delta(\xi-\xi_0) = [\,j/(1+\zeta_0^2)]\,\Sigma_{nm}\,(-1)^m\,[(n+m)!/(n-m)!]\ E_{nm}\,P_n^m(\xi)\,Q_n^m(j\zeta_1)e^{im\varphi}$

To review: we computed the normal difference across the spheroid at the point charge location and set this ~ equal first to the surface charge density that charge represents, and second to our Smythian form derivative difference, and we got the above! Now it should be an easy matter to computer the E_{nm}. In my form, we have both positive and negative m integers in the m sum, whereas in the doc last referenced we were using $\cos(m[\varphi-\varphi_m])$. I will stick with my current notation.

4. Use orthogonality twice to compute the coefficients E_{nm}

We use first apply $\int d\varphi\, e^{-im\varphi}$ to both sides of the above and use the azimuthal orthogonality relation,

$$\int d\varphi\, e^{-im'\varphi}\, e^{im\varphi} = 2\pi\, \delta_{m',m}$$

to get

$$(q/\varepsilon)\,(h_2/h_3h_1)\, e^{-im'\varphi_0}\, \delta(\xi - \xi_0) = [\, j/(1+\zeta_0^2)]\, \Sigma_{nm}\, (-1)^m\, [(n+m)!/(n-m)!]\; E_{nm}\, P_n^m(\xi)\, Q_n^m(j\zeta_1)\, 2\pi\, \delta_{m',m}$$

$$= [\, 2\pi j/(1+\zeta_0^2)]\, \Sigma_n\, (-1)^{m'}\, [(n+m')!/(n-m')!]\; E_{nm'}\, Q_n^m(j\zeta_1)\, P_n^{m'}(\xi)$$

Next, we apply $\int d\xi\, P_n^{m'}(\xi)$ to both sides and use the associated Legendre orthogonality relation, (see ODE/Legendre docs)

$$\int_{-1}^{1} dz\, P_n^m(z) P_k^m(z) = 2\delta_{n,k}\, (2n+1)^{-1}\, (n+m)!/(n-m)! \qquad n,k = m, m+1, m+2 \ldots\ldots \infty$$

to get

$$(q/\varepsilon)\,(h_2/h_3h_1)\, e^{-im'\varphi_0}\, P_n^{m'}(\xi_0) =$$

$$[\, 2\pi j/(1+\zeta_0^2)]\, \Sigma_n\, (-1)^{m'}\, [(n+m')!/(n-m')!]\; E_{nm'}\, Q_n^{m'}(j\zeta_1) 2\delta_{n,n'}\, (2n'+1)^{-1}\, (n'+m')!/(n'-m')!$$

$$= [\, 4\pi j/(1+\zeta_0^2)]\, (-1)^{m'}\, [(n'+m')!/(n'-m')!]^2\; E_{n'm'}\, Q_{n'}^{m'}(j\zeta_1)\, (2n'+1)^{-1}$$

and now remove all the primes

$$(q/\varepsilon)\,(h_2/h_3h_1)\, e^{-im\varphi_0}\, P_n^m(\xi_0) = [\, 4\pi j/(1+\zeta_0^2)]\, (-1)^m\, [(n+m)!/(n-m)!]^2\; E_{nm}\, Q_n^m(j\zeta_1)\, (2n+1)^{-1}$$

and now we know E_{nm}. From above we had $(h_2/h_3h_1) = [(1+\zeta^2)\, c_1]^{-1}$ now at ζ_0 so:

$$(q/\varepsilon)\, [(1+\zeta_0^2)\, c_1]^{-1} e^{-im\varphi_0}\, P_n^m(\xi_0) = [\, 4\pi j/(1+\zeta_0^2)]\, (-1)^m\, [(n+m)!/(n-m)!]^2\; E_{nm}\, Q_n^m(j\zeta_1)\, (2n+1)^{-1}$$

$$(q/\varepsilon)\, e^{-im\varphi_0}\, P_n^m(\xi_0) = [\, 4\pi j c_1\,]\, (-1)^m\, [(n+m)!/(n-m)!]^2\; E_{nm}\, Q_n^m(j\zeta_1)\, (2n+1)^{-1}$$

$$E_{nm} = (q/4\pi j c_1 \varepsilon)\, (-1)^m\, [(n-m)!/(n+m)!]^2 e^{-im\varphi_0}\, P_n^m(\xi_0)/\, Q_n^m(j\zeta_1)$$

