On-axis Green's Function for an Oblate Spheroid META PhL 1.11.10

See 2-page Overview at the start of the raw doc.

Only the "on-axis" Green's Function for the Oblate Spheroid (and plate) is found here. In each section I try in these meta notes to show the logic flow and quote only the main results. I follow this TOC which is identical to that of the raw document. This doc is 7 pages, the raw one is 26 pages.

0. History of my work on this problem: 1

2. On-Axis Green's Function for an Oblate Spheroid 2

(a) Limits of the Final Result 3

(b) Charge Distribution on the Spheroid (and circular plate) 3

3. Now we have to do our "pillbox construction". 4

(a) Finding the surface charge density σ in the pillbox for a point charge 4

4. Question: how do we take our pillbox "to the north pole" ? Finding G. 5

5. Taking the circular plate limit of the oblate spheroidal Green's Function 6

(a) Taking the flat plate limit of our spheroidal on-axis Green's Function result 6

(b) Review of the potential of an on-axis point charge in oblate coordinates 7

(c) Subtracting off the potential of the point charge to get Smythe's Problem 85 p 209 result 7

6. Some limits of Smythe's result. 7

(a) Verify Smythe's large ζ limit of Qn(jζ) 8

(b) This same problem appears in Smythe third edition 8

(c) Why does Smythe's result include only even terms? 8

0. History of my work on this problem:

I point our here how I mistakenly thought $Q_n(0+) = 0$ and then describe some problems that caused. It caused me to get a result different from Smythe's Problem 85 result, for example, and a result that had internal inconsistencies that were quite mysterious.

1. Introduction.

In attacking Smythe's Problem 85,

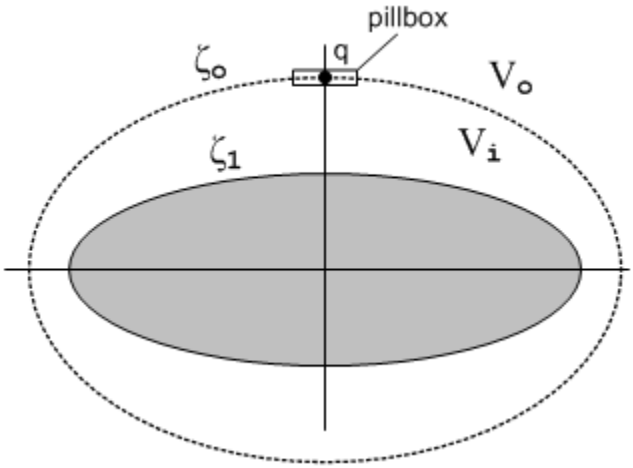
85. An earthed conducting disk of radius a whose equation is $\zeta = 0$ is influenced by a point charge q at $1, \zeta_0, 0$. Show that the potential due to the induced charge on the disk is

$$V_i = \frac{q}{2\pi^2 \epsilon a} \sum_{n=0}^{\infty} (4n + 1) Q_{2n}(j\zeta_0) Q_{2n}(j\zeta) P_{2n}(\xi)$$

the only way I know how to proceed is to compute the on-axis Green's Function for an oblate spheroid, then take the limit where the spheroid becomes a plate. I need to compute the potential in this situation (ie, the "Green's Function"), then I can subtract off the point charge potential to get the potential of just the charge that got induced onto the spheroid. Then I can answer Smythe's question.

2. On-Axis Green's Function for an Oblate Spheroid

Here is a view of our oblate spheroid, seen edge on, with the Green's on-axis point charge q:



In order that we have V = 0 on the metal inner spheroid and have continuity at the ζ_0 boundary and have decaying large ζ behavior, I arrive at this "Smythian form" for the potential in the two regions shown,

$$V_o(\zeta,\xi) = \Sigma_n E_n \, [P_n(j\zeta_1)Q_n(j\zeta_0) - Q_n(j\zeta_1)P_n(j\zeta_0)] \, Q_n(j\zeta) \, P_n(\xi)$$
$$V_i(\zeta,\xi) = \Sigma_n E_n \, [P_n(j\zeta_1) \, Q_n(j\zeta) - Q_n(j\zeta_1)P_n(j\zeta) \,] \, Q_n(j\zeta_0) \, P_n(\xi)$$

where E_n are some to-be-determined coefficients. I then apply ∂_ζ to both expressions, subtract, then evaluate the result at $\zeta = \zeta_0$. A Wronskian appears and the result simplifies to this form,

$$\partial_\zeta V_i(\zeta_0,\xi) - \partial_\zeta V_o(\zeta_0,\xi) = j \, (\, 1+\zeta_0^2)^{-1} \, \Sigma_n \, E_n \, Q_n(j\zeta_1) \, P_n(\xi)$$

