On-axis Green's Function for Oblate Spheroid PhL 1.8.10→1.11.10

Note: Smythe Problem 85 concerns exactly this Green's Function, but in the spheroid → disk limit. The problem gives the potential of the charge induced on the disk by the on-axis Green's point charge.

Overview (9.29.10, 2 pages) (see also 6 page META review doc!) 1

0. History of my work on this problem: 3

1. Introduction. 4

2. On-Axis Green's Function for an Oblate Spheroid 4

(a) Limits of the Final Result 9

(b) Charge Distribution on the Spheroid (and circular plate) 9

3. Now we have to do our "pillbox construction". 12

(a) Finding the surface charge density σ in the pillbox for a point charge. 12

(b) Applying Gauss's Law to the pillbox 13

Application #1: Pillbox on spherical surface 15

Application #2: Pillbox on surface of an oblate spheroid. 15

4. Question: how do we take our pillbox "to the north pole" ? Finding G. 16

Reduction of a delta function product to a product of *fewer* delta functions at singular points. 16

5. Taking the circular plate limit of the oblate spheroidal Green's Function 20

(a) Taking the flat plate limit of our spheroidal on-axis Green's Function result 20

(b) Review of the potential of an on-axis point charge in oblate coordinates 21

(c) Subtracting off the potential of the point charge to get Smythe's Problem 85 p 209 result. 22

6. Some limits of Smythe's result. 23

(a) Verify Smythe's large ζ limit of Qn(jζ) 25

(b) This same problem appears in Smythe third edition 26

(c) Why does Smythe's result include only even terms? 27

Overview (9.29.10, 2 pages) (see also 6 page META review doc!)

In **Section 0** I review my error of using the zeros of the on-the-cut Q functions in place of those for the normal Q off-cut functions which appear in oblate coordinates. All Q zeros are quoted here. Before I discovered this error, I failed to get Smythe's Problem 85 result. I even looked at his second edition to see if the result had been changed. It had not.

In **Section 1** I quote Smythe's Problem 85 and show that (interpret his text) his expression is for the induced part of the potential which is the on-axis Green's function for an oblate spheroid. I use 1,2,3 = ξ,ζ,φ where ζ labels spheroids, and ξ is the effective polar z, where ξ = 1 on axis. I show a cross section picture of a spheroid.

In **Section 2,** I put a Green's point charge on axis (ξ_0=1,ζ_0,φ_0=x) where ζ_0 marks the dotted "math" spheroid on which this charge is located. By requiring continuity on this dotted boundary and V = 0 when ζ=ζ_1 (metal spheroid label), I propose this Smythian form (*):

$$V_o(\zeta,\xi) = \Sigma_n E_n \; [P_n(j\zeta_1)Q_n(j\zeta_0) - Q_n(j\zeta_1)P_n(j\zeta_0)] \; Q_n(j\zeta) \; P_n(\xi)$$
$$V_i(\zeta,\xi) = \Sigma_n E_n \; [P_n(j\zeta_1) \; Q_n(j\zeta) \; - Q_n(j\zeta_1)P_n(j\zeta) \;] \; Q_n(j\zeta_0) \; P_n(\xi)$$

I assume that the pillbox condition has the form $\partial_\zeta V_i(\zeta_0,\xi) - \partial_\zeta V_o(\zeta_0,\xi) = G\,\delta(\xi-1)$ where G is TBD. I compute the derivatives for the LHS, use the Wronskian which appears, and then use $P_n(\xi)$ orthogonality to find the coefficients E_n. In Section 4 I later show that $G = q/[2\pi c_1\varepsilon] / (1+\zeta_0^2)$, so I install all this stuff to get this **Final Form for the on-axis Green's function of an oblate spheroid:**

$$V_o(\zeta,\xi) = q/[4\pi j c_1\varepsilon] \; \Sigma_n(2n+1) \; [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\} \; Q_n(j\zeta_0) - P_n(j\zeta_0)] \; Q_n(j\zeta) \; P_n(\xi)$$
$$V_i(\zeta,\xi) = q/[4\pi j c_1\varepsilon] \; \Sigma_n(2n+1) \; [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\} \; Q_n(j\zeta) \; - P_n(j\zeta) \;] \; Q_n(j\zeta_0) \; P_n(\xi)$$

In **Subsection (a)** I take the limit of the above when $\zeta \to \infty$. In this limit $r = \zeta c_1$ where r is the spherical radial coordinate. From the form of the resulting V we immediately conclude that

$$q_{induced}/q = - \cot^{-1}\zeta_0 / \cot^{-1}\zeta_1 \qquad \to \qquad (2/\pi) \cot^{-1}\zeta_0 \text{ for disc where } \xi_1 \to 0$$

where ζ_0 labels the Green's charge location and ζ_1 labels the metal spheroid.

In **Subsection (b)** I compute $\partial_\xi V_i$ to get the σ on the spheroid, then take $\xi_1 \to 0$ to get σ on the disc:

$$\sigma = - q/[4\pi c_1^2] \; [\sqrt{1+\zeta_1^2}\,\sqrt{\xi^2+\zeta_1^2}\,]^{-1} \; \Sigma_n \; (2n+1) \; [Q_n(j\zeta_0)/Q_n(j\zeta_1) \;] \; P_n(\xi) \qquad \text{// spheroid}$$

$$\sigma_\pm = - q/[4\pi a^2](\; 1/\sqrt{1-(\rho/a)^2}\;) \; \Sigma_n \; (2n+1) \; [Q_n(j\zeta_0)/Q_n(0+) \;] \; P_n(\pm\sqrt{1-(\rho/a)^2}\;) \qquad \text{// disc}$$

$Q_n(0+) = (-1)^{(n+1)/2} \; (n-1)!! / n!!$ n odd

$Q_n(0+) = (-1)^{n/2} \; (-j\pi/2) \; (n-1)!! / n!!$ n even

The factor $1/\sqrt{1-(\rho/a)^2}$ shows how the charge density builds up at the edge of the disc, becoming infinite but integrable there, something we always see in this kind of problem. The charge ratio front/back is explained by the "scattering" argument where we get term cancellation on the back side so σ there will be smaller than on the front. Even the disk result is pretty messy.

In **Section 3** I show that for a more general location of the pillbox, we have this ζ pillbox condition:

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon) \; (h_2/h_3h_1) \; \delta(\varphi-\varphi') \; \delta(\xi - \xi')$$

In **Section** 4 I show how we take this pillbox to the north pole as a limit, which gives G quoted above.

In **Section 5 (a)** I take the disc limit $\zeta_1 \to 0$ of the Final Form on-axis Green's function above to get

$$V(\zeta,\xi) = q/[4\pi^2 c_1\varepsilon] \; \Sigma_n(2n+1) \; [\{1+(-1)^n\} \; Q_n(j\zeta_<) - P_n(j\zeta_<)] \; Q_n(j\zeta_>) \; P_n(\xi) \qquad \text{// disc}$$

$$= g(\zeta, \xi \mid \zeta_0, 1) \qquad \text{// point charge is at } \zeta_0, 1 \qquad \zeta_> = \max(\zeta, \zeta_0) \qquad \zeta_< = \min(\zeta, \zeta_0)$$

Even though we are in the simple "on axis" location, and even though the disc is a pretty simple geometry, this result is still a single sum of messy functions. We could convert to cylindrical coordinates using the following (2. Oblate Spheroidal...doc), but things just get messier ($r^2 = \rho^2 + z^2$, $a = c_1$)

$$\zeta^2 = (1/2a^2)\ \{\ (r^2-a^2) + \sqrt{(r^2-a^2)^2 + 4z^2}\ \}$$
$$-\xi^2 = (1/2a^2)\ \{\ (r^2-a^2) - \sqrt{(r^2-a^2)^2 + 4z^2}\ \}$$

I suspect there is a simple form for this result, but I don't know what it is! The complexity here of both the potential and the σ discouraged me from trying to solve the charged bowl problem by inverting the on axis Green's function of a disk. At first sight, this sounds like a good thing to do, since you replace the complicated bowl with a problem that seems more "planar". But discs are not very nice things!

In **Section 5 (b)** I quote the 1/R expansion in oblate coordinates for a point charge on axis at ζ_0.

In **Section 5 (c)** I subtract this 1/R thing from my Green's function for the disk shown above, and I find

$$V_i = V_o = q/[4\pi j c_1 \varepsilon]\ \Sigma_{n,\,even}\ (2n+1)\ [(2j/\pi)\ Q_n(j\zeta_0)]\ Q_n(j\zeta)\ P_n(\xi)$$

$$= q/[2\pi^2 a\varepsilon]\ \Sigma_{n=0}^{\infty}\ (4n+1)\ Q_{2n}(j\zeta_0)\ Q_{2n}(j\zeta)\ P_{2n}(\xi)$$

which is precisely Smythe's Problem 85 claimed result for the potential of the induced charge on the disc.

In **Section 6** I verify Smythe's large ζ limit of $Q_n(j\zeta)$ using Bateman, and I explain the trivial reason why the above result has only even terms: a planar disc of charge must generate an even potential.

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0. History of my work on this problem:

I found this fact in Smythe,

$$(n \text{ even}) \qquad Q_n(0) = 0 \qquad (1)$$
$$(n \text{ odd}) \qquad Q_n(0) = (-1)^{\frac{1}{2}(n+1)} \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(n-1)}{1\cdot 3\cdot 5\cdot\cdots\cdot n} \qquad (2)$$

// on the cut Q's

which is correct for on-the-cut Q_n functions, but not for off-the-cut Q_n functions used in $Q_n(j\zeta)$ analysis. For off-the-cut Q_n, the zeros are different for even n, and in particular are not zero:

$$(n \text{ even})\ Q_n(j\cdot 0) = -\frac{1}{2} j\pi P_n(j\cdot 0) = (-1)^{\frac{1}{2}n} \frac{-j\pi}{2} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot n} \qquad (3)$$
$$(n \text{ odd})\ \ Q_n(j\cdot 0) = (-1)^{\frac{1}{2}(n+1)} \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(n-1)}{1\cdot 3\cdot 5\cdot\cdots\cdot n} \qquad (4)$$

// regular Q's

By thinking $Q_n(j0+) = 0$ for evens, I created incorrect "forms" for potentials, and this led to paradoxes, such as expressions for V_i and V_o for the induced charge that had a change in functional form at the math boundary $\zeta=\zeta_0$, though nothing exists on that boundary to cause such a form change. Before I

realized my error, I was convinced that Smythe's stated result for his Problem 85 was wrong, but now I think it is right. Another problem I had was an inability to take the flat disk limit of the oblate spheroid, I was getting infinities from $1/Q_n(0)$ for half my terms, and zeros for the other. Once the error was detected, the various mysteries went away, and the correct result was obtained.