5. Statement of the final result

Therefore, here is our Green's Function result:

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(j\zeta_1)Q_n^m(j\zeta_0) - Q_n^m(j\zeta_1)P_n^m(j\zeta_0)] Q_n^m(j\zeta) P_n^m(\xi) e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(j\zeta_1) Q_n^m(j\zeta) - Q_n^m(j\zeta_1)P_n^m(j\zeta)] Q_n^m(j\zeta_0) P_n^m(\xi) e^{im\varphi}$$

$$E_{nm} = (q/4\pi jc_1\varepsilon)(-1)^m(2n+1) [(n-m)!/(n+m)!]^2 P_n^m(\xi_0) e^{-im\varphi 0}$$

Install to get

$$V_o(\zeta,\xi,\varphi) = (q/4\pi jc_1\varepsilon)\Sigma_{nm} (-1)^m (2n+1) [(n-m)!/(n+m)!]^2$$
$$[\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\}Q_n^m(j\zeta_0) - P_n^m(j\zeta_0)] Q_n^m(j\zeta) P_n^m(\xi_0) P_n^m(\xi) e^{im(\varphi-\varphi 0)}$$

$$V_i(\zeta,\xi,\varphi) = (q/4\pi jc_1\varepsilon)\Sigma_{nm} (-1)^m (2n+1) [(n-m)!/(n+m)!]^2$$
$$[\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\} Q_n^m(j\zeta) - P_n^m(j\zeta)] Q_n^m(j\zeta_0) P_n^m(\xi_0) P_n^m(\xi) e^{im(\varphi-\varphi 0)}$$

If we now define $\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$, we can write this as a single line:

$$V(\zeta,\xi,\varphi) = (q/4\pi jc_1\varepsilon)\Sigma_{nm} (-1)^m (2n+1) [(n-m)!/(n+m)!]^2$$
$$[\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] Q_n^m(j\zeta_>) P_n^m(\xi_0) P_n^m(\xi) e^{im(\varphi-\varphi 0)}$$

This is **our final result in all its glory**, which we now restate: q is at $(\zeta_0,\xi_0,\varphi_0)$

$$g(\zeta,\xi,\varphi \mid \zeta_0,\xi_0,\varphi_0) = (q/4\pi jc_1\varepsilon)\Sigma_{nm} (-1)^m (2n+1) [(n-m)!/(n+m)!]^2 *$$
$$[\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] Q_n^m(j\zeta_>) P_n^m(\xi_0) P_n^m(\xi) e^{im(\varphi-\varphi 0)}$$

$\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$ $\qquad$ ζ_1 = surface on which g = 0

Note that this result is symmetric under $\zeta,\xi,\varphi \leftrightarrow \zeta_0,\xi_0,\varphi_0$.

Note added 2.14.10. When I got this answer, I did not wonder if it was real or complex. Ignoring the complex nature of the φ functions, it is certainly not obvious that the g I have here obtained is a real function. If it is complex, then it cannot be the true solution to this problem. This issue did not occur to me until I went on to the "bloid" problem. Now since m and n are both integers, it is likely that this answer really is in fact "real". In fact I will now show this is true:

$$C \equiv [\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] Q_n^m(j\zeta_>) / j$$

$$C^* \equiv [\{P_n^m(-j\zeta_1)/Q_n^m(-j\zeta_1)\}Q_n^m(-j\zeta_<) - P_n^m(-j\zeta_<)] Q_n^m(-j\zeta_>) / (-j)$$