I then conjecture that the difference has this form, where G is some constant I have to figure out,

$$\partial_\zeta V_i(\zeta_0,\xi) - \partial_\zeta V_o(\zeta_0,\xi) = G \, \delta(\xi-1)$$

The idea here is that the derivative of V must match everywhere except at the location of the Green's point charge q as shown in the above picture, which is at $\xi = 1$. At that location we are going to apply Gauss's Law to the pillbox and we will end up with G.

Just assuming at this point that some G exists to make the above true, we then get

$$G \, \delta(\xi-1) = j \, (\, 1+\zeta_0^2)^{-1} \, \Sigma_n \, E_n \, Q_n(j\zeta_1) \, P_n(\xi)$$

We next use the orthogonality of the P_n to project out the coefficients, and find that

$$E_n = (G/j) \, (\, 1+\zeta_0^2) \, (2n+1)/ \, [\, 2 \, Q_n(j\zeta_1) \,]$$

I then sneak a future look at the result for G from the next section, so I can finish off solving the problem. That result is (from Section 3 below)

$$G = q/[2\pi c_1 \varepsilon] / (1+\zeta_0^2)$$

where c_1 is the focal length of the spheroid and $V = 1/(4\pi\varepsilon r)$ for a point charge in sphericals (for the meaning of ε). The factors $(1+\zeta_0^2)$ cancel and we get

$$E_n = = q/[4\pi j c_1 \varepsilon] (2n+1)/ Q_n(j\zeta_1)$$

which we then install into our "Smythian form" to get the **complete solution to our problem:**

$$V_o(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_n(2n+1) [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\} Q_n(j\zeta_0) - P_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi)$$
$$V_i(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_n(2n+1) [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\} Q_n(j\zeta) - P_n(j\zeta)] Q_n(j\zeta_0) P_n(\xi)$$

Notice that the form above shows matching at $\zeta = \zeta_0$ and that we still have $V_i = 0$ at $\zeta = \zeta_1$ (ie, on the metal). The terms all satisfy oblate Laplace. So the key thing is that we get the pillbox done right.

If we now define $\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$, we can write this as a single line:

$$V(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_n(2n+1) [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\} Q_n(j\zeta_<) - P_n(j\zeta_<)] Q_n(j\zeta_>) P_n(\xi)$$

(a) Limits of the Final Result

The limit $\zeta_1 \to 0$ makes the spheroid ζ_1 become a circular plate (see Section 5 below).

The limit of large ζ is also of interest. We find:

$$V_o(\zeta,\xi) == q/[4\pi\varepsilon] [1 - \cot^{-1}\zeta_0 / \cot^{-1}\zeta_1] (1/r)$$

where $r = c_1\zeta$. This tells us that our (Green's point charge q plus spheroid) appear, from far away, as a point charge of the following size

$$q_{eff} = q(1 - \cot^{-1}\zeta_0 / \cot^{-1}\zeta_1)$$

This means that the total induced charge on the spheroid must be

$$q_{spheroid} = (-q) * \text{fraction} \qquad \text{fraction} = \cot^{-1}\zeta_0 / \cot^{-1}\zeta_1$$

This general subject is discussed more at the end of the raw notes.

(b) Charge Distribution on the Spheroid (and circular plate)

We (re) compute the derivative of the potential and evaluate it at the metal ζ_1,

$$\partial_\zeta V_i(\zeta_1,\xi) = (1+\zeta_1^2)^{-1}\ q/[4\pi c_1\varepsilon]\ \Sigma_n\ (2n+1)\ [Q_n(j\zeta_0)/Q_n(j\zeta_1)]\ P_n(\xi)$$

We then compute the surface charge density correcting as usual for the scaling factor $dn = h_2 d\zeta$,

$$\sigma = -\varepsilon\ \partial_n V_i = -(\varepsilon/h_2)\ \partial_\zeta V_i \qquad\qquad h_2 = c_1\sqrt{\xi^2+\zeta_1^2}\ /\sqrt{1+\zeta_1^2}$$

and our result is

$$\sigma = -q/[4\pi c_1^2]\ [\sqrt{1+\zeta_1^2}\sqrt{\xi^2+\zeta_1^2}\ ]^{-1}\ \Sigma_n\ (2n+1)\ [Q_n(j\zeta_0)/Q_n(j\zeta_1)]\ P_n(\xi)$$

