Jackson Chapter 3 Notes PhL 1.28.03

Chapter 3: Solving spherical and cylindrical electrostatic problems.

This chapter talks about the technology needed for working in the (r,θ,ϕ) and (ρ,z,ϕ) worlds. In each world, the radial equation implies certain special functions (Legendre and Bessel). The final section treats the flat metal charged disk as an example of Cauchy mixed boundary conditions.

3.1 Laplace Equation in (r,θ,ϕ). Assume (3.2) as separated form, find the three separated ODE equations. ϕ is easy and causes quantized integral parameter m. The r/θ separation constant is written $l(l+1)$ and it is later shown why l must be an integer. So now we have the three equations, and we have solved one of them. The radial equation is trivial and the solution is r^l or $1/r^{l+1}$ as in (3.8). So only the θ equation requires work, and this is Legendre.

3.2 Legendre Land. The θ equation is written with $x = \cos\theta$, but here only in the case m=0. Assume +x power series, get recursion for coefficients, discover that to get convergence at $x = \pm 1$ you need truncation, and that in turn makes l be integral. The solutions for given l are the famous $P_l(x)$. The second kind solutions don't fall out here because the power series assumes convergence at x=0. The $P_l(x)$ form an ortho set, all the properties are given and some are derived in detail. Page 59 shows the expansion and coefficient formulas, the expansion and projection as I like to call them.

3.3 Azimuthal Symmetry problems in (r,θ,ϕ). This makes m=0 so we then have the $P_l(x)$ and the absolutely most general solution is (3.33), I agree. This makes it easy to solve near a sphere with an arbitrary $V(\theta)$ potential on it, specialize to 3.34 and you are done as in 3.35, at least as a series. We now do the $\pm V$ hemispheres for a second time and get (3.37), a more organized answer than last time.

A nice trick is this. Suppose you know $\phi(z)$ only on the symmetry axis where expansion reduces to 3.38 since $P_l(1) = 1$, the forward direction. Then use that data to get the coefficients, and add back the P_l and you have your full answer! We found the axis solution exactly for the $\pm V$ problem, so we can do that here as a third way to solve this same problem!

Next, 1/R is expanded in $P_l(\cos\gamma)$ as shown in (3.41) and a proof is given. Recall the $r_<$ notation trick! His proof is just to look on the z axis and get a series there, then add P_l back in.

Another sample problem: find ϕ near a ring of charge, page 63. Again, we know it easily on the axis. Expand and answer is (3.46) on the axis still, then add $P_l(\cos\theta)$ and 3.48 is all yours. I liked it last time, and I still like it.

3.4 The $P^l_m(x)$ and $Y_{lm}(\theta,\phi)$ Worlds. You need this for <u>no azimuthal symmetry</u>. All just the basic nuts and bolts. Our same most general expansion is now written as (3.61). Jackson is trying to throttle the rate at which complexity builds up, not hitting the poor students with m right off the bat.

3.5. The Y_{lm} Addition Theorem. The thing is shown in (3.62). The important point is that this let's you *separate* the variables of the two vectors which are making the $\cos\gamma$ angle. When you then stick this expansion into our 1/R Legendre expansion, you get the glorious (3.70), which provides separation in all three variables, at the cost of a messy double sum. Still, it lets you do something when you might otherwise be blocked. You can bet this result will play a role in the multipole field expansion!