We know that $P_n^m(-z) = P_\nu^m(z) (-1)^n$ and $Q_n^m(-z) = Q_\nu^m(z) (-1)^{n+1}$ when m and n are integers, this from Leg prop doc. Therefore

$$C^* = [\{(-1)^n P_n^m(j\zeta_1)/ (-1)^{n+1}Q_n^m(j\zeta_1)\} (-1)^{n+1}Q_n^m(j\zeta_<) - (-1)^n P_n^m(j\zeta_<)] (-1)^{n+1}Q_n^m(j\zeta_>)/(-j)$$

$$= [\{(-1)^n P_n^m(j\zeta_1)/ Q_n^m(j\zeta_1)\} Q_n^m(j\zeta_<) - (-1)^n P_n^m(j\zeta_<)] (-1)^{n+1} Q_n^m(j\zeta_>)/(-j$$

$$= [\{P_n^m(j\zeta_1)/ Q_n^m(j\zeta_1)\} Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] (-1)^1 Q_n^m(j\zeta_>)/(-j$$

$$= - [\{P_n^m(j\zeta_1)/ Q_n^m(j\zeta_1)\} Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] (-1)^1 Q_n^m(j\zeta_>)/(-j)$$

$$= + C$$

So for our spheroid problem, we really are getting a real solution.

6. Verify the on-axis limit.

Now lets try to take the limit $\xi_0 \rightarrow 1$ (charge at north pole) and see if we recover our on-axis result. We use $P_n^m(\xi_0) \rightarrow P_n^m(1) = \delta_{m,0}$ to get

$$V(\zeta,\xi,\varphi) = (q/4\pi j c_1 \varepsilon)\Sigma_n (2n+1) [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\} Q_n^m(j\zeta_<) - P_n(j\zeta_<)] Q_n(j\zeta_>) P_n(\xi_0) P_n(\xi)$$

and yes, this does replicate our on-axis result exactly. I know this result is correct because I had to derive it and use it to get Smythe's Problem 85 result.

7. Compute the limit for q in the equatorial plane ξ_0=0

This means q is at $(\zeta_0,\xi_0=0,\varphi_0)$ so we have $P_n^m(\xi_0) = P_n^m(0)$. First, here is our general result

$$g(\zeta,\xi,\varphi \mid \zeta_0,\xi_0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm} (-1)^m (2n+1) [(n-m)!/(n+m)!]^2 *$$

$$[\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] Q_n^m(j\zeta_>) P_n^m(\xi_0) P_n^m(\xi) e^{im(\varphi-\varphi_0)}$$

$\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$ ζ_1 = surface on which g = 0

From my Properties of Legendre doc we have [Bateman CS convention]

$P_n^m(x=0) = (-1)^{(n+m)/2} (n+m-1)!!/ (n-m)!!$ // n+m = even
$P_n^m(x=0) = 0$ // n+m = odd

We then get:

$$g(\zeta,\xi,\varphi \mid \zeta_0,0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm,\,n+m\ \text{even}} (-1)^{(n-m)/2} (2n+1) [(n-m)!/(n+m)!]^2 [(n+m-1)!!/ (n-m)!!]$$

$$[\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)] Q_n^m(j\zeta_>) P_n^m(\xi) e^{im(\varphi-\varphi_0)}$$

which is certainly not very pleasant! If your Green's charge is to the side of the spheroid, this is what you get. Charge position is ζ_0 and φ_0, spheroid is ζ_1, and $\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$

Failed Extra: Let's try this using all gamma functions. We then have this general result for all n,m:

$$P_n^m(x=0) = 2^m\sqrt{\pi} \; / \; [\, \Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2)\,]$$

Recall that our Wronskian form was this:

$$W[\, P_n^m(z), Q_n^m(z)] = (-1)^m\, (n+m)!\, /\, (n-m)! \quad *\; 1/(1-z^2)$$

but we know we can also write this (Bateman) like so:

$$2^{2m}\; (1-z^2)^{-1}\, \Gamma(1 + m/2 + n/2)\, \Gamma(1/2 + m/2 + n/2)\, /\, [\Gamma(1 - m/2 + n/2)\, \Gamma(1/2 - m/2 + n/2)]$$

which basically tells us that:

$$(n+m)!\, /\, (n-m)! = (-1)^m\, 2^{2m}\, \Gamma(1 + m/2 + n/2)\, \Gamma(1/2 + m/2 + n/2)\, /\, [\Gamma(1 - m/2 + n/2)\, \Gamma(1/2 - m/2 + n/2)]$$

so when we square this thing, we get

$$[(n+m)!/(n-m)!]^2 = 2^{4m} \; *$$

$$\Gamma^2(1 + m/2 + n/2)\, \Gamma^2(1/2 + m/2 + n/2)\, /\, [\Gamma^2(1 - m/2 + n/2)\, \Gamma^2(1/2 - m/2 + n/2)]$$

which is certainly a lot uglier. Now replace m with -m to get

$$[(n-m)!/(n+m)!]^2 = 2^{-4m} \; *$$

$$\Gamma^2(1 - m/2 + n/2)\, \Gamma^2(1/2 - m/2 + n/2)\, /\, [\Gamma^2(1 + m/2 + n/2)\, \Gamma^2(1/2 + m/2 + n/2)]$$