1. Introduction.

First, let's look at this Smythe problem on page 209

> **85.** An earthed conducting disk of radius a whose equation is $\zeta = 0$ is influenced by a point charge q at $1, \zeta_0, 0$. Show that the potential due to the induced charge on the disk is

$$V_i = \frac{q}{2\pi^2 \epsilon a} \sum_{n=0}^{\infty} (4n+1) Q_{2n}(j\zeta_0) Q_{2n}(j\zeta) P_{2n}(\xi)$$

I remember looking at this problem long ago and being very confused. I could not even understand what the physical picture was. But now I think I might stand a chance, having done other oblate problems. Smythe's oblate coordinate ordering is 1,2,3 = ξ,ζ,φ. So his point charge is on some spheroid labeled by ζ_0, but ξ = 1 means "on axis", where I would always use his page 163 version of the coordinate system (this is a side view across the oblate spheroid showing ξ=1 as the north pole.)

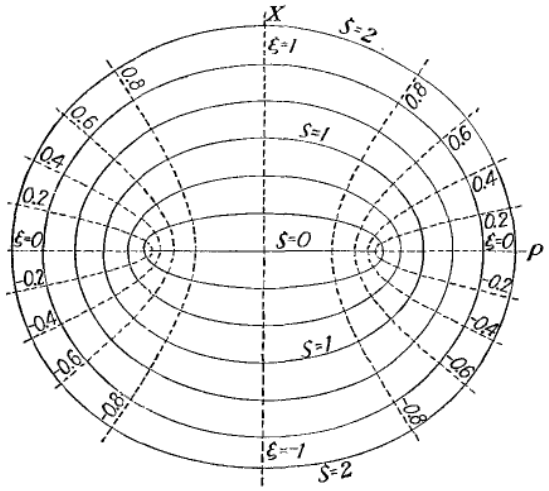


FIG. 5.273.

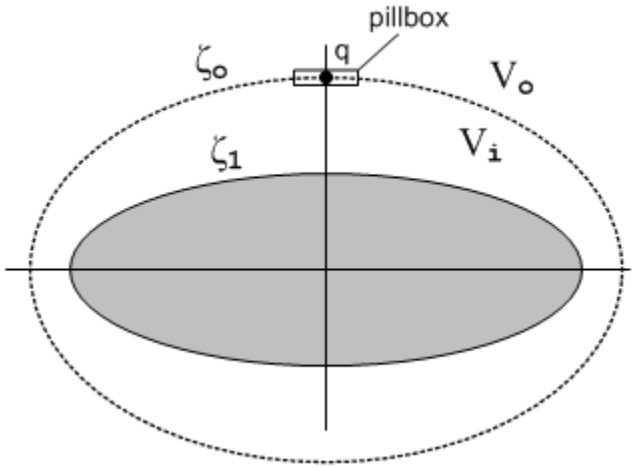
and you see ξ = 1 as sort of the "z axis". The angle φ then has no meaning, he sets it to 0.

So this problem is in fact the Green's Function problem for the disk, but on-axis only. I think this is a problem I have in fact been after for a while, thinking it might be related to a spherical bowl problem by inversion, but let's forget about that for now.

BUT be careful. Smythe is asking for the potential due to just of the induced charge on the spheroid, whereas the Green's function will include the effect of the point charge. So when we get our answer, we are going to have to subtract out the potential of the point charge!

2. On-Axis Green's Function for an Oblate Spheroid

Here is a view of our oblate spheroid, seen edge on :



Let's start trying to do this for an unsquashed oblate spheroid metal with some ζ_1, and later we can try to take the limit $\zeta_1 \rightarrow 0$. I will try to use "the Smythe method". I imagine a math boundary out at $\zeta = \zeta_0$ where the point charge lies. I will make a form for outside that point, and for inside that point, then try to do the "cone problem" pillbox trick right at the charge. So how about, where we set m = 0 since azimuthally symmetric, this form for outside:

$$V_o(\zeta,\xi) = \Sigma_n A_n Q_n(j\zeta) P_n(\xi)$$

We are forced to $P_n(\xi)$ since we have to include $\xi = -1$ in our variable range. We are forced to $Q_n(j\zeta)$ since we are going to go off to infinity. Now between the metal spheroid and the one at ζ_0 we could have both, so I might then say (I must say, in fact)

$$V_i(\zeta,\xi) = \Sigma_n [B_n Q_n(j\zeta) + C_n P_n(j\zeta)] P_n(\xi)$$

But away from our point charge, we have to have the two solutions match, that is,

$$\Sigma_n A_n Q_n(j\zeta_o) P_n(\xi) = \Sigma_n [B_n Q_n(j\zeta_0) + C_n P_n(j\zeta_0)] P_n(\xi)$$

I think this then requires that

$$A_n Q_n(j\zeta_o) = B_n Q_n(j\zeta_0) + C_n P_n(j\zeta_0)$$

I know that we don't want to set $A_n = B_n$ and $C_n = 0$, because then $V_o = V_i$ everywhere and we won't be able to do our pillbox trick. So let's just go with these forms for now:

$$V_o(\zeta,\xi) = \Sigma_n A_n Q_n(j\zeta) P_n(\xi)$$
$$V_i(\zeta,\xi) = \Sigma_n [B_n Q_n(j\zeta) + C_n P_n(j\zeta)] P_n(\xi)$$

We do know, however, that the inside potential vanishes on the metal spheroid at ζ_1:

$$V_i(\zeta_1,\xi) = \Sigma_n [B_n Q_n(j\zeta_1) + C_n P_n(j\zeta_1)] P_n(\xi) = 0$$

This then tells us that

$$B_n Q_n(j\zeta_1) + C_n P_n(j\zeta_1) = 0$$

so we can write

$$V_o(\zeta,\xi) = \Sigma_n A_n Q_n(j\zeta) P_n(\xi)$$
$$V_i(\zeta,\xi) = \Sigma_n B_n [Q_n(j\zeta) - \{Q_n(j\zeta_1)/ P_n(j\zeta_1)\} P_n(j\zeta)] P_n(\xi)$$

But then can mult and divide by $P_n(j\zeta_1)$ and just redefine our coefficients: $B_n/ P_n(j\zeta_1) = D_n$

$$V_o(\zeta,\xi) = \Sigma_n A_n Q_n(j\zeta) P_n(\xi)$$
$$V_i(\zeta,\xi) = \Sigma_n D_n [P_n(j\zeta_1)Q_n(j\zeta) - Q_n(j\zeta_1)P_n(j\zeta)] P_n(\xi)$$

then the vanishing of V_i at ξ_1 is "built into" the form. This is how Smythe would have done it. Now we can require that $V_o = V_i$ at the boundary between the regions if we stay away from our point charge.

$$V_o(\zeta_0,\xi) = \Sigma_n A_n Q_n(j\zeta_0) P_n(\xi)$$
$$V_i(\zeta_0,\xi) = \Sigma_n D_n [P_n(j\zeta_1)Q_n(j\zeta_0) - Q_n(j\zeta_1)P_n(j\zeta_0)] P_n(\xi)$$

and this tells us that we have a condition

$$A_n Q_n(j\zeta_0) = D_n [P_n(j\zeta_1)Q_n(j\zeta_0) - Q_n(j\zeta_1)P_n(j\zeta_0)]$$

so we can then write our two forms as

$$V_o(\zeta,\xi) = \Sigma_n D_n [P_n(j\zeta_1)Q_n(j\zeta_0) - Q_n(j\zeta_1)P_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi)/ Q_n(j\zeta_0)$$
$$V_i(\zeta,\xi) = \Sigma_n D_n [P_n(j\zeta_1)Q_n(j\zeta) - Q_n(j\zeta_1)P_n(j\zeta)] P_n(\xi)$$

But now define $E_n = D_n / Q_n(j\zeta_0)$ and we then have our final "form" for this problem:

$$V_o(\zeta,\xi) = \Sigma_n E_n [P_n(j\zeta_1)Q_n(j\zeta_0) - Q_n(j\zeta_1)P_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi) \qquad (*)$$
$$V_i(\zeta,\xi) = \Sigma_n E_n [P_n(j\zeta_1) Q_n(j\zeta) - Q_n(j\zeta_1)P_n(j\zeta)] Q_n(j\zeta_0) P_n(\xi)$$

Now THIS is "the form" that Smythe would have started his writeup with. You can see by inspection now that $V_i = 0$ at $\zeta = \zeta_1$ (metal), and you can also see by inspection that we have continuity at $\zeta = \zeta_0$. So this is the correct starting point!

