A brief review of green Jackson electrostatics. PhL 6.27.10

I already have brief summaries of each of the three chapters, but here I want a higher level view which I hope will include a list of actual problems solved as examples. These three chapters are about 100 pages.

Chapter 1 (Introduction)

talks generally about **Coulomb's Law**, then **Gauss's Law**, and we have $\nabla\bullet\mathbf{E} = 4\pi\rho$ **Poisson Equation** which in mks would be $\nabla\bullet\mathbf{E} = \rho/\varepsilon$. We have the potential φ written as an integral of ρ/R and comments about work. Single and double **layers** get mentioned, as in Stak. The equation for φ is $\nabla^2\varphi = -4\pi\rho$ with that integral just stated as the "solution". We then get **Green's identities and theorems**, and out pops this:

$$\varphi = \int \rho/R + (1/4\pi) \int_S [(\partial_n\varphi)/R - \varphi\partial_n(1/R)] = \text{basic} + \text{Neumann effect} + \text{Dirichlet effect}$$

In **Neumann and Dirichlet problems**, one of the three terms is missing. You can actually think about G_N and G_D as two kinds of Green's functions. Stak does this as well in different notation. Interpret the two terms as surface charge layer ($\partial_n\varphi$) and surface dipole layer φ. Jackson says $\varphi = 0$ outside surface S which I only understand in the case $\varphi = 0$ on the surface. Jackson lists off the various BC cases in his page 17 table. Using Green's identity #2 we then see

$$\varphi = \int \rho G + (1/4\pi) \int_S [(\partial_n\varphi)G - \varphi\, \partial_n G)]$$

where $\nabla^2 G = -4\pi\delta$, so this is the idea that **if you can figure out G**, you can solve any Dirichlet or Neumann problem by inspection. Chapter ends with energy discussion. Notice that if you have some charges inside a grounded surface S for which you have the Green's function G, only $\int \rho G$ survives. He does not mention this as $\varphi = G\rho$ being the solution of $L\varphi = \rho$ in operator sense.

So this is sort of a "theory" chapter. There are no sample problems being solved here. Rough on the poor student perhaps.

Chapter 2 (BV I).

We get the **image method** for **plane** and **sphere** Green's problems. We compute the charge on the plane, and the attractive force. Superposition for a single metal object is mentioned. A third image example is a sphere with **4 image charges** set up to emulate a sphere in a uniform E field and we learn σ induced on this sphere. (This is really a mixed BC with V = 0 on sphere and $\partial_z V = k$ on two of the great cube walls. You could solve it by Smythian form with $P_1^0(z)$ atom only, as with Smythe's hole in plate problem. But here images do the trick.)

Next comes the long **inversion method** discussion.

Jackson then returns to the sphere Green's problem and restates G in spherical coordinates (sum of two images), and computes $\partial_n G$ as well. He then uses the latter to write the φ integral for the **sphere Dirichlet problem**, which is the Poisson kernel thing in 3D that Stak did. A Dirichlet example is then the

two half-spheres at V and -V, but you can't do the integral! Notice that we have no special functions introduced yet!

So that is next: **orthogonal functions** and **expansions** treated generally, with **Fourier Series** as example, which becomes the **Fourier Transform** for infinite interval.

Next is **separation of variables in Cartesians** and we do a simple box problem with $\varphi \neq 0$ on one wall, and sinh functions appear as usual.

So this chapter does the image and inversion methods of finding potentials for situations, then does some examples with the sphere and a few other minor things, then more "theory" on orthogonals, and finally solving Laplace doing separation of variables, but only in Cartesians.

Chapter 3 (BV II).

We open with **sphericals** and separation in them which leads to the associated Legendre equation, and to the usual r^n and r^{-n-1} radial solution. Jackson starts with m = 0 and rolls out the usual Legendre (m=0) properties. Then he tries some azisym problems so can stick with m=0, and we then get our first Smythian Form. He then resolves the **±V sphere halve**s interior problem, but does not write down a general form for the A_n Smythian coefficients, just states the first few of the series. He then comes up with the 1/R expansion in $P_n(z)$ functions. Next azisym example is the potential of a **charged ring**. So these m=0 problems have solutions which are single sums (n, since m=0) of spherical atoms.

