Jackson Chapter 2 Notes PhL 1.28.03

I notice that Jackson is in 3rd edition 1998 at $94 Amazon. It seems to me he has added a lot more applications, in Portis style, and has gotten rid of the entire multipole expansion chapter 16 and replaced it with some stuff about radiation damping. Maybe he decided the multipole was just too hard! Otherwise, the 3rd edition has the same chapters and basic contents as my 1st edition.

Chapter 2: Green's Function solutions for Metal Sphere, and other things

Images is really the Green's thing in disguise. Why charges can't leave a charged sphere. Inversion. Then orthogonal functions and separation of variables.

2.1 Method of Images. The idea here is that you consider a point charge near some metal conductor, and you know that in this case, $\phi(x) = qG(x,x')$ where x' is the location of the charge. Think of image charges as creating the F(x,x') discussed earlier. You then carefully select the image or images to make $G = \phi = 0$ on your conductor surface. If you can find a set of image charges that do the trick, then you have found A solution for ϕ in a volume bounded by a surface on which ϕ is known. Since we know that in this case (ϕ known on a closed surface, Dirichlet) the solution is unique, then the point charge + image charge configuration must be THE solution to the problem. A less fancy way to think of image charges is that you select them to make field lines perpendicular to the metal surface so there is no tangential E field. The usual situation is a point charge above an infinite plane, then the image has opposite charge and is an equal distance behind the surface. As usual, the field implied in the image problem "on the other side of the boundary" is fictitious and has no bearing on the physical problem.

2.2 Point Charge near Grounded Metal Sphere . I never realized until just now the significance of this problem and how it relates to something I recently was wondering about. Here is the logic flow. First, Jackson puts a point charge q distance y from the center of a grounded sphere of radius a, and puts it outside so $y > a$. He shows that you can get $\phi = G = 0$ on the sphere by assuming one image charge inside the sphere whose size is $-q(a/y)$ and whose distance from sphere center is a^2/y . The potential solution is then given by the first 2 terms of (2.8), and you can then go and compute whatever you want. In this solution, the grounded sphere brings in from ground a total charge of $q' = -q(a/y)$, as Gauss's law tells you. Again, we see that the image charge is simulating what the surface charge distribution really does.

2.3 Point Charge near charged, insulated metal sphere. Suppose you add to the above solution a sphere with total charge $Q1 = Q - [-q(a/y)]$. The total solution then has Q on the sphere and solution (2.8) with the third term. This is then the solution you get by putting a point charge outside a charged insulated sphere.

But why is this interesting? The potential is the sum of three terms, and if you do $\nabla\phi$, you get the electric field (F(x)) at all points in space, and it is a function of a, y, and Q. However, if you want to know the force on the point charge q, you only consider the last two terms in (2.8)and set x=y, and this then gives (2.9). The point charge feels a force from the Q at sphere center, and from the image charge which lies under the sphere surface below the point charge. The distance between the point charge and the image charge is given by $y - a^2/y$. As the point charge approaches the surface, the image charge approaches from the other side and it's force dominates. The image charge always has a sign opposite q. If Q and q have opposite signs, then q is attracted by the central Q and by the image charge. The more interesting case is when Q and q have the same sign. Then q is repelled by the central Q, but attracted by

the image charge. The two forces are equal when q lies about $(a/2)\sqrt{q/Q}$ above the sphere surface in the case Q>>q. For an electron over a macroscopic sphere of some reasonable charge, this is a very small distance. As q approaches the surface, the image charge wins out (the reforming surface charge, that is to say) and the force becomes infinite right at the surface. So here is the point: an individual electron on a charged sphere is held on by an infinite restoring force if it tries to leave. No doubt if you work with a non-idealized metal surface, you will find the force is finite, and a finite energy can get an electron over the equilibrium hump and this is called the work function. Not too long ago I made a bogus argument about why electrons did not leave a charged metal object in which I referred to the 1 eV chemical idea. I thought the electron was held by its chemical binding stability energy. That might be true, but this image charge force seems much more significant.

This whole discussion could have been tried with regard to a charge over a flat metal surface, but there it does not work so easily. You could imagine an insulated large round plate that is grounded and then you get the image charge solution assuming very large radius. But then you think of that plate as having charge Q in addition and you compute ϕ of this, things have log divergence, not very nice. You have to put in a cutoff radius R, but I think the whole thing can be made to work. For radius R of the plate, I think this is the potential the point charge sees from the image and the plate:

$$\phi(y) = -q/(2y) + (Q+q)/(\pi R^2)\,[\sqrt{R^2 + y^2} - y]$$

If I set $\phi'(y) = 0$ with large Q, I find that $y = R\sqrt[3]{\pi q/2Q}$ is the equilibrium point. Yes, it is quite ugly. The problem here is that distances do matter, you cannot just ignore far away things and say that locally the surface is flat and we will use the "flat solution". Jackson solution diverges for $a \to \infty$ as well. So there sphere is a very instructive case indeed that shows clearly the principle involved. In other cases, the same general idea will probably be true, but much harder to compute.

2.4 Point charge near a metal sphere of fixed potential. This is identical to the previous problem except you replace Q+aq/y with Va in the third term. We are just superposing another sphere on the original problem. We have now changed the y dependence of everything a little by doing this.

2.5 Metal Sphere in a uniform E field. Uses image charges at $\pm R \to \pm \infty$ to simulate the E field and finds that the sphere acts as a dipole object. So this example uses 2 image charges and no point charge to start with. Resulting σ on the sphere is a very simple $\cos\theta$ form suggesting dipole.

2.6. Method of Inversion. Equation (2.17) with (2.18) makes the basic symmetry theorem. You can take the known solution of one problem and "invert it" to apply to another problem. An example is that you can solve the sphere above a plane in this way as in Figure 2.10. There are tricky points here, I have not studied it. I think this is a special case of the more general conformal mapping ideas that Jackson gave us notes on and probably has added to his later editions? [see enhanced notes elsewhere]

2.7 Green's function for a Sphere. We outlined the idea in the last chapter: find G that vanishes on a sphere, then use that in (1.42) to find ϕ given some value of ϕ on the sphere. We already know the right G from our sphere image solution, it is (2.22) which becomes (2.23) in polar. The derivative is (2.24). Since there is no ρ in the closed region (which, by the way, is exterior here to the sphere), only one term of the three in (1.42) survive and it is shown in (2.25).

2.8 Application to two hemispheres at +V and -V. Jam everything in, get (2.27) which, I am happy to see, "cannot be integrated in closed form". He can get the result on-axis, and he can power-series expand things and he gets something that looks like multipole, but that idea has not been set up yet.

2.9. Orthogonal Functions and Expansions. Covers the basic idea of expanding f(x) in terms of some orthogonal functions $\phi_n(x)$, the orthonormality condition, the completeness condition. Fourier Series and Integral are stated. I am pretty solid on this stuff.

2.10 Separation of Variables. The problem is a cube with 5 faces at $\phi=0$ and some V(x,y) on the top surface. The solution is to use cartesians, separate variables, find some basis functions that satisfy the 5 zero conditions, then force the last one by summing with coefficients as in (2.63) with coeffs in the next equation. The comment made is that you often end up "making your own" custom set of ortho-functions for a particular class of problems.

11 problems, I have done 4 it would appear.