We are going to need radial derivatives, so that will be easy enough,

$$\partial_\zeta V_o(\zeta,\xi) = j \Sigma_n E_n [P_n(j\zeta_1)Q_n(j\zeta_0) - Q_n(j\zeta_1)P_n(j\zeta_0)] Q_n'(j\zeta) P_n(\xi)$$

$$\partial_\zeta V_i(\zeta,\xi) = j \sum_n E_n [P_n(j\zeta_1)Q_n'(j\zeta) - Q_n(j\zeta_1)P_n'(j\zeta)] Q_n(j\zeta_0)P_n(\xi)$$

and we are going to want this quantity:

$$\partial_\zeta V_i(\zeta,\xi) - \partial_\zeta V_o(\zeta,\xi) = j \sum_n E_n P_n(\xi) *$$

$$\{ [P_n(j\zeta_1)Q_n'(j\zeta) - Q_n(j\zeta_1)P_n'(j\zeta)] Q_n(j\zeta_0) - [P_n(j\zeta_1)Q_n(j\zeta_0) - Q_n(j\zeta_1)P_n(j\zeta_0)] Q_n'(j\zeta) \}$$

But we are going to want this quantity at $\zeta = \zeta_0$ which marks the spheroid holding our point charge, so

$$\partial_\zeta V_i(\zeta_0,\xi) - \partial_\zeta V_o(\zeta_0,\xi) = j \sum_n E_n P_n(\xi) *$$

$$\{ [P_n(j\zeta_1)Q_n'(j\zeta_0) - Q_n(j\zeta_1)P_n'(j\zeta_0)] Q_n(j\zeta_0) - [P_n(j\zeta_1)Q_n(j\zeta_0) - Q_n(j\zeta_1)P_n(j\zeta_0)] Q_n'(j\zeta_0) \}$$

The first and third terms are equal and cancel, so we get

$$\partial_\zeta V_i(\zeta_0,\xi) - \partial_\zeta V_o(\zeta_0,\xi) = j \sum_n E_n P_n(\xi) *$$

$$\{ - Q_n(j\zeta_1)P_n'(j\zeta_0) Q_n(j\zeta_0) + Q_n(j\zeta_1)P_n(j\zeta_0) Q_n'(j\zeta_0) \}$$

$$\partial_\zeta V_i(\zeta_0,\xi) - \partial_\zeta V_o(\zeta_0,\xi) = j \sum_n E_n P_n(\xi) * Q_n(j\zeta_1)\{ -P_n'(j\zeta_0) Q_n(j\zeta_0) + P_n(j\zeta_0) Q_n'(j\zeta_0) \}$$

$$= j \sum_n E_n P_n(\xi) * Q_n(j\zeta_1) W(P_n(j\zeta_0), Q_n(j\zeta_0))$$

But this is exactly the Wronskian that appeared in our Neumann problem solution, except here we have m = 0 which makes the result very simple,

$$W(P_n(j\zeta_0), Q_n(j\zeta_0)) = [(-1)^m (n+m)! / (n-m)! (1/(1+\zeta_0^2))] = (1+\zeta_0^2)^{-1}$$

so we now have

$$\partial_\zeta V_i(\zeta_0,\xi) - \partial_\zeta V_o(\zeta_0,\xi) = j \sum_n E_n P_n(\xi) * Q_n(j\zeta_1) (1+\zeta_0^2)^{-1}$$

$$= j (1+\zeta_0^2)^{-1} \sum_n E_n Q_n(j\zeta_1) P_n(\xi) \qquad (**)$$

which is certainly a pleasantly simple result.

The next step, then, is to construct our little "pill-box" and thereby learn how to write the LHS of the above as a distribution. For now let's just **conjecture** there is some constant G (it may depend on ζ_0) such that the pillbox result is the following:

$$\partial_\zeta V_i(\zeta_0,\xi) - \partial_\zeta V_o(\zeta_0,\xi) = G \delta(\xi-1)$$

If this is correct, we will then have

$$G \delta(\xi-1) = j (1+\zeta_0^2)^{-1} \sum_n E_n Q_n(j\zeta_1) P_n(\xi)$$

We will then use our P_n orthogonality

$$\int_{-1}^{1} d\xi\, P_n(\xi)P_k(\xi) = 2\delta_{n,k}\,(2n+1)^{-1}$$

to find that

$$\int_{-1}^{1} d\xi\, P_k(\xi)\, G\, \delta(\xi-1) = j\,(1+\zeta_0^2)^{-1}\,\Sigma_n\, E_n\, Q_n(j\zeta_1)\, 2\delta_{n,k}\,(2n+1)^{-1}$$

$$P_k(1)\, G = j\,(1+\zeta_0^2)^{-1}\, E_k\, Q_k(j\zeta_1)\, 2\,(2k+1)^{-1}$$

$$E_k = G\, P_k(1)\,(1/j)\,(1+\zeta_0^2)\,(2k+1)/\,[\,2\, Q_k(j\zeta_1)\,]$$

$$E_n = G\, P_n(1)\,(1/j)\,(1+\zeta_0^2)\,(2n+1)/\,[\,2\, Q_n(j\zeta_1)\,]$$

Then we shall insert this into our "forms" above

$$V_o(\zeta,\xi) = \Sigma_n\, E_n\, [P_n(j\zeta_1)Q_n(j\zeta_0) - Q_n(j\zeta_1)P_n(j\zeta_0)]\, Q_n(j\zeta)\, P_n(\xi)$$
$$V_i(\zeta,\xi) = \Sigma_n\, E_n\, [P_n(j\zeta_1)\, Q_n(j\zeta) - Q_n(j\zeta_1)P_n(j\zeta)\,]\, Q_n(j\zeta_0)\, P_n(\xi)$$

and this then will be the on-axis Green's Function for a metal oblate spheroid ξ_1 where the point charge is located on axis at ξ_0. It will turn out below in **Section 4** that,

$$G = q/[2\pi c_1\varepsilon] \,/\, (1+\zeta_0^2)$$

and of course we know that $P_n(1) = 1$, so then

$$E_n = q/[2\pi c_1\varepsilon] \,/\, (1+\zeta_0^2)\, P_n(1)\,(1/j)\,(1+\zeta_0^2)\,(2n+1)/\,[\,2\, Q_n(j\zeta_1)\,]$$

$$= q/[4\pi j c_1\varepsilon]\,(2n+1)/\, Q_n(j\zeta_1)$$

We can now insert this into our forms above to get our **Final Result:**

$$V_o(\zeta,\xi) = q/[4\pi j c_1\varepsilon]\,\Sigma_n(2n+1)\, [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\}\, Q_n(j\zeta_0) - P_n(j\zeta_0)]\, Q_n(j\zeta)\, P_n(\xi)$$
$$V_i(\zeta,\xi) = q/[4\pi j c_1\varepsilon]\,\Sigma_n(2n+1)\, [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\}\, Q_n(j\zeta) - P_n(j\zeta)\,]\, Q_n(j\zeta_0)\, P_n(\xi)$$

The above is, I believe, the correct Green's Function for a point charge q located on-axis at the location $(\xi=1,\zeta=\zeta_0,\varphi=X)$ in the presence of a grounded conducting oblate spheroid of focal distance c_1 and label ζ_1 which is centered at the origin.

Notice that the form above shows matching at $\zeta = \zeta_0$ and that we still have $V_i = 0$ at $\zeta = \zeta_1$ (ie, on the metal). The terms all satisfy Laplace. So the key thing is that we got the pillbox thing right.

(a) Limits of the Final Result

There are of course limits of the above tentative result that we would like to take.

The limit $\zeta_1 \to 0$ gives us limit as spheroid ζ_1 becomes a circular plate (see Section 5 below).

The limit of large ζ should also be interesting. Let's just do that right here using these facts stolen from Section 5 below,

$$Q_n(j\zeta) \xrightarrow[\zeta \to \infty]{} \frac{2^n (n!)^2}{(2n+1)!}(-j)^{n+1}\frac{1}{\zeta^{n+1}} \qquad (4.1)$$

leading term: $Q_o(j\zeta) \to (-j)/\zeta$ $\qquad$ $Q_0(j\zeta) = -j \cot^{-1}\zeta$

So in the far distance (large ζ) , we keep only the n = 0 term in V_0 which gives this result:

$$\begin{aligned}
V_o(\zeta,\xi) &= q/[4\pi j c_1 \varepsilon] \; [\{P_0(j\zeta_1)/Q_0(j\zeta_1)\}\, Q_0(j\zeta_0) - P_0(j\zeta_0)]\, Q_0(j\zeta)\; P_0(\xi) \\
&= q/[4\pi j c_1 \varepsilon] \; [\{1/Q_0(j\zeta_1)\}\, Q_0(j\zeta_0) - 1]\, Q_0(j\zeta) \\
&= q/[4\pi j c_1 \varepsilon] \; [\{Q_0(j\zeta_0)/Q_0(j\zeta_1)\} - 1]\, Q_0(j\zeta) \\
&= q/[4\pi j c_1 \varepsilon] \; [\{\cot^{-1}\zeta_0 / \cot^{-1}\zeta_1\} - 1]\, Q_0(j\zeta) \\
&= q/[4\pi c_1 \varepsilon] \; [\{\cot^{-1}\zeta_0 / \cot^{-1}\zeta_1\} - 1]\, Q_0(j\zeta)/j \\
&= q/[4\pi c_1 \varepsilon] \; [1 - \cot^{-1}\zeta_0 / \cot^{-1}\zeta_1]\, [jQ_0(j\zeta)] \\
&= q/[4\pi c_1 \varepsilon] \; [1 - \cot^{-1}\zeta_0 / \cot^{-1}\zeta_1]\, \cot^{-1}\zeta \\
&= q/[4\pi c_1 \varepsilon] \; [1 - \cot^{-1}\zeta_0 / \cot^{-1}\zeta_1]\, (1/\zeta) \\
&= q/[4\pi \varepsilon] \; [1 - \cot^{-1}\zeta_0 / \cot^{-1}\zeta_1]\, (1/r)
\end{aligned}$$

where any of these results only applies for large ζ . The last result shows that we can interpret our combined +q charge and oblate spheroid as a "point charge" of this magnitude

$$q_{\mathbf{eff}} = q(\, 1 - \cot^{-1}\zeta_0 / \cot^{-1}\zeta_1\,)$$

This means that the total induced charge on the spheroid must be

$$q_{\mathbf{spheroid}} = (-q) * \text{fraction} \qquad \text{fraction} = \cot^{-1}\zeta_0 / \cot^{-1}\zeta_1$$

where recall ζ_0 is the location of the point charge, and ζ_1 describes the spheroid, and so $\zeta_0 > \zeta_1$. When we go to the flat plate limit $\zeta_1 = 0$, this fraction becomes $(2/\pi)\cot^{-1}\zeta_0$. This subject is discussed at some length in Section 6 below.