Nothing blows up here. The main interest is the ξ dependence, but we can only see what that is for each term, though the leading factor causes an increase at the oblate equator as you would expect. The sum here is a little like scattering theory. In the "forward direction", meaning on the side of the spheroid facing the point charge, we have $P_n(1) = 1$ so all the terms are additive and we get a "large" result. On the back of the spheroid we have $P_n(-1) = (-1)^n$ so we get interference and a smaller result.

In the limit where the spheroid squashes down to a circular metal plate, $\zeta_1 = 0$ and the above becomes

$$\sigma_{plate} = -q/[4\pi c_1^2]\ [\ \xi\ ]^{-1}\ \Sigma_n\ (2n+1)\ [Q_n(j\zeta_0)/Q_n(0+)\ ]\ P_n(\xi)$$

$$Q_n(0+) = (-1)^{(n+1)/2}\ (n-1)!!\ /\ n!! \qquad n \text{ odd}$$
$$Q_n(0+) = (-1)^{n/2}\ (-j\pi/2)\ (n-1)!!\ /\ n!! \qquad n \text{ even}$$

which we can write in Cartesian coordinates as

$$\sigma_{plate\pm} = -q/[4\pi a^2]\left(\ 1/\sqrt{1-(\rho/a)^2}\ \right)\ \Sigma_n\ (2n+1)\ [Q_n(j\zeta_0)/Q_n(0+)\ ]\ P_n\left(\pm\sqrt{1-(\rho/a)^2}\ \right)$$

where now ± refers to the front or back surface of the plate. Again, the back surface will have less σ due to the interference of terms.

The factor $1/\sqrt{1-(\rho/a)^2}$ shows how the charge density builds up at the edge of the plate, becoming infinite but integrable there, something we always see in this kind of problem. The charge ratio front/back again depends on the "scattering" argument given above where we get term cancellation on the back side so σ there will be smaller than on the front. For a tiny plate, back and front are about the same. For a huge plate (wall) the back σ is zero. So generally we are in between these extremes.

Notice that we now know both the total charge q_{eff} on the spheroid / plate, and we know exactly how it is distributed.

3. Now we have to do our "pillbox construction".

(a) Finding the surface charge density σ in the pillbox for a point charge.

We use general orthogonal curvilinear coordinates here. If we have a point charge q on a surface q_1 = constant, I show that we can represent it by either a 3D or 2D charge density as follows:

$$\rho = q\ 1/(h_1h_2h_3) * \delta(q_1-q_1') \delta(q_2-q_2') \delta(q_3-q_3')$$

$$\sigma = q(1/h_2h_3) * \delta(q_2-q_2') \delta(q_3-q_3')$$

where these are of course distributional symbolic functions. By applying Gauss's Law using a pillbox like that shown above in our drawing (but at an arbitrary location on the surface), I show that

$$\partial_n V_i - \partial_n V_o = \sigma/\varepsilon \qquad \Rightarrow \qquad \partial_{q1} V_i - \partial_{q1} V_o = h_1\sigma/\varepsilon$$

so we get our crucial key result which is

$$\partial_{q1} V_i - \partial_{q1} V_o = (q/\varepsilon)\ (h_1/h_2h_3)\ \delta(q_2-q_2')\ \delta(q_3-q_3')$$

Application # 1: I show that in spherical coordinates with a point charge on the spherical surface at r = a this says

$$\partial_r V_i - \partial_r V_o = (q/\varepsilon)\ (1/a^2)\ \delta(z-z')\ \delta(\varphi-\varphi')$$

which agrees with a result I got doing Smythe cones in some other doc.

