Jackson Chapter 1 Notes PhL 1.28.03

Chapter 1: Introduction to Electrostatics

This is a pretty heavy-duty introduction! It sets up the entire mechanism for computing potential ϕ with the Dirichlet or Neumann boundary conditions using Green's Functions.

1.1 Coulomb's Law.

1.2 Electric Field. Of a point charge, then many point charges, then integral over ρ. Mention of the "statcoulomb" with mks put in appendix. Review of delta function technology.

1.3 Gauss's Law. Proves this for a point charge in the integral sense, result (1.11)

1.4 Differential Form of Gauss's Law. Jackson just quotes "the divergence theorem" without comment. At least Portis proves it. I remember being hit flat in the face by this strange new theorem. So $\nabla \bullet \mathbf{E} = 4\pi\rho$.

1.5 The Potential. Defines it for general ρ by saying $\mathbf{E} = -\nabla\phi$ and then ϕ is what it is. Since $\mathbf{E}$ has this form, we are guaranteed by vector identity to have $\nabla \mathbf{x} \mathbf{E}=0$. Relation to work. ϕ as line integral of $\mathbf{E}$.

1.6 Surface charge σ and the Dipole Layer. Shows how to compute it using pillbox and get $4\pi\sigma = \Delta E_{normal.}$ The **dipole layer** is defined as two σ layers separated by d such that $\sigma d = D$ as $d \rightarrow 0$, all functions of x. So you deal only with the D(x) result. Now $\phi = -\int D\, d\Omega$ and $\Delta\phi = 4\pi D$ across the layer.

1.7. Poisson and Laplace Equations. Poisson has ρ on the right, Laplace does not. Finds that Green's is 1/R see (1.31).

1.8 Green's Theorem Applied to ϕ. In the divergence theorem set $A = \phi\nabla\psi$ and you get the "first identity". Swap the functions, subtract, and you get the "second identity" we usually just call "Green's Theorem" (1.35). So we have just derived this from the divergence theorem. Now set $\psi = 1/R$ and $\phi = \phi$. Note that $\nabla^2\psi = 4\pi\delta$ while $\nabla^2\phi = 4\pi\rho$, and voila!, out pops the underpinning equation of electrostatics (1.36). It tells you how to find the potential given some ρ and some boundary conditions on a closed surface.

Interpretation is very important and very tricky here!

If point **x** is *outside* the closed surface, the derivation shows that the LHS of (1.36) is zero and so the two terms on the right must exactly cancel. In this case, you are getting no statement at all about what ϕ is inside or outside the surface. If the surface is just a mathematical one around some charge distribution ρ, all you can say is that that the right term surface term will cancel the ρ term on the RHS. Now the first term is the infinite-surface solution for the potential ϕ that we are familiar with, due to charges ρ. The second term can be *interpreted* as arising from a mysterious distribution of charge σ and dipole layer D on the surface that makes a potential which exactly cancels the first term. Suppose we <u>really could</u> put this distribution of real charge and D on such a surface, gluing it into free space somehow. Then if we went outside (now with an infinite surface beyond *us)*, we would add the potentials from the original ρ term and then from the *real* surface σ and D terms, these would cancel, and we would *conclude* that $\phi = 0$

everywhere outside the surface. We can then reverse this situation by saying the following. Suppose we have a cavity in grounded metal which matches our surface. We *know* that $\phi = 0$ everywhere outside the surface! Then we can conclude that the σ and D in this case *really exist* on the inside surface of the cavity and these real things are what cancels the ρ-generated potential! { I am not quite sure what the D term means, although I know what D means; the σ term means real surface charge. }

If point x is *inside* the closed surface, we get an equation for ϕ which is the usual ρ term plus the surface terms. If we have a mathematical surface only, which supports no charge or D, then the surface term will give exactly 0 (assuming there are no metal surfaces anywhere). But if we are in a metal cavity and we have real σ and D on the metal surface arranged as described above so they cancel our ρ potential on the outside, then this equation tells us how to find the complete ϕ inside the cavity, and in this case the second term will NOT be zero.