(b) Charge Distribution on the Spheroid (and circular plate)

The main result we need to find this out we take from above,

$$\partial_\zeta V_i(\zeta,\xi) = j \; \Sigma_n \, E_n \; [P_n(j\zeta_1)Q_n'(j\zeta) - Q_n(j\zeta_1)P_n'(j\zeta)\;]\, Q_n(j\zeta_0)P_n(\xi)$$

into which we insert our coefficients

$$E_n = q/[4\pi j c_1 \varepsilon]\ (2n+1)/\ Q_n(j\zeta_1)$$

so that

$$\partial_\zeta V_i(\zeta,\xi) = q/[4\pi c_1 \varepsilon]\ \Sigma_n\ (2n+1)\ [P_n(j\zeta_1)Q_n'(j\zeta) - Q_n(j\zeta_1)P_n'(j\zeta)\ ]\ [Q_n(j\zeta_0)/Q_n(j\zeta_1)\ ]\ P_n(\xi)$$

and then of course we want to know this on the metal surface $\zeta = \zeta_1$ so we get

$$\partial_\zeta V_i(\zeta_1,\xi) = q/[4\pi c_1 \varepsilon]\ \Sigma_n\ (2n+1)\ [P_n(j\zeta_1)Q_n'(j\zeta_1) - Q_n(j\zeta_1)P_n'(j\zeta_1)\ ]\ [Q_n(j\zeta_0)/Q_n(j\zeta_1)\ ]\ P_n(\xi)$$

and we recognize our Wronskian which from above is $(1+\zeta_1^2)^{-1}$ so we then have

$$\partial_\zeta V_i(\zeta_1,\xi) = (1+\zeta_1^2)^{-1}\ q/[4\pi c_1 \varepsilon]\ \Sigma_n\ (2n+1)\ [Q_n(j\zeta_0)/Q_n(j\zeta_1)\ ]\ P_n(\xi)$$

We know the surface charge is negative, so we expect $\partial_\zeta V_i(\zeta_1,\xi)$ to be positive.

As we discuss in the next pillbox section,

$$-\partial_n V_i = \sigma/\varepsilon \qquad \text{and} \qquad \partial_n = (1/h_2)\,\partial_\zeta$$

so we find that

$$\sigma = -(\varepsilon/h_2)\,\partial_\zeta V_i$$

$$h_2 = c_1^2\zeta\left[\frac{x^2}{c_1^4\zeta^4} + \frac{\rho^2}{c_1^4(1+\zeta^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1+\zeta^2}\right)^{\frac{1}{2}}$$

where h_2 is evaluated on the metal, meaning $\zeta = \zeta_1$. So we have

$$h_2 = c_1\sqrt{\xi^2+\zeta_1^2}\ /\sqrt{1+\zeta_1^2}$$

Then we have

$$\sigma = -\varepsilon\sqrt{1+\zeta_1^2}\ /\ c_1\sqrt{\xi^2+\zeta_1^2}\ )\ (1+\zeta_1^2)^{-1}\ q/[4\pi c_1\varepsilon]\ \Sigma_n\ (2n+1)\ [Q_n(j\zeta_0)/Q_n(j\zeta_1)\ ]\ P_n(\xi)$$

$$= -q/[4\pi c_1^2]\ [\sqrt{1+\zeta_1^2}\sqrt{\xi^2+\zeta_1^2}\ ]^{-1}\ \Sigma_n\ (2n+1)\ [Q_n(j\zeta_0)/Q_n(j\zeta_1)\ ]\ P_n(\xi)$$

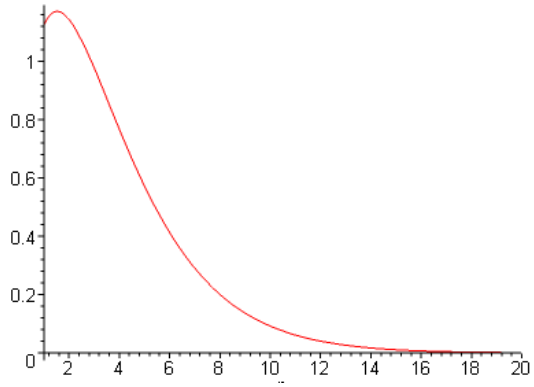
Nothing in general is blowing up here. The main interest is the ξ dependence, but we can only see what that is for each term, though the leading factor causes an increase at the equator as you would expect. The sum here is a little like scattering theory. In the "forward direction", meaning on the side facing the point charge, we have $P_n(1) = 1$ so all the terms are additive and we get a "large" result. On the back of the spheroid we have $P_n(-1) = (-1)^n$ so we get interference and a smaller result.

I can't really tell details of the ξ distribution of charge because you have to include some reasonable number of terms to build the result. For example, here is plot of the coefficient of $P_n(\xi)$ for n = 1..20

```
restart;
zeta0 := 5:
zeta1 := 3:
f := LegendreQ(n,0,evalf(I*zeta0))/LegendreQ(n,0,evalf(I*zeta1));
```

$$f := \frac{\text{LegendreQ}(n, 5.\ I)}{\text{LegendreQ}(n, 3.\ I)}$$

```
plot ((2*n+1)*Re(evalf(f)), n = 1..20);
```



So to get a reasonable plot of σ(ξ), you would have to include maybe 10 terms in this example. As you increase ξ_0 and make new plots, you see that fewer terms are needed as you move away from the spheroid with your point charge. Eventually you only need the n=0 term and then there is no front/back difference at all.

What about the plate limit $\zeta_1 \to 0$? We quote from below that, for $Q_n(j\zeta_1)$, we need to use

$Q_n(0+) = (-1)^{(n+1)/2}\ (n-1)!!\ /\ n!!$ n odd

$Q_n(0+) = (-1)^{n/2}\ (-j\pi/2)\ (n-1)!!\ /\ n!!$ n even

and in this limit our result would be

$$\sigma_{plate} = -\ q/[4\pi c_1^2]\ [\ \xi\ ]^{-1}\ \Sigma_n\ (2n+1)\ [Q_n(j\zeta_0)/Q_n(0+)\ ]\ P_n(\xi)$$

For the plate, coordinate ξ tells you where you are radially on the plate.

$\rho = c_1[(1 + \zeta^2)(1 - \xi^2)]^{\frac{1}{2}}$ $\rho = c_1\sqrt{1-\xi^2}$

$\xi^2 = 1 - (\rho/a)^2$ $\xi = \sqrt{1 - (\rho/a)^2}$ $a = c_1$

so

$$\sigma_{plate} = -\ q/[4\pi a^2]\left(\ 1/\sqrt{1 - (\rho/a)^2}\ \right)\ \Sigma_n\ (2n+1)\ [Q_n(j\zeta_0)/Q_n(0+)\ ]\ P_n\left(\sqrt{1 - (\rho/a)^2}\ \right)$$

The factor $1/\sqrt{1 - (\rho/a)^2}$ shows how the charge density builds up at the edge of the plate, becoming infinite but integrable there, something we always see in this kind of circular conducting plate problem. The charge ratio front/back again depends on the "scattering" argument given above where we get term cancellation on the back side so it will be smaller.

We shall now look carefully at the Pillbox Condition which gives the result shown above for G.

3. Now we have to do our "pillbox construction".

(a) Finding the surface charge density σ in the pillbox for a point charge.

Let's do this for general curvilinear coordinates I will call q_1, q_2, q_3 having scale factors h_1 h_2 h_3. I should be putting the q indices up, but I will put them down for convenience. Assume that the coordinate 1 is perpendicular to our thin pillbox which lies surrounding a piece of surface. Here are some facts we know:

$dA_1 = h_2h_3\, dq_2dq_3$ = area of either end of the pillbox (or of its center layer where σ lies)

$dV = h_1h_2h_3\, dq_1dq_2dq_3$ = volume of the pillbox.

The perpendicular dimension of our pillbox must therefore be h_1dq_1. Our pillbox here is really a curvilinear parallelepiped, very small and very flat, compressed down onto the surface it straddles. Since the coordinates are orthogonal, in the limit it becomes a right-angles rectangular solid.

Assume we have a point charge q inside the pill box at location **r**'. We then know that

$$\rho = q\, \delta(\mathbf{r}-\mathbf{r}') = q/(h_1h_2h_3) * \delta(q_1-q_1')\, \delta(q_2-q_2')\, \delta(q_3-q_3')$$

This is the key fact! We know that δ(**r**-**r**') works the opposite of dV, so that

$$q = \int dV\, \rho = \int h_1h_2h_3\, dq_1dq_2dq_3\, q/(h_1h_2h_3) * \delta(q_1-q_1')\, \delta(q_2-q_2')\, \delta(q_3-q_3') = q$$

$$= \int \rho\, h_1h_2h_3\, dq_1dq_2dq_3$$

[The charge happens to have the name q which is the same as the name of the coordinates. I hope this won't confuse me later.]

Now we also know that

$$q = \int dA_1\, \sigma \qquad dA_1 = h_2h_3\, dq_2dq_3 \qquad \Rightarrow \qquad q = \int \sigma\, h_2h_3\, dq_2dq_3$$

where here we are integrating over a layer-cake internal frosting-layer which contains σ, which lies along the center "plane", let us say, of our pillbox. Perhaps σ varies, so we have an integral.

So at this point we have these two equations:

$$q = \int dq_2 \int dq_3\, \sigma\, h_2h_3$$

$$q = \int dq_2 \int dq_3 \int dq_1\, \rho\, h_1h_2h_3$$

In order for these to be equal we should (can) have

$$\sigma\, h_2h_3 = \int dq_1\ \rho\, h_1h_2h_3 = \int dq_1\ \{ q/(h_1h_2h_3) * \delta(q_1-q_1') \delta(q_2-q_2') \delta(q_3-q_3')\}\ h_1h_2h_3$$

$$= q \int dq_1\, \delta(q_1-q_1') \delta(q_2-q_2') \delta(q_3-q_3') = q\, \delta(q_2-q_2') \delta(q_3-q_3')$$

Therefore we learn through this tortuous exercise that

$\sigma = q(1/h_2h_3) * \delta(q_2-q_2') \delta(q_3-q_3')$ = the charge density in our pillbox for a point charge

and of course we also know that

$\rho = q\ 1/(h_1h_2h_3) * \delta(q_1-q_1') \delta(q_2-q_2') \delta(q_3-q_3')$

Here is a faster way we could have found the result for σ. Imagine the 2D space spanned by the "tangent vectors" of the two coordinates q_2 and q_3 at the point charge location **r'**. See "tensors and curvilinear" on this. In this subspace, our volume charge density would be ρ = σ, and our volume element would be $dV = h_2h_3\, dq_2dq_3 = dA$ and we would describe our point charge as $\rho = \sigma = (1/h_2h_3)\, \delta(q_2-q_2') \delta(q_3-q_3')$ and we would have our answer.