Application # 2: For our oblate spheroid example our surface perp coordinate is $q_2 = \zeta$ (Smythe's choice is that $q_1, q_2, q_3 = \xi, \zeta, \varphi$). The above result is then

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\ (h_2/h_3h_1)\ \delta(\varphi-\varphi')\ \delta(\xi - \xi')$$

where

$$h_1 = c_1^2\xi\left[\frac{x^2}{(-c_1^2\xi^2)^2} + \frac{\rho^2}{c_1^4(1-\xi^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1-\xi^2}\right)^{\frac{1}{2}} \qquad (6)$$

$$h_2 = c_1^2\zeta\left[\frac{x^2}{c_1^4\zeta^4} + \frac{\rho^2}{c_1^4(1+\zeta^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1+\zeta^2}\right)^{\frac{1}{2}} \qquad (7)$$

$$h_3 = \frac{\rho d\phi}{d\phi} = \rho = c_1[(1+\zeta^2)(1-\xi^2)]^{\frac{1}{2}} \qquad (8)$$

4. Question: how do we take our pillbox "to the north pole" ? Finding G.

Our problem now is that our point charge is not at some "general location" but is at a very singular location with respect to the oblate spheroidal coordinates. In this verbose section, I review the general idea of how one handles this situation, advised by Stakgold, and I conclude that

$$\lim_{\xi' \to 1} [\ \delta(\xi-\xi')\ \delta(\varphi-\varphi')\ ] = (1/2\pi)\ \delta(\xi-1)$$

Using this with the result just given above in Application #2, we get

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/2\pi\varepsilon)(h_2/h_3h_1)\,\delta(\xi-1) = (q/2\pi\varepsilon)\,[\,c_1(1+\zeta_0^2)]^{-1}\,\delta(\xi-1)$$

This is how, then, we learn the value of G mentioned in Section 1 above

$$G = (q/2\pi\varepsilon)(h_2/h_3h_1) = G = q/[2\pi c_1\varepsilon]/(1+\zeta_0^2)$$

5. Taking the circular plate limit of the oblate spheroidal Green's Function

(a) Taking the flat plate limit of our spheroidal on-axis Green's Function result

My oblate result above from Section 2 above is this:

$$V_o(\zeta,\xi) = q/[4\pi j c_1\varepsilon]\,\Sigma_n(2n+1)\,[\{P_n(j\zeta_1)/Q_n(j\zeta_1)\}\,Q_n(j\zeta_0) - P_n(j\zeta_0)]\,Q_n(j\zeta)\,P_n(\xi)$$
$$V_i(\zeta,\xi) = q/[4\pi j c_1\varepsilon]\,\Sigma_n(2n+1)\,[\{P_n(j\zeta_1)/Q_n(j\zeta_1)\}\,Q_n(j\zeta) - P_n(j\zeta)\,]\,Q_n(j\zeta_0)\,P_n(\xi)$$

and we want to take the limit $\xi_1 \rightarrow 0$. The only appearance of ζ_1 is in the $\{P_n(j\zeta_1)/Q_n(j\zeta_1)\}$ factor. We know (after some work and consternation) that

$$P_n(0)/Q_n(0+) = 0 \qquad n \text{ odd}$$
$$P_n(0)/Q_n(0+) = 2j/\pi \qquad n \text{ even}$$

and the above form then becomes

$$V_o(\zeta,\xi) = q/[4\pi j c_1\varepsilon]\,\Sigma_{n,even}\,(2n+1)\,[(2j/\pi)\,Q_n(j\zeta_0) - P_n(j\zeta_0)]\,Q_n(j\zeta)\,P_n(\xi) \qquad (*)$$
$$- q/[4\pi j c_1\varepsilon]\,\Sigma_{n,odd}\,(2n+1)\,P_n(j\zeta_0)\,Q_n(j\zeta)\,P_n(\xi)$$

$$V_i(\zeta,\xi) = q/[4\pi j c_1\varepsilon]\,\Sigma_{n,even}\,(2n+1)\,[(2j/\pi)\,Q_n(j\zeta) - P_n(j\zeta)\,]\,Q_n(j\zeta_0)\,P_n(\xi)$$
$$- q/[4\pi j c_1\varepsilon]\,\Sigma_{n,odd}(2n+1)P_n(j\zeta)\,Q_n(j\zeta_0)\,P_n(\xi)$$

Note added 1.25.10: We might write the above factor as

$$P_n(0)/Q_n(0+) = (j/\pi)\,\{1+(-1)^n\}$$

Then the Green's Function result is :

$$V_o(\zeta,\xi) = q/[4\pi^2 c_1\varepsilon]\,\Sigma_n(2n+1)\,[\{1+(-1)^n\}\,Q_n(j\zeta_0) - P_n(j\zeta_0)]\,Q_n(j\zeta)\,P_n(\xi)$$
$$V_i(\zeta,\xi) = q/[4\pi^2 c_1\varepsilon]\,\Sigma_n(2n+1)\,[\{1+(-1)^n\}\,Q_n(j\zeta) - P_n(j\zeta)\,]\,Q_n(j\zeta_0)\,P_n(\xi)$$