We have a seeming paradox here. We want the surface terms to be 0 if we have x on the inside of a *purely mathematical* surface, but we want the surface terms to be non-zero and exactly cancel the ρ term if we are on the outside of the surface, because in this case the theorem says the LHS = 0. So how can the same surface terms be 0 in one case and non-zero in the other case! The answer is that the surface term integral really is different depending on whether x is inside or outside. I did not understand this until I did the following example case, and then it became clear.

Example: A point charge q in the center of a sphere of radius a. In this case, **r'** is the integration variable that wanders over the surface and we can replace da' with $a^2\, d\Omega'$. We choose the z' axis along the direction of vector **r**. We then replace da' with $2\pi a^2\, dx'$ where $x' = \cos\theta'$. Here are some intermediate steps in computing the surface integrals:

$$\partial/\partial n' = \partial/\partial r' \qquad R^2 = a^2 + r^2 - 2ar\cos\theta' = A + Bx' \qquad A = a^2 + r^2 \;\; B = -2ar$$

and $\partial/\partial n'(1/R) = -(a - rx')/R^3$ at the surface

We know with a purely math surface that $\phi = q/r'$ at the surface, so

$$\partial/\partial n'(\phi) = -q/a^2 \qquad \text{at the surface}$$

So the complete surface business is this:

$$(1/4\pi)\, 2\pi a^2 \int dx' \{ (-q/a^2)/R - (q/a)[-(a - rx')/R^3] \}$$

where of course R = R(x'), and integrate -1 to 1. There are three terms to integrate here. If we plug this integral symbolically into Maple, we get a pretty big mess which involves $\sqrt{A+B}$ and $\sqrt{A-B}$. We know that A > B, so that is not the problem. This is where the situation a > r or a > r makes a huge difference. We have:

$$\sqrt{A+B} = \sqrt{(a-r)^2} = \text{sign}(a-r)*(a-r) = |a-r| \qquad \text{// critical thing is sign(a-r) !}$$
$$\sqrt{A-B} = \sqrt{(a+r)^2} = (a+r) \qquad \text{// no confusion here}$$

When r < a (meaning our point x is inside the surface), the entire surface integration gives 0, showing us that our answer is just $\phi = q/r$ from the ρ term. On the other hand, when r > a (meaning we are on the outside), then the big mess comes out being -q/r and exactly cancels the ρ term!

The main point is to understand that, although the surface integral "looks the same" in both the x inside and x outside cases, it is not the same!

Note: Regarding the integrand $\sqrt{A+Bx'}$, it has a branch cut to the left of -1 which lies outside our integration range. We are working on the real analytic sheet where we take the + square root, the other sheet has the - square root, we never go there. When we have to deal with the endpoint value $\sqrt{(a-r)^2}$, we must therefore take the positive square root which is |a-r|. There is no more to it than that!

This same situation occurs in a more trivial situation. Imagine computing the potential inside a sphere of uniform surface charge σ. We do exactly as above and we get $\int dx'\ 1/R$ and we get an answer that looks like {|a-r| - (a+r)}/2ar. If r>a, we get a 1/r answer, and if r<a we get a constant 1/a answer -- the potential is constant inside the sphere. At exactly r=0 both answers are the same.

Now we come to another main point. In general, for x inside, we should just regard (1.36) as a complicated "integral equation" that we have to solve for ϕ. But we look at the equation and wonder: is it possible that we can take some forced values for ϕ and ϕ' on the surface, and then use the equation to compute ϕ everywhere inside (also knowing ρ of course)? If so, that would be a powerful tool, and this is the subject of the next section.

1.9. Comments on Solutions in the Dirichlet, Neumann and Cauchy cases. It turns out that if you try to force both ϕ and ϕ' on a closed surface (Cauchy), you have "overspecified" the problem and there is no solution generally speaking. However, if you specify *either* ϕ (Dirichlet, the usual case) *or* ϕ' (Neumann) on the closed surface, then in fact this *does* completely determine ϕ inside, but you have to do some work to find the solution, which we will do below.

A table on page 17 summarizes things, and I put now some older notes right here.