NOTE: Our above result for σ may have to be modified at singular points of the coordinate system. We have assumed we are at some "general point" in space where things are not singular.

(b) Applying Gauss's Law to the pillbox

The reason we are so interested in "σ" above ill be seen in a moment. Consider now some equations relating to Gauss's Law [we could set ε to different things in different systems of units. At least we have "something there" that lets us do this.]

$\nabla\bullet\mathbf{E} = \rho/\varepsilon \quad \mathbf{E} = -\nabla V \qquad \rho = q\, \delta(\mathbf{r}-\mathbf{r'})$

$\int_V dV \nabla\bullet\mathbf{E} = \int_S d\mathbf{S}\bullet\mathbf{E}$ // divergence theorem = Gauss's Law when the vector is **E**

$\text{LHS} = \int_V dV \nabla\bullet\mathbf{E} = \int_V dV\, \rho/\varepsilon = \int_V dr\, q\, \delta(\mathbf{r}-\mathbf{r'})/\varepsilon = q/\varepsilon$

$\text{RHS} = \int_S d\mathbf{S}\bullet\mathbf{E} = -\int_S d\mathbf{S}\bullet\nabla V = -\int_S dS\, \partial_n V$

We can express the LHS ("the charge enclosed") in the following manner, for the limiting situation:

$\text{LHS} = (\sigma\, dA)/\varepsilon$

since σdA is "the charge enclosed" which is of course just q. We can then write the RHS as

$$\text{RHS} = -\int_S dS\, \partial_n V = -[\ (\partial_n V_o)dA - (\partial_n V_i)dA\ ] = [\ (\partial_n V_i) - (\partial_n V_o)]\ dA$$

Here we have assumed that the normal (Cartesian) vector **n** is pointing "out", so $\hat{\mathbf{n}} = \hat{\mathbf{q}}_1$. The potential V has the functional form called V_o on the "outside" of the surface, and V_i on the "inside". So it is on the outside that d**S** = d**A** lines up with $\hat{\mathbf{n}}$ and gives the positive contribution to $\int_S dS\, \partial_n V$. In general, both terms shown on the far right will be positive. Just inside the point charge, $(\partial_n V_i)$ is very positive because we are approaching an infinite (q>0) point charge. Just outside, $(\partial_n V_o)$ is very negative because we are moving away from this same point charge. So we do not have nearly equal and opposite terms here, both terms including their signs are very positive.

Now, if we set LHS = RHS above we get

$$[\ (\partial_n V_i) - (\partial_n V_o)]\ dA = (\sigma\ dA)/\varepsilon$$

or

$$\partial_n V_i - \partial_n V_o = \sigma/\varepsilon \qquad (*)$$

THIS is why we were so interested in an expression for σ in the previous section! Since the dA's cancel, we can avoid worrying about the exact form of dA.

Before taking the final step, we note that

$dn = h_1 dq_1 = ds_1$ = the "radial" distance scale factor situation.

$$\Rightarrow \quad \partial_n = (1/h_1)\, \partial_{q1}$$

We can thus restate our result (*) above as

$$\partial_{q1} V_i - \partial_{q1} V_o = h_1 \sigma/\varepsilon$$

Now recall from above that, for our point charge situation, we had

$$\sigma = q(1/h_2 h_3) * \delta(q_2 - q_2')\, \delta(q_3 - q_3')$$

Therefore, we conclude that

$$\partial_{q1} V_i - \partial_{q1} V_o = (q/\varepsilon)\, (h_1/h_2 h_3)\, \delta(q_2 - q_2')\, \delta(q_3 - q_3')$$

This is our **Big Result** that I have been having so much trouble with! Again, we take as a warning that this may need repair at singular points of the coordinate system.

Comments: (just repeating)

(1) notice that we never have to worry about the details of what dA actually looks like in curvilinear coordinates. It cancels out on both sides of our equation to give (*).

(2) the "units system" is entirely contained in (q/ε). Everything else is units-independent.

Application #1: Pillbox on spherical surface closing off our cone in our cone problem of another document, where I first started pondering this important pillbox component of "Smythe's Method" .

Our coordinates are

$q_1, q_2, q_3 = r,z,\varphi$

$h_1, h_2, h_3 = 1,r,r$ // see page 178 of M&M, but using z instead of θ

[I will try to remember to confirm this h_i claim in an Appendix at the end of this doc.] Our desired surface is in fact perp to coordinate $q_1 = r$ in this application example. Our end result is then this

$$\partial_{q1}V_i - \partial_{q1}V_o = (q/\varepsilon)\,(h_1/h_2h_3)\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

$$\partial_r V_i - \partial_r V_o = (q/\varepsilon)\,(1/a^2)\,\delta(z-z')\,\delta(\varphi-\varphi') \qquad \text{since } r = a \text{ at our pillbox}$$

This agrees with the result I got earlier in a different doc (by less clear means) where I said

$$[\,\partial_r V_i - \partial_r V_e\,] = (1/\varepsilon a^2)\,(1/\,\Delta\Omega_\beta\,) \rightarrow (1/\varepsilon a^2)\,\delta(z-z_\beta)\delta(\varphi-\varphi_\beta)$$

Application #2: Pillbox on surface of an oblate spheroid.

I am going to carefully make use of Smythe's slightly illogical 1,2,3 definitions here so we will have

$q_1, q_2, q_3 = \xi, \zeta, \varphi$ ζ = "the radial coordinate"

In our previous examples we always had "1" as the perpendicular coordinate, but here we have "2". Therefore we can do a cyclic rotation on our result:

$$\partial_{q1}V_i - \partial_{q1}V_o = (q/\varepsilon)\,(h_1/h_2h_3)\,\delta(q_2-q_2')\,\delta(q_3-q_3') \qquad \text{// pre cyclic}$$

$$\partial_{q2}V_i - \partial_{q2}V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(q_3-q_3')\,\delta(q_1-q_1') \qquad \text{// post cyclic}$$

[Everything is positive, so there is no confusion really about this cyclic order business.] So

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi')\,\delta(\xi-\xi') \qquad \text{// post cyclic}$$

and from Smythe we have for the three h_i scale factors,

$$h_1 = c_1^2\xi\left[\frac{x^2}{(-c_1^2\xi^2)^2} + \frac{\rho^2}{c_1^4(1-\xi^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1-\xi^2}\right)^{\frac{1}{2}} \qquad (6)$$

$$h_2 = c_1^2\zeta\left[\frac{x^2}{c_1^4\zeta^4} + \frac{\rho^2}{c_1^4(1+\zeta^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1+\zeta^2}\right)^{\frac{1}{2}} \qquad (7)$$

$$h_3 = \frac{\rho d\phi}{d\phi} = \rho = c_1[(1+\zeta^2)(1-\xi^2)]^{\frac{1}{2}} \qquad (8)$$

By the way, the area of our pill box cross section for this problem is

$dA_2 = h_3h_1dq_3dq_1 = h_3h_1d\varphi\, d\xi$

4. Question: how do we take our pillbox "to the north pole" ? Finding G.

In the problem we are now working on, our point charge is "on axis" so we have to get things to the north pole. For our general oblate spheroidal pillbox we have

$\sigma = q\,(1/h_3h_1) * \delta(q_3-q_3')\,\delta(q_1-q_1')$ // general curvilinear coordinates

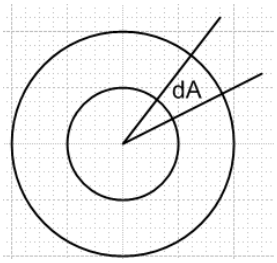
so

$\sigma(\xi,\varphi,\zeta) = q\,(1/h_3h_1) * \delta(\varphi-\varphi')\,\delta(\xi-\xi')$ // oblate spheroidal coordinates

so

$\partial_n V_i - \partial_n V_o = \sigma/\varepsilon = (q/\varepsilon)\,(1/h_3h_1) * \delta(\varphi-\varphi')\,\delta(\xi-\xi')$ // pillbox condition

Here is a picture of the situation at the north pole



Our point charge is inside dA and as $\xi' \to 1$, this approaches the north pole and becomes a little wedge of area there. One idea I have now is to "diffuse" the charge density in dA over the entire little annular ring. Then in the limit of "north pole", it does not matter whether we use dA or the ring.

Reduction of a delta function product to a product of *fewer* delta functions at singular points.

Well, in the end, we are looking for something like this:

$\lim_{\xi'\to 1}\sigma(\xi,\varphi,\zeta) = \lim_{\xi'\to 1}[\,q(1/h_3h_1) * \delta(\varphi-\varphi')\,\delta(\xi-\xi')] = q\,f(\zeta)\,\delta(\xi-1)$

and so the subject here is how a "two coordinate delta function" can be a "one coordinate" delta function in the limit. Luckily for me, I have seen this discussed in Stakgold, and I will now scan for it. One example we say in Stak in Section p 14 was this

$$\lim_{\alpha\to 0} [\ \pi^{-3/2} \exp(-r^2/\alpha^2) / \alpha^3\ ] \ = \ \delta(x_1)\delta(x_2)\delta(x_3)$$

But this is not what we want, but I think we are in the right ball park. I think the Example on Stak page 21 might be relevant. He is talking spherical coordinates and notes that things are singular on the z axis and we have to ponder what to do. Aha? I have spotted what I want on page 17 of my raw Stak Ch 5 notes:

$$\delta^3(\mathbf{x}) \ = 1/[r^2\sin\theta] * \delta(r-r')\ \delta(\theta-\theta')\ \delta(\varphi-\varphi')$$

$$\delta(\mathbf{x} - \mathbf{x}') \ = \delta(r - r')\ \delta(\theta) / [2\pi r^2\sin\theta] \qquad \text{when } x' \text{ has } \theta' = 0 \ \text{ (on } z' \text{ axis)}$$

$$\delta^3(\mathbf{x}) = \delta(r)/[4\pi r^2]$$

The first line is the general case. The second line is what happens if **x'** = on z' axis. In this case, a triplet of deltas becomes a doublet of them! And finally, if **x'** = 0, we have the last line where the triplet has been reduced to a single delta function! So yes, this is definitely the right ball park, and I have luckily been "trained" in this area, though I now forget the details. Unfortunately, Stak in his discussion on p 21-22 uses the obscure names u_1, u_2, u_3 for r,θ,φ, so you have to "translate things" to get what I state above.