3.6 Cylindrical Symmetry and Bessel World. We start over, separate in z,ρ,ϕ. We get two constants as before k^2 and υ^2. The ϕ and z equations are both trivial giving expos as in 3.76 implying $\upsilon = m$ = integer. The ρ equation then ends up as Bessel's Equation where the k constant is absorbed into the argument $J_\upsilon(k\rho)$, so $x = k\rho$ in this world. We get various J properties. For *integral* υ we need the second kinders $N_\upsilon(k\rho)$ which are the Neumann's. The Hankels are also defined (third kind). Various properties of things are given, very useful and here all in one place. Now when you try to make orthonormal functions over some range 0,a for ρ, you end up using $J_\upsilon(k\rho)$ with k = zeros/a. This is needed to make parts go away in the proof or orthogonality. This is called Fourier-Bessel series, I have never used it in any of my doings. Jackson finally mentions some other versions of Bessel series with names Kapteyn, Schlomilch, and Watson's Bessel book is where to look for this stuff. We assumed non-phase expo for z, and this led to soft Bessels in the other direction. If we assume soft expo phasor form in z, then we get the hard other kind of Bessels, the I and K functions, so-called "modified Bessel functions". Hard means they blow up in some direction, soft means sine like.

3.7 Cylinder/Bessel problems. Cylinder with ϕ=0 everywhere but prescribed to a function on one end. Then an application of the Fourer-Bessel series. Skip this stuff, use when you need it! I don't think we will be doing our scattering problem in cylindricals, but if so, come back here.

3.8 Doing a general Green's analysis in r,θ,ϕ. We did the Green's deal in Dirichlet for a sphere (find G such that G=0 on a sphere) and the image charge trick made short work of it. But suppose you have some other spherically symmetrical situation, such as concentric spheres. The general form of G is (3.118) with (3.119), that is, a double harmonic expansion with some radial green's function g(r,r') that we have to figure out for our geometry. For the single sphere that function is shown in 3.114, which we can think of as the 1/R factor (in its fancy expansion) plus F as the expansion of the image charge in a similar fashion. For the concentric spheres case, the resulting g is the much messier (3.122), and the full Green's is shown in 3.125. For this case, you would need an infinite set of image charges to get the same result, Jackson claims. Remember, you are trying to make G=0 on both spheres at the same time! This section also gives some useful delta function stuff in the spherical coordinates, page 79. This is a pretty tough row to hoe! Each sample problem is a whole paper you could write.

3.9 Spherical Problems. We did Spherical setup work above, then we did cylindrical setup work, and then cylinder problems, but we never did spherical problems in the general case, so here we are now. This is getting closer to my main interest theme right now which is applying multipole to plane wave scattering, but at the moment we are back in Electrostatic Land and we are about to do something here along these lines.

Here, we are first reminded of the general Dirichlet 2-term result (3.126). The first term is the ρ term if you have any, the second term is integrating a prescribed ϕ over your surface with dG/dn as a factor. In 3.127 we compute this dG/dn for sphere(b), then 3.128 gives the non-ρ term of the 3.126. So in the following two examples, we use a grounded sphere and practice doing the ρ term. In the general case, you can superpose this solution with solution to a sphere with some general ϕ on its surface. For example, you could do a ring-charge inside the now-infamous ±V sphere.

Example 1: put a ring of charge inside a grounded metal sphere. In this case of course the surface integral is 0, and we are going to practice doing the ρG integral term. This ρ is given with delta functions which in turn kill off the dV' integration in the ρ term integral (ϕ is just 2π) giving 3.130. Note that one delta is forcing $\cos\theta' = 0$ since charge ring is in that plane, hence then original $Y_{lm}(\theta',\phi')$ within G gets

pinned to $Y_{lm}(0,-)$ and m=0 due to symmetry and that is why you see $P_l(0)$ sitting in 3.130. The other P_l is of course from the other Y factor in G. So 3.130 is the solution to our problem, a sum over ℓ.

Example 2: put a diameter line charge inside a grounded metal sphere. We have a new ρ with its deltas, and we take note of the $1/r^2$ in these forms. The result is 3.133. Since ρ in this case only has delta in θ', we still have the r'-integration to do and the final result is 3.136. Jackson notes that this thing actually diverges on the z axis (it should, that is where the charge is), BUT manages to be 0 on the sphere, meeting the ϕ=0 requirement there (grounded sphere).