Now consider the combination

$$[(n-m)!/(n+m)!]^2\, P_n^m(0) =$$

$$2^{-4m}\, \Gamma^2(1 - m/2 + n/2)\, \Gamma^2(1/2 - m/2 + n/2)\, /\, [\Gamma^2(1 + m/2 + n/2)\, \Gamma^2(1/2 + m/2 + n/2)]$$

$$*\; 2^m\sqrt{\pi} \; / \; [\, \Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2)\,]$$

$$= 2^{-3m}\, \Gamma^2(1 - m/2 + n/2)\, \Gamma^2(1/2 - m/2 + n/2)\, /\, [\Gamma^2(1 + m/2 + n/2)\, \Gamma^2(1/2 + m/2 + n/2)]$$

and we get no cancellation, so don't bother to continue.

8. Compute the limit for circular plate $\zeta_1 = 0$

This means $\zeta_1 = 0$ so we will need

$\{P_n^m(0+)/Q_n^m(0+)\} = ?$

From my Legendre properties doc we have

$P_n^m(z=0\pm) = (-1)^{(n+m)/2}\,(n+m-1)!!/(n-m)!!\; e^{\mp\, i n m/2}$ // n+m = even

$P_n^m(z=0\pm) = 0$ // n+m = odd

$Q_n^m(z=0\pm) = (\pi/2)\,(\mp i)^{n+1}\,(n+m-1)!!/(n-m)!!$ // n+m = even

$Q_\nu^m(z=0\pm) = (\mp i)^{n+1}\,(n+m-1)!!/(n-m)!!$ // n+m = odd

and we can now take the ratios:

$P_n^m(0+)/Q_n^m(0+) =$ // n+m = even

$[(-1)^{(n+m)/2}\,(n+m-1)!!/(n-m)!!\; e^{-i n m/2}] \,/\, [(\pi/2)\,(-i)^{n+1}\,(n+m-1)!!/(n-m)!!]$

$= (2/\pi)\;(-1)^{(n+m)/2}\,e^{-i n m/2}\,(+i)^{n+1} = (2/\pi)\,(-1)^{(n+m)/2}\,(+i)^{n+1-m}$

$= (2/\pi)\,(i)^{n+m+n+1-m} = (2/\pi)\,(+i)^{2n+1}$

$P_n^m(0+)/Q_n^m(0+) = 0$ // n+m =odd

To summarize

$\{P_n^m(0+)/Q_n^m(0+)\} = (2/\pi)\,(+i)^{2n+1}$ n+m even

$= 0$ n_m odd

Our general Green's result is this:

$g(\zeta,\xi,\varphi \mid \zeta_0,\xi_0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm}\,(-1)^m\,(2n+1)\,[(n-m)!/(n+m)!]^2\;*$

$[\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)]\,Q_n^m(j\zeta_>)\,P_n^m(\xi_0)\,P_n^m(\xi)\,e^{im(\varphi-\varphi 0)}$

$\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$ ζ_1 = surface on which g = 0

which we can now rewrite for $\zeta_1 = 0$ (plate) as [Green's charge at ζ_0, ξ_0, φ_0]

$g(\zeta,\xi,\varphi \mid \zeta_0,\xi_0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm,\,n+m\ even}\,(-1)^m\,(2n+1)\,[(n-m)!/(n+m)!]^2\;*$

$[(2/\pi)\,(+i)^{2n+1}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)]\,Q_n^m(j\zeta_>)\,P_n^m(\xi_0)\,P_n^m(\xi)\,e^{im(\varphi-\varphi 0)}$

$\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$ ζ_1 = surface on which g = 0