So I am now going to study Stak's example again very carefully. I did this quite well in my notes where I "verify" the equalities above. Very good, it all comes back and makes sense. [See page 17 of my raw Chap 5 notes!]

Let's back up to here $[q_1, q_2, q_3 = \xi, \zeta, \varphi\ ]$

$$\delta(\mathbf{r}-\mathbf{r}') \ = \ 1/(\ h_1h_2h_3) * \delta(q_1-q_1')\ \delta(q_2-q_2')\ \delta(q_3-q_3')$$

$$\delta(\mathbf{r}-\mathbf{r}') \ = \ 1/(\ h_1h_2h_3) * \delta(\xi-\xi')\ \delta(\zeta - \zeta')\ \delta(\varphi-\varphi')$$

Now we are interested in the case where r' lies on the ξ' axis. We want to see something like this":

$$\delta(\mathbf{r}-\mathbf{r}') \ = \ g\ \ \delta(\xi-1)\ \delta(\zeta - \zeta')$$

and we want to find what g is by our "verification method". We use f(**r**) as our "test function".

$$f(\mathbf{r}') = \int\int\int dxdydx\ \delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r}) \ = \ \int\int\int\ h_1h_2h_3\ d\xi\ d\zeta\ d\varphi\ \ \delta(\mathbf{r}-\mathbf{r}')\ f(\mathbf{r})$$

$$= \ \int\int\int\ h_1h_2h_3\ d\xi\ d\zeta\ d\varphi\ \ g\ \ \delta(\xi-1)\ \delta(\zeta - \zeta')\ f(\xi,\zeta,\varphi)$$

$$= \ \int\int\int\ d\xi\ d\zeta\ d\varphi\ \ \delta(\xi-1)\ \delta(\zeta - \zeta')\ f(\xi,\zeta,\varphi)\ h_1h_2h_3\ g$$

Do the dξ integral first. Since the range is (-1,1) you might wonder whether we pick up all of the delta function or just half of it. Looking at Stak, I see in his case δ(θ) he had the same issue, and he picked up the entire delta function. So I will do that here. We then have

$$= \int\int d\xi\, d\varphi\, \delta(\zeta - \zeta')\, f(1,\zeta,\varphi,)\, h_1 h_2 h_3\, g$$

Now once again we recall

$$h_1 = c_1^2\xi\left[\frac{x^2}{(-c_1^2\xi^2)^2} + \frac{\rho^2}{c_1^4(1-\xi^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1-\xi^2}\right)^{\frac{1}{2}} \qquad (6)$$

$$h_2 = c_1^2\zeta\left[\frac{x^2}{c_1^4\zeta^4} + \frac{\rho^2}{c_1^4(1+\zeta^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1+\zeta^2}\right)^{\frac{1}{2}} \qquad (7)$$

$$h_3 = \frac{\rho d\phi}{d\phi} = \rho = c_1[(1+\zeta^2)(1-\xi^2)]^{\frac{1}{2}} \qquad (8)$$

which shows that none of the h_i includes the φ variable. And we know that f(1,ζ,φ,) = f(1,ζ,0,) because a function on the z axis is the same for all angles φ. So we continue the above. Let's assume for the moment that g is not a function of φ as well, which I think is a reasonable thing to assume. Then continue

$$= \int d\xi\, d\varphi\, \delta(\zeta - \zeta')\, f(1,\zeta,0)\, h_1 h_2 h_3\, g$$

$$= \int d\xi\, \delta(\zeta - \zeta')\, f(1,\zeta,0)\, h_1 h_2 h_3\, g \int d\varphi$$

$$= 2\pi \int d\xi\, \delta(\zeta - \zeta')\, f(1,\zeta,0)\, h_1 h_2 h_3\, g$$

$$= f(1,\zeta',0)\, [2\pi\, h_1 h_2 h_3\, g\,]$$

where now the h's and g are evaluated at ζ = ζ' as well as ξ = 1. But f(1,ζ',0) = f(**r'**) since this is exactly how you write a point on the positive symmetry axis. So we have then shown that

$$f(\mathbf{r'}) = f(\mathbf{r'})\, [2\pi\, h_1 h_2 h_3\, g\,]$$

and we conclude that we must have

$$g = 1/[2\pi\, h_1 h_2 h_3]$$

so our rule must be

$$\delta(\mathbf{r}-\mathbf{r'}) = 1/[2\pi\, h_1 h_2 h_3]\ \delta(\xi-1)\, \delta(\zeta - \zeta')$$

Digression on Sphericals: Let's see if this gives the right spherical coordinates answer.

$q_1, q_2, q_3 = r,z,\varphi$

$h_1, h_2, h_3 = 1,r,r$ // see page 178 of M&M, but using z instead of θ

$h_1h_2h_3 = r^2$

$\Rightarrow \delta(\mathbf{r}-\mathbf{r}') = 1/[2\pi r^2]\ \delta(z-1)\ \delta(r - r')$

and from above (somewhere) we had

$\delta(\mathbf{x} - \mathbf{x}') = \delta(r - r')\ \delta(\theta) / [2\pi r^2 \sin\theta]$ when x' has θ' = 0 (on z' axis)

so I think we are cooking with gas here.

<u>Back to oblate spheroidals.</u> So far then we have:

$\delta(\mathbf{r}-\mathbf{r}') = 1/[2\pi\ h_1h_2h_3]\ \delta(\xi-1)\ \delta(\zeta - \zeta')$ // **r**' on the ξ = 1 axis

$\delta(\mathbf{r}-\mathbf{r}') = 1/(\ h_1h_2h_3) * \delta(\xi-\xi')\ \delta(\zeta - \zeta')\ \delta(\varphi-\varphi')$ // general location

If we set the two quantities on the right equal, we get

$1/[2\pi\ h_1h_2h_3]\ \delta(\xi-1)\ \delta(\zeta - \zeta') = 1/(\ h_1h_2h_3) * \delta(\xi-\xi')\ \delta(\zeta - \zeta')\ \delta(\varphi-\varphi')$

$\lim_{\xi'\to 1}\ [\ \delta(\xi-\xi')\ \delta(\varphi-\varphi')\ ] = (1/2\pi)\ \delta(\xi-1)$

From above we had

$\sigma(\xi,\varphi,\zeta) = q\ (1/h_3h_1) * \delta(\varphi-\varphi')\ \delta(\xi - \xi')$ // oblate spheroidal coordinates

so

$\partial_n V_i - \partial_n V_o = \sigma/\varepsilon = (q/\varepsilon)\ (1/h_3h_1) * \delta(\varphi-\varphi')\ \delta(\xi - \xi')$ // pillbox condition

Therefore, in our north pole limit we are finding that

$$\partial_n V_i - \partial_n V_o = \sigma/\varepsilon = (q/\varepsilon)\ (1/h_3h_1) * \delta(\varphi-\varphi')\ \delta(\xi - \xi')$$
$$= (q/\varepsilon)\ (1/h_3h_1)\ (1/2\pi)\ \delta(\xi-1)$$

In Section X above we conjectured that

$\partial_\zeta V_i(\zeta_0,\xi) - \partial_\zeta V_o(\zeta_0,\xi) = G\ \delta(\xi-1)$

so we conclude that

$G = (q/2\pi\varepsilon)\ (h_2/h_3h_1) = (q/2\pi\varepsilon)\ [\ c_1\ (1+\zeta_0^2)]^{-1}$

where the second factor I get from hand calculation and it agrees with a result from my previous document where I had $(1+\zeta^2)\, c_1\, h_2 = h_1 h_3$. So

$$G = q/[2\pi c_1 \varepsilon] \,/\, (1+\zeta_0^2)$$

5. Taking the circular plate limit of the oblate spheroidal Green's Function

(a) Taking the flat plate limit of our spheroidal on-axis Green's Function result

My oblate result above was this:

$$V_o(\zeta,\xi) = q/[4\pi j c_1 \varepsilon]\, \Sigma_n (2n+1)\, [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\}\, Q_n(j\zeta_0) - P_n(j\zeta_0)]\, Q_n(j\zeta)\, P_n(\xi)$$
$$V_i(\zeta,\xi) = q/[4\pi j c_1 \varepsilon]\, \Sigma_n (2n+1)\, [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\}\, Q_n(j\zeta) - P_n(j\zeta)\,]\, Q_n(j\zeta_0)\, P_n(\xi)$$

and I wanted to take the limit $\xi_1 \rightarrow 0$.

History Note. I was seeing $P_n(0)/Q_n(0)$ in my limit. But then I was confused by this entry in Smythe's book (and I now realize that the n-even Q result is for on-the-cut Q and is wrong for application here, while all the other results are right).

$$(n \text{ odd}) \qquad P_n(0) = 0$$
$$(n \text{ even}) \qquad P_n(0) = (-1)^{\frac{1}{2}n} \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n}$$

$$(n \text{ even}) \qquad Q_n(0) = 0 \qquad (1)$$
$$(n \text{ odd}) \qquad Q_n(0) = (-1)^{\frac{1}{2}(n+1)} \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n} \qquad (2)$$

which seems to say that $P_n(0)/Q_n(0)$ is zero for odd integers and infinite for even integers, so I was getting complete garbage as a limit. I think, however, that when dealing with the $j\zeta$ type argument, we want to be using the "regular" P and Q functions, not the ones "on the cut". This means that as you come down toward 0 from $j\zeta$, you come onto the top of the cut of Q and then I expect some imaginary limit there.