In statics, you have the wave-equation with k=0 ($\lambda = \infty$) with ρ as the driving source [known as Poisson], as in (1.28), and the general solution is (1.17) where 1/R is the Green's function or propagator (think exp(ikR)/R with k=0). This is the "particular solution", and you can add solutions to $\nabla^2\phi = 0$ [known as Laplace] , as in (1.36), in order to find a complete solution that matches some boundary conditions. The two boundary surface terms can be interpreted as from an effective surface charge and "dipole layer", page 15. [see discussion above!]

In the boundary conditions, you are either specifying the normal E field at the surface (Neumann), or you specify ϕ itself (Dirichlet). If you specify either one on a closed boundary, you get a unique and correct solution, but if you try to specify both (Cauchy) , it is "too much" and the only solution is 0. Specifying any of these three on an open (partial) surface is "not enough" to get a solution. This is shown in the first column of the table on page 17.

Notice the second column which applies to "hyperbolic" ODE's like the wave equation with $k \neq 0$. In this case, specifying *anything* on a closed surface is too much, and the only chance you have is doing Cauchy (ie, specifying ϕ and ϕ') on an open partial surface. In Kirchhoff scalar diffraction theory (field ψ) applied to an aperture, you are in effect trying to specify ψ and $\partial\psi/\partial n$ on a *closed* surface (in the hole, you are assuming you have the incident field unaltered, on the screen you assume 0, on the great sphere you assume 0). Thus you are trying to do Cauchy on the wave equation, and that is known to only give a 0 solution. The fix is that in the aperture, things are not quite unaltered!

The third column applies to ODE's like the heat equation which I have never really studied! Jackson does not prove these things, but refers us to famous historic sources like Morse & Feshbach, and Sommerfeld himself.

1.10. How to solve for ϕ with boundary conditions using the G method. The basic problem is that you would like to eliminate one or the other of the two surface terms mentioned above. Earlier we used Green's Theorem with $\psi = 1/R$ and $\phi = \phi$. If we instead use $\psi = G(x.x')$, we get (1.42). If we could somehow solve this for G(x,x'), obviously we can find ϕ. If we can somehow arrange for G to both solve the Poisson equation AND to be 0 on a surface, then we will have eliminated the first surface term, and we can then get our answer by knowing only ϕ on the surface (that is, we don't have to preknow ϕ', nor do we want to know it if we know ϕ there). So this is method to solve a Dirichlet problem!

On the other hand, suppose we could find a G with G'=0 on the surface. That would appear to eliminate the second term in (1.42). HOWEVER, this leads to an inconsistency! It is easy to show that the surface integral of G' must be -4π, so you can never have G'=0. The fix is to set G' = -4π/Area = a constant. If you do this, then the second surface term in (1.42) does not quite vanish and you end up with an extra term 1/Area * $\int \phi \, dA$ which is the average of ϕ on the surface. This is the "extra" first term you see in (1.46) . If we agree to only do Neumann problems working in the space between a finite surface and an infinite one, which together make up our surface S, then this average ϕ term vanishes. This is tricky, though, because you have to remember that your true "interior" region is now "exterior" to the inner surface.

Finally, Jackson in (1.40) writes a general form for the Green's function which solves (1.39). the usual 1/R is the particular solution, but F can be in principle any solution of Laplace! Our game is going to be finding F(x,x') that gives us stuff like G = 0 on some surface. The term F can be directly associated with the induced surface charges in a problem with conductors.

1.11 Electrostatic potential energy and energy of the electric field. Jackson starts by doing the simple not-equal sum between a set of charges to add up the potential energy as in (1.50). For continuous this appears as 1.52 and he notes that now we have included the diagonal or self-energy stuff. But 1.52 gives 1.53 where you integrate $\rho\phi$, and then you fiddle easily to show 1.54 which we know well, that the density is $|E|^2 / 8\pi$. He shows with an example what must always be true, that the self energy terms will make sure the energy density can never be negative, though the non-self-energy component can in fact be negative. He mentions that you can use the virtual displacement idea to see how energy changes and thus do a force computation.

We then have 13 problems. I seem to have done 7 of them during the course!