3.10 Green's for (ρ,z,ϕ). Write the usual equating defining G. Expand both sides. The RHS is obvious, the LHS expansion of G shown in 3.140 is justified because we say that the cylindrical solutions had $\exp(ikx)\exp(im\phi)$, where we have now used the I, K sign of k^2. We have thus "separated variables" in G and have $g(\rho.\rho')$ to figure out. In a typical tour-de-force, Jackson shows that the solution to the radial ρ equation is $g(\rho,\rho') = 4\pi\ I_m(k\rho_<)\ K_m(k\rho_>)$. Recall that I is the one that is best-behaved at ρ=0 and no surprise that it is connected with $\rho_<$. So we can jam this into 3.140 to get the cylinder Green's G. But we know that this is 1/R as well, so we get the 3.148 expansion of 1/R. This result is pretty messy! Jackson then shows that if you compute ln(1/R) from this messy thing, you get the relatively simple expansion shown as 3.152. Very nice. No examples however.

3.11 The Eigenfunction Method. This is a quantum-mechanics Schrodinger-equation approach. Take your ODE as in 3.153 and find some eigensolutions as in 3.154 which have some desired boundary conditions, such as vanishing at the walls of a box that you want to study. The greens must at once be of the general form 3.157 expanding onto the eigenfunction basis. Goof around and end up with 3.160. One point not quite made clearly is that you might have λ=0 in 3.153 (as in the Laplace equation), but you will still of course have non-zero eigenvalues in 3.154. In particular, for that mentioned box in cartesians, your eigenfunctions are 3.166 and λ_n as in 3.165, although there is no $\lambda = k^2$ in the Laplace. That is why we see just λ_n^2 in the denominator of the final G for this situation in 3.167, call this example 1. For example 2, we take the same λ=0 Laplace equation, but instead of a box we do infinite space. In this case we get that $\lambda_n = k^2$ which is now going to be a continuous eigenvalue, and 3.160 becomes 3.164. This is an interesting way to interpret this nice form for 1/R. Example 3 is to do the thing in cylindricals, and we then get G = 1/R expressed as 3.168.

Comments on this method? The idea is to decide ahead of time on some boundary conditions, then solve for eigenfunctions by insisting on these boundary conditions for all the eigenfunctions, and by inserting some λ_n into your original equation, which really comes from separating the variables, even if λ = 0 in the full equation for G. That is, we get a separate "wave equation" for each variable-separated function. The full original equation still has λ=0. We know what each equation looks like from our earlier work, and only one of the three variables will be tough. Notice from page 48 that the sum of the effective k^2 values has to be zero, so if two of them are phasor type, the third must be hard expo type. You cannot have all three be phasor type because they would then add up to a negative value, not 0.

3.12 An Example of mixed (Cauchy) boundary conditions. The problem is a charged, insulated metal disk, sounds simple enough! We take as our closed surface an infinite hemisphere on one side of the disk. We know that ϕ = constant on the disk and ϕ=0 on the infinite half-sphere, but we don't know ϕ in the disk plane outside the disk! However, we do know $\partial\phi/\partial z$ outside the disk, the along-axis E field, and that must be zero by symmetry. So here we have a Cauchy situation that really was not mentioned in

Jackson's table. We really do have a closed surface, and we have ϕ prescribed on some of it, and ϕ' on the rest of it.

OK, next since we know there is no ϕ dependence we specialize the cylindrical expansion to get 3.170, but we don't yet know what f(k) is. We quickly get the two "integral equations" for f(k) shown in 3.173 from the mixed boundary conditions. Jackson is forced to just tell us that the solution is 3.176 without a derivation, but of course we can verify that it works. This gives 3.177 for the full solution. The integral there can be done to give the final form 3.178, a nice compact closed-form solution to this problem with no special functions required! Jackson computes σ on each side of the disk (same on both sides of course) and finds a spike at the edge as you would expect. It is as if there were some kind of centrifugal force pushing the charge out to the edge to maintain no radial E field. Jackson finally notes that the problem can be exactly solved in elliptical coordinates, something he treats does in this book.

Fourteen problems, of which I have done 5. Stupidly, I did not keep my problem sets, is that possible?