Continuing Along . We later see in Smythe p 146

$$(n \text{ even})\ Q_n(j \cdot 0) = -\frac{1}{2} j\pi P_n(j \cdot 0) = (-1)^{\frac{1}{2}n} \frac{-j\pi}{2} \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \qquad (3)$$
$$(n \text{ odd})\ Q_n(j \cdot 0) = (-1)^{\frac{1}{2}(n+1)} \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n} \qquad (4)$$

I have verified all these results in my "gamma function page.doc":

$P_n(0) = \quad 0 \qquad\qquad\qquad\qquad$ n odd

$P_n(0) = \quad (-1)^{n/2}(n-1)!! / n!! \qquad$ n even

$Q_n(0+) = (-1)^{(n+1)/2} (n-1)!! / n!! \qquad$ n odd

$Q_n(0+) = (-1)^{n/2} (-j\pi/2) (n-1)!! / n!! \qquad$ n even

$Q_n(0+) = (-j\pi/2) P_n(0) \qquad\qquad$ n even

$P_n(0) / Q_n(0+) = (-j\pi/2)^{-1} = 2j/\pi$

So a revised look at the situation suggests that

$P_n(0)/Q_n(0+) = 0 \qquad$ n odd

$P_n(0)/Q_n(0+) = 2j/\pi \qquad$ n even

My flat plate limit of the oblate spheroidal Green's Function result then becomes

$$V_o(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_n (2n+1) [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\} Q_n(j\zeta_0) - P_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi)$$

$$= q/[4\pi j c_1 \varepsilon] \Sigma_{n,even} (2n+1) [(2j/\pi) Q_n(j\zeta_0) - P_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi)$$

$$+ q/[4\pi j c_1 \varepsilon] \Sigma_{n,odd} (2n+1) [- P_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi)$$

$$V_i(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_n (2n+1) [\{P_n(j\zeta_1)/Q_n(j\zeta_1)\} Q_n(j\zeta) - P_n(j\zeta)] Q_n(j\zeta_0) P_n(\xi)$$

$$= q/[4\pi j c_1 \varepsilon] \Sigma_{n,even} (2n+1) [(2j/\pi) Q_n(j\zeta) - P_n(j\zeta)] Q_n(j\zeta_0) P_n(\xi)$$

$$+ q/[4\pi j c_1 \varepsilon] \Sigma_{n,odd} (2n+1) [- P_n(j\zeta)] Q_n(j\zeta_0) P_n(\xi)$$

Summarize this result one more time (**Final Result** for Green's Function of circular plate)

$$V_o(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_{n,even} (2n+1) [(2j/\pi) Q_n(j\zeta_0) - P_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi)$$
$$- q/[4\pi j c_1 \varepsilon] \Sigma_{n,odd} (2n+1) P_n(j\zeta_0) Q_n(j\zeta) P_n(\xi)$$

$$V_i(\zeta,\xi) = q /[4\pi j c_1 \varepsilon] \Sigma_{n,even} (2n+1) [(2j/\pi) Q_n(j\zeta) - P_n(j\zeta)] Q_n(j\zeta_0) P_n(\xi)$$
$$- q/[4\pi j c_1 \varepsilon] \Sigma_{n,odd} (2n+1) P_n(j\zeta) Q_n(j\zeta_0) P_n(\xi)$$

This is "more complicated" than the original oblate spheroidal result only because we have to separate out the even and odd terms in each of our potential expressions since $P_n(0)/Q_n(0+)$ is different for even/odd.

(b) Review of the potential of an on-axis point charge in oblate coordinates

Now, we need to subtract off the potential of the point charge q in order to answer Smythe's question. Luckily, we have the general result for the potential of point charge as follows: (from my Sm oblate doc)

$$\zeta < \zeta_0 \qquad V_i = \sum_{n=0}^{\infty} \sum_{m=0}^{n} M_{mn} P_n^m(j\zeta) P_n^m(\xi) \cos m(\phi - \phi_0) \qquad (3)$$

$$\zeta > \zeta_0 \qquad V_o = \sum_{n=0}^{\infty} \sum_{m=0}^{n} N_{mn} Q_n^m(j\zeta) P_n^m(\xi) \cos m(\phi - \phi_0) \qquad (4)$$

$$M_{mn} = j(2 - \delta_m^0) q (-1)^m \frac{2n+1}{4\pi\epsilon c_1} \left[\frac{(n-m)!}{(n+m)!}\right]^2 Q_n^m(j\zeta_0) P_n^m(\xi_0) \qquad (1)$$

$$N_{mn} = j(2 - \delta_m^0) q (-1)^m \frac{2n+1}{4\pi\epsilon c_1} \left[\frac{(n-m)!}{(n+m)!}\right]^2 P_n^m(j\zeta_0) P_n^m(\xi_0) \qquad (2)$$

Remember, this is for a point charge located in space at position (ζ_0, ξ_0, φ_0). But we want *our* charge to be at $\xi_0 = 1$ where φ_0 won't matter. In this case we know that V_i and V_o must be independent of φ, but how does this come about from the above formulas? We know that

$P_n^m(z) = (z^2-1)^{m/2} \partial_m P_n(z)$ m = 1,2,3....

$\Rightarrow P_n^m(1) = \delta^m_0$

so this kills off all the coefficients M_{mn} and N_{mn} with $m \neq 0$. We can then transcribe the above to read

$V_i = \Sigma_n M_{0n} P_n(j\zeta) P_n(\xi)$
$V_o = \Sigma_n N_{0n} Q_n(j\zeta) P_n(\xi)$

$M_{on} = j\, q\, (2n+1)/(4\pi\varepsilon c_1) * Q_n(j\zeta_0)$ // $P_n(1) = 1$
$N_{on} = j\, q\, (2n+1)/(4\pi\varepsilon c_1) * P_n(j\zeta_0)$

So write this out as

$V_i = jq/(4\pi\varepsilon c_1) * \Sigma_n (2n+1) Q_n(j\zeta_0) P_n(j\zeta) P_n(\xi)$
$V_o = jq/(4\pi\varepsilon c_1) * \Sigma_n (2n+1) P_n(j\zeta_0) Q_n(j\zeta) P_n(\xi)$

(c) Subtracting off the potential of the point charge to get Smythe's Problem 85 p 209 result.

I now want to take the Green's function result and subtract from it the point charge result. Thus,

$V_o(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_{n,\text{even}} (2n+1) [(2j/\pi) Q_n(j\zeta_0) - P_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi)$
$- q/[4\pi j c_1 \varepsilon] \Sigma_{n,\text{odd}} (2n+1) P_n(j\zeta_0) Q_n(j\zeta) P_n(\xi)$
$+ q/(4\pi\varepsilon j c_1) * \Sigma_n (2n+1) P_n(j\zeta_0) Q_n(j\zeta) P_n(\xi)$

$V_i(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_{n,\text{even}} (2n+1) [(2j/\pi) Q_n(j\zeta) - P_n(j\zeta)] Q_n(j\zeta_0) P_n(\xi)$
$- q/[4\pi j c_1 \varepsilon] \Sigma_{n,\text{odd}} (2n+1) P_n(j\zeta) Q_n(j\zeta_0) P_n(\xi)$

$+q/(4\pi\varepsilon j c_1) * \Sigma_n (2n+1) Q_n(j\zeta_0) P_n(j\zeta) P_n(\xi)$

I converted the last line in each group from - to + by moving the j downstairs. If we ignore the ugly first line of each group, we see that the odd terms of the point charge contribution exactly cancel the odd terms above them, leaving only the even terms of the point charge. So we rewrite as

$V_o(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_{n,even} (2n+1) [(2j/\pi) Q_n(j\zeta_0) - P_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi)$
$+q/(4\pi\varepsilon j c_1) \Sigma_{n,even} (2n+1) P_n(j\zeta_0) Q_n(j\zeta) P_n(\xi)$

$V_i(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_{n,even} (2n+1) [(2j/\pi) Q_n(j\zeta) - P_n(j\zeta)] Q_n(j\zeta_0) P_n(\xi)$
$+q/(4\pi\varepsilon j c_1) \Sigma_{n,even} (2n+1) Q_n(j\zeta_0) P_n(j\zeta) P_n(\xi)$

The good news is that both potentials have only even terms, something we want. Now a stunning thing happens. In each group, the last term of the first line cancels the second line !!! Our results are now:

$V_o(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_{n,even} (2n+1) [(2j/\pi) Q_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi)$

$V_i(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_{n,even} (2n+1) [(2j/\pi) Q_n(j\zeta)] Q_n(j\zeta_0) P_n(\xi)$

And we now see that both terms are the same, we have this result:

$V_i(\zeta,\xi) = V_o(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] *(2j/\pi) \Sigma_{n,even} (2n+1) Q_n(j\zeta_0) Q_n(j\zeta) P_n(\xi)$

$= q/[2\pi^2 a\varepsilon] \Sigma_{n,even} (2n+1) Q_n(j\zeta_0) Q_n(j\zeta) P_n(\xi)$

$= q/[2\pi^2 a\varepsilon] \Sigma_{n=0}^{\infty} (4n+1) Q_{2n}(j\zeta_0) Q_{2n}(j\zeta) P_{2n}(\xi)$

and we now look again at Smythe's statement of the problem,

85. An earthed conducting disk of radius a whose equation is $\zeta = 0$ is influenced by a point charge q at $1, \zeta_0, 0$. Show that the potential due to the induced charge on the disk is

$$V_i = \frac{q}{2\pi^2 \epsilon a} \sum_{n=0}^{\infty} (4n+1) Q_{2n}(j\zeta_0) Q_{2n}(j\zeta) P_{2n}(\xi)$$

Unbelievable !!! I finally have obtained exactly his result after very many hours and misguided steps.

6. Some limits of Smythe's result.

Here is Smythe's result for the potential of the induced charge (point charge was at ζ_0).

$$V_i = \frac{q}{2\pi^2 \epsilon a} \sum_{n=0}^{\infty} (4n+1) Q_{2n}(j\zeta_0) Q_{2n}(j\zeta) P_{2n}(\xi)$$

Notice that we do NOT have V = 0 on the circular plate which is at $\zeta = 0$ if we just consider the induced charge on the spheroid, so that is *not* a limit which applies to the above.

Now what about looking at the induced charge on the plate from "far away". I realize just now that I had a misconception about what to expect here. I was thinking that the induced charge on the plate must be -q if the point charge that induced it was +q. But that is completely wrong, and here is a descriptive answer why.