9. Compute limit for circular plate AND q in the equatorial plane:

Start with the equatorial plane result which is

$$g(\zeta,\xi,\varphi \mid \zeta_0,0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm,\,n+m\ \text{even}} (-1)^{(n-m)/2} (2n+1)\,[(n-m)!/(n+m)!]^2\,[(n+m-1)!!/(n-m)!!]$$

$$[\{P_n^m(j\zeta_1)/Q_n^m(j\zeta_1)\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)]\,Q_n^m(j\zeta_>)\,P_n^m(\xi)\,e^{im(\varphi-\varphi_0)}$$

Then insert our round plate result to get

$$g(\zeta,\xi,\varphi \mid \zeta_0,0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm,\,n+m\ \text{even}} (-1)^{(n-m)/2} (2n+1)\,[(n-m)!/(n+m)!]^2\,[(n+m-1)!!/(n-m)!!]$$

$$[\{(2/\pi)\,(+i)^{2n+1}\}Q_n^m(j\zeta_<) - P_n^m(j\zeta_<)]\,Q_n^m(j\zeta_>)\;P_n^m(\xi)\,e^{im(\varphi-\varphi_0)}$$

$\zeta_> = \max(\zeta,\zeta_0)$ and $\zeta_< = \min(\zeta,\zeta_0)$ $\qquad$ ζ_1 = surface on which g = 0

This is more complicated than I was expecting.

Just above the surface of the plate, we know $\zeta_> = \zeta_0$ and $\zeta_< = \zeta$ so we can write

$$g(\zeta,\xi,\varphi \mid \zeta_0,0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm,\,n+m\ \text{even}} (-1)^{(n-m)/2} (2n+1)\,[(n-m)!/(n+m)!]^2\,[(n+m-1)!!/(n-m)!!]$$

$$[\{(2/\pi)\,(+i)^{2n+1}\}Q_n^m(j\,\zeta) - P_n^m(j\,\zeta)]\,Q_n^m(j\,\zeta_0)\;P_n^m(\xi)\,e^{im(\varphi-\varphi_0)}$$

The charge density on the plate will involve $d/d\xi$ and then set $\zeta_0 = 0$,

$$\partial_\xi g(\zeta,\xi,\varphi \mid \zeta_0,0,\varphi_0) = (q/4\pi j c_1 \varepsilon)\Sigma_{nm,\,n+m\ \text{even}} (-1)^{(n-m)/2} (2n+1)\,[(n-m)!/(n+m)!]^2\,[(n+m-1)!!/(n-m)!!]$$

$$[\{(2/\pi)\,(+i)^{2n+1}\}Q_n^m(j\,\zeta) - P_n^m(j\,\zeta)]\,Q_n^m(j\,\zeta_0)\;P_n^{m\,\prime}(\xi)\,e^{im(\varphi-\varphi_0)}$$

These are all amazingly complicated results for such a simple problem! Is it possible that results are somehow simpler in cylindrical coordinates for this plate problem?

Note added 3.9.10. One problem with the above formula is that the radial distance to a point on the iris is $\rho = c_1\sqrt{1+\zeta^2}$, so we have these horrible $Q_n^m(j\zeta)$ functions sitting in the above formula for the charge density with $\zeta = \zeta(\rho)$ as shown. I don't see any possible way that we can "get rid of" $Q_n^m(j\zeta)$ in the above formula. We don't have ζ being small or large or anything useful. The above formula might even be fully correct for the change density (apart from the missing scale factor) but as such, it is a complex double-sum expansion of a function which we know from the Smythe problem is in reality pretty simple.

Aside: Here is a proof that charge density is 0 outside the plate.

The plane outside the plate is the bloid with $\xi = 0$. But from our Legendre data doc we have

$P_n^m{}'(0) = -P_n^{m+1}(0)$

Our sum includes only terms with n+m = even. But we also have this result from my doc

$P_n^m(x=0) = (-1)^{(n+m)/2}\,(n+m-1)!!/\,(n-m)!!$ // n+m = even
$P_n^m(x=0) = 0$ // n+m = odd

and we rewrite the second line as

$P_n^{m+1}(x=0) = 0$ // n+m+1 = odd
$P_n^{m+1}(x=0) = 0$ // n+m = even

Amazing! This seems to kill off ALL terms and we get $\partial_\xi g(\zeta,\xi,\varphi \mid \zeta_0,0,\varphi_0) = 0$. This agrees with the fact that there is no charge outside the plate!