If we now define $\zeta_> = \max(\zeta, \zeta_0)$ and $\zeta_< = \min(\zeta, \zeta_0)$, we can write this as a single line:

$$V(\zeta,\xi) = q/[4\pi^2 c_1\varepsilon]\,\Sigma_n(2n+1)\,[\{1+(-1)^n\}\,Q_n(j\zeta_<) - P_n(j\zeta_<)]\,Q_n(j\zeta_>)\,P_n(\xi)$$

= g(ζ, ξ | ζ_0, 1) // point charge is at ζ_0, 1

A single infinite sum is regarded as "pretty good" in Green's World. For the 2D problem of the Green's Function for a point charge *inside* a grounded metal unit circle, the result is Stak p II.160 6.119,

$$g(r, \varphi \mid r_0, \varphi_0) = -\frac{1}{2\pi} \log r_> - \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\cos n(\varphi - \varphi_0)}{n} (r_<^n)(r_>^n - r_>^{-n}). \tag{6.119}$$

where we see the corresponding 2D functions

(b) Review of the potential of an on-axis point charge in oblate coordinates

Since we are soon going to subtract off the potential of the Green's point charge q, we want to know the potential of this charge in oblate spheroidal coordinates. Smythe showed this result for general positioning of the point charge relative to the origin. For the charge on-axis, the result is simply this

$V_i = jq /(4\pi\varepsilon c_1) * \Sigma_n (2n+1) Q_n(j\zeta_0) P_n(j\zeta) P_n(\xi)$ // from the point charge q
$V_o = jq /(4\pi\varepsilon c_1) * \Sigma_n (2n+1) P_n(j\zeta_0) Q_n(j\zeta) P_n(\xi)$

(c) Subtracting off the potential of the point charge to get Smythe's Problem 85 p 209 result.

If we subtract the two lines shown in (b) from the quantities shown in (a) as (*), we find that the two residual potentials are exactly the same,

$V_i = V_o = q/[4\pi j c_1 \varepsilon] \Sigma_{n,\text{even}} (2n+1) [(2j/\pi) Q_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi)$

$= q/[2\pi^2 a\varepsilon] \Sigma_{n=0}^{\infty} (4n+1) Q_{2n}(j\zeta_0) Q_{2n}(j\zeta) P_{2n}(\xi)$

and this is exactly the result Smythe displays in his Problem 85 :

85. An earthed conducting disk of radius a whose equation is $\zeta = 0$ is influenced by a point charge q at 1, ζ_0, 0. Show that the potential due to the induced charge on the disk is

$$V_i = \frac{q}{2\pi^2 \epsilon a} \sum_{n=0}^{\infty}{}' (4n + 1) Q_{2n}(j\zeta_0) Q_{2n}(j\zeta) P_{2n}(\xi)$$

6. Some limits of Smythe's result.

Here I discuss the appearance of this induced charge on the plate from far away. I show that the total charge on the plate is not -q as I once carelessly thought, but is some fraction of -q,

fraction = $(2/\pi) [jQ_0(j\zeta_0)] = (2/\pi) \cot^{-1}(\zeta_0)$

which is consistent with our comments in Section 1 above.

(a) Verify Smythe's large ζ limit of $Q_n(j\zeta)$.

I did this using Bateman.

(b) This same problem appears in Smythe third edition

When I earlier thought Smythe's result was wrong, I wanted to see if the answer had been changed in the third edition 1968, after 18 years had passed since the 1950 second edition. It had not changed. This made me realize that I had done surely something wrong.

(c) Why does Smythe's result include only even terms?

The answer is that when the point charge q is removed but the induced plate charge is first glued down and the metal converted to an insulator (or to nothing at all), the charge that was $\sigma_{\mathtt{front}}$ and $\sigma_{\mathtt{back}}$, different values, becomes a single layer of charge $\sigma = \sigma_{\mathtt{front}} + \sigma_{\mathtt{back}}$ and this makes a potential that is symmetric on the two sides of the plate. Thus, the potential sum cannot have odd $P_{\mathtt{n}}(\xi)$ terms since for odd n $P_{\mathtt{n}}(-\xi) = -P_{\mathtt{n}}(\xi)$. The front of the plate has the $\xi > 0$ values, and the back $\xi < 0$.