Suppose you have +q sitting on a tiny metal sphere which is 2 feet from an origin which is surrounded by a of a great metal sphere of radius 1000 feet. We know that -q is induced on the great sphere's inner surface, because if we apply Gauss's law to the entire picture, the "charge enclosed" is 0. Now suppose we add at our origin a tiny metal plate 1 mm in radius which is grounded by a thin wire to the great sphere. The plate's normal vector points to our +q charge. We can ignore this wire operationally as follows. We have the wire present to let charge move between the plate and the great sphere to make the plate be at V = 0 like the great sphere. When current stops flowing, we remove the wire and we then have three items only: the original tiny sphere with +q on it; our 1 mm plate at V = 0 which is 2 feet from the +q sphere. And the great sphere at V = 0.

Now we know that if we make the plate be .000001 mm in diameter, we can neglect it, so there must still be -q on the great sphere. When the plate is 2 mm, some tiny part of the -q from the great sphere is now on our plate. If we make the plate infinitely large so it is a plane, then we know from images that the entire charge -q has moved to the plate. So in the intermediate situation, we know that for a moderately sized plate, *some fraction* of the -q that was on the great sphere before the plate was added to the scene (with wire, then wire removed) is now on the plate. The plate has some total negative charge (-q) x fraction, and the larger the plate and the closer it is to the charge, the larger that fraction will be. In general, then, the charge on the plate is NOT -q.

Now take the large ζ limit of Smythe's formula. We can take this limit using

$$Q_n(j\zeta) \xrightarrow[\zeta \to \infty]{} \frac{2^n (n!)^2}{(2n+1)!} (-j)^{n+1} \frac{1}{\zeta^{n+1}} \qquad (4.1)$$

where the leading term is this:

$Q_0(j\zeta) \rightarrow (-j)/\zeta$

and we then get

$V_i = (q/2\pi^2 \varepsilon a)\, Q_0(j\zeta_0) * (-j)/\zeta * P_0(\xi) = (q/2\pi^2 \varepsilon a \zeta)\, [-jQ_0(j\zeta_0)]$

We can think of this as (knowing that $r = a\,\zeta$ far away)

$V_i = q_{plate} / [\, 4\pi\varepsilon\, r\,] = q_{plate} / [\, 4\pi\varepsilon\, a\, \zeta\,]$

Comparing we find that

$q_{plate} = (q/2\pi^2 \varepsilon a \zeta)\, [-jQ_0(j\zeta_0)] * 4\pi\varepsilon\, a\, \zeta = -q\,(2/\pi)\, [jQ_0(j\zeta_0)]$

So we claim then that the fraction of the great sphere's charge that has moved to the plate is

fraction = $(2/\pi)\ [jQ_0(j\zeta_0)]\ =\ (2/\pi)\ \cot^{-1}(\zeta_0)$

If we move the plate far away from our +q charge by increasing ζ_0 , we are in effect making the plate smaller (great sphere is at R = ∞). In this limit of large ζ_0 we have

fraction = $(2/\pi)[\ j\ (-j)/\zeta_0\ ]\ =\ (2/\pi\zeta_0)$

which we are happy to see is REAL, and it approaches 0 .

We in fact know the exact function here, since we know that (Smythe p 145)

$$Q_n(j\zeta) = -j\cot^{-1}\zeta P_n(j\zeta) - \frac{2n-1}{1\cdot n}P_{n-1}(j\zeta) - \frac{2n-5}{3(n-1)}P_{n-3}(j\zeta) - \cdots \quad (2)$$

$Q_0(j\zeta) = -\ j\ \cot^{-1}\zeta$

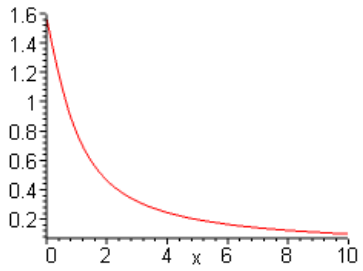
$[\ j\ Q_0(j\zeta)] = \cot^{-1}\zeta_0$

Therefore we have

fraction = $(2/\pi)\ \cot^{-1}\zeta_0$

We know at ζ_0 this fraction is 1, and this is our image charge limit. And things just drop off as I plotted elsewhere: "The function $\cot^{-1}(\zeta)$ starts at π/2 and drops off "

```
plot(arccot(x),x=0..10);
```



So normalize this to 1 and interpret x as ξ_0 and then we have a plot of our "fraction".

(a) Verify Smythe's large ζ limit of Qn(jζ) .

Use Bateman p 122 (5) which says

$Q_\nu(z) = 2^{-\nu-1}\ \sqrt{\pi}\ \Gamma(\nu+1)/\Gamma(\nu+3/2)\ z^{-\nu-1}\ *\ 1$

$$Q_n(z) = 2^{-n-1} \sqrt{\pi}\ \Gamma(n+1)/\Gamma(n+3/2)\ z^{-n-1}$$

But Schaum p 102 (16.9) with x = n+1 tells us that

$$2^{2(n+1)-1}\,\Gamma(n+1)\,\Gamma(n+3/2) = \sqrt{\pi}\ \Gamma(2n+2) = \sqrt{\pi}\,(2n+1)!$$

$$2^{2n+1}\,\Gamma(n+1)\,\Gamma(n+3/2) = \sqrt{\pi}\,(2n+1)!$$

$$1/\,\Gamma(n+3/2) = 2^{2n+1}\,\Gamma(n+1)/\,[(2n+1)!\,\sqrt{\pi}\,]$$

Then we get

$$Q_n(z) = 2^{-n-1} \sqrt{\pi}\ \Gamma(n+1)/\Gamma(n+3/2)\ z^{-n-1}$$

$$= 2^{-n-1} \sqrt{\pi}\ \Gamma(n+1)\{\,2^{2n+1}\,\Gamma(n+1)/\,[(2n+1)!\,\sqrt{\pi}\,]\}\ z^{-n-1}$$

$$= 2^n\ \ (n!)^2/\,[(2n+1)!] * z^{-n-1}$$

Then we get

$$Q_n(j\zeta) = 2^n\,(n!)^2/\,[(2n+1)!] * (j\zeta)^{-n-1}$$

$$= 2^n\,(n!)^2/\,[(2n+1)!] * (j)^{-n-1}\ \ 1/\zeta^{n+1}$$

$$= 2^n\,(n!)^2/\,[(2n+1)!] * (1/j)^{n+1}\ \ 1/\zeta^{n+1}$$

$$= 2^n\,(n!)^2/\,[(2n+1)!] * (-j)^{n+1}\ \ 1/\zeta^{n+1}$$

$$Q_n(j\zeta) \xrightarrow[\zeta \to \infty]{} \frac{2^n (n!)^2}{(2n+1)!}(-j)^{n+1}\frac{1}{\zeta^{n+1}} \qquad (4.1)$$

so Smythe has this limit exactly right.

(b) This same problem appears in Smythe third edition

This took some smart scanning of a blocked Google document. I was looking at a time I thought his answer was wrong and he might have updated it in the third edition of 1968.

First edition:	1939	
Second edition:	1950	// what I have
Third edition:	1968	
paperback	1989	"A Summa Book" Hemisphere Publishing

Page 233

by a point charge q at 1, ζ_0, 0. Show that the potential due to the induced charge on the disk is

$$V_i = \frac{q}{2\pi^2 \epsilon a} \sum_{n=0}^{\infty} (4n+1) Q_{2n}(j\zeta_0) Q_{2n}(j\zeta) P_{2n}(\xi)$$

It really is the third edition I am looking at , viz

2 pages matching **third edition** in this book

Page iii

Third Edition

Page xiii

PREFACE TO THE THIRD EDITION v

PREFACE TO THE SECOND EDITION VII

PREFACE TO THE FIRST EDITION IX

So if it is wrong, it did not get corrected in that edition. [But it is right!]

(c) Why does Smythe's result include only even terms?

The result for the field due just to the induced plate charge is this:

$$V = q/[2\pi^2 a\varepsilon] \Sigma_{n=0}^{\infty} (4n+1) Q_{2n}(j\zeta_0) Q_{2n}(j\zeta) P_{2n}(\xi)$$

Here is my interpretation of this result. First, when the charge is present and we have a metal conductor, the potential is not just even terms. But think how the situation arises that creates the above potential. You first have the point charge +q and the spheroid or plate. You let things stabilize, then you somehow "glue down" the charge to the surface of the conductor. Then you change that conductor to a plastic insulator with this same charge glued down to it. Only then can we remove the +q point charge. The charge glued to our plastic insulating spheroid stays put because it was glued down. It continues to generate exactly the potential it did before as its "contribution" to the total potential.

Now, in the limit that this plastic spheroid with charged glued to it is an infinitely thin disk, we have to think about the charge. This is no longer a metal disk with a conducting interior which had different surface charges on the two sides. This is basically a mathematical disk with that same charge that was on both sides now being a single layer of surface charge.

While the metal was there, you could "support" different charge densities on the two sides of the thin plate. But now that the metal is gone, there is only one charge density $\sigma = \sigma_{front} + \sigma_{back}$ even though $\sigma_{front} \neq \sigma_{back}$. Since we have a single later of charge σ, the electric field on either side of the plate, front or back, must be exactly the same!! That is the point. This is why there are only even terms in the potential. Any odd term would cause front/back difference in the potential.

Basically if you have a circular super-thin insulator with charge glued onto it, everything has to be the same on the two sides: electric field and potential. Again, you cannot therefore have odd n in the potential since $P_n(\xi)$ for odd n does $P_n(-\xi) = -P_n(\xi)$.

Think of the extreme limit of this problem where the plate is an entire plane. When the charge is present, we know that the side of the metal plane facing the charge has a certain peaked charge pattern that integrates to -q, and the back side of the plane has no charge at all, so $\sigma_{\texttt{front}} \neq 0$ and $\sigma_{\texttt{back}} = 0$. Now glue the charge down and remove the point charge. The potential is now suddenly symmetric on the two sides of the plane. Even if you think of it as still metal, the charge is "glued down", and if you compute the field at two mirror points front and back, you get the same magnitude for field and potential. It's not like the metal plate "blocks" the E field from coming through. This would be a bad misunderstanding of "shielding". A conductor gets an induced charge to cancel something on the other side, as in the above case where the charge is still present. There is no field behind the wall because the wall shields the point charge on the other side so that the sum of their potentials is 0 behind the wall. But once you remove the point charge, there is no longer cancellation. I just found that making the metal become plastic helped think about this issue, since charge normally can flow on a metal's surface (tangentially).