He then does the **associated Legendre properties** and **spherical harmonics** and sets his convention, and then we have the general **Smythian Form** in terms of full spherical atoms. He does not mention that this fails for an origin on a piece of metal. We then get the P function addition theorem written out several ways, and finally the spherical 1/R expansion.

We then switch to **cylindricals** and obtain the atoms for this case. This leads to Bessel function details, and he uses N for Neumann function. He does the SL problem on (0,a) and finds the orthogonal Bessel functions there (**Bessel-Fourier Series**), analogous to the Legendre orthogonality for its SL problem. But he does not talk about SL problems! The I and K functions are defined and properties given. The first **cylindrical problem** is a cylinder with top at some specified V, rest at 0, so this is **Dirichlet**. He does this by stating his Smythian form and then solving for the two coefficients. He closes this section extending the interval to get the **Hankel Transform**.

Now he jumps back to sphericals. Next, he rethinks the spherical Green's using the **reduced Green's method** (partial eigenfunction expansion) where that 1D radial Green's needs to satisfy some BC's. An example now appears, a **spherical shell between two spheres**. and he gets the **Green's function** for the shell, double sum. Next comes a **ring of charge inside a grounded sphere**. He uses $\varphi = \int \rho G$ to solve this problem, using G for the sphere. He then repeats this analysis for a **line segment of charge inside the grounded sphere**.

Next we are back to cylindricals. He develops the I-K form of the **cylindrical 1/R expansion**, by again using the reduced Green's function method, starting off with a Smythian form involving an integral over k. As I note in my 1/R doc, this leads to various integrals and sum rules which he quotes for this case.

Then we are back to general theory and the **full eigenfunction expansion method** for finding a Green's function. He first does this with Helmholtz in Cartesians with no surfaces around which gives the usual 1/R form as a sum of 3D $e^{-ik\bullet r}$ plane waves, which is just a 3D Fourier integral of 1/R. He uses the term "infinite space Green's function" where Stak would talk about a "fundamental solution". He then re-

does this inside a **cubic box** and gets the Green's function there as a triple sum. But then he goes back to Smythian forms and gets an alternate form as a double sum expansion with sin and sinh stuff.

The chapter closes with the **charged disk** discussion. This is his only mixed BC example, and we jive around with the dual integral equations and gets the fancy result.

Comments

One might summarize by saying that we know how to solve several distinct kinds of electrostatic problems: Green's Function, Dirichlet with possible charges, Neumann with possible charges, Mixed with possible charges, and charges with no BCs. All examples in these chapters fall into those boxes. We did these things in Cartesian, Spherical, and Cylindrical coordinates. There was no mention of other coordinate systems, and no mention of potential theory relevant for other applications like fluid flow or diffusion. Here then is my list of examples:

2	Green's for plane	Green's	Cartesian	images
	Green's for sphere	Green's	spherical	images
	metal sphere in E field	mixed	spherical	images
	general sphere Dirichlet	Dirichlet	spherical	
	the ±V half spheres	Dirichlet	spherical	
	box V≠0 on one wall	Dirichlet	Cartesian	
3	the ±V half spheres revisited	Dirichlet	spherical	
	charged ring	charges with no BC	spherical	
	cylinder V≠0 on one end	Dirichlet	cylindrical	
	infinite radius cylinder abstract	Hankel transform	cylindrical	
	shell between two spheres	Green's	spherical	
	charge ring inside grounded sphere	Dirichlet with charges	spherical	
	charged line inside grounded sphere	Dirichlet with charges	spherical	
	cubic box	Green's	Cartesian	

Jackson was forced to include data on the Legendre and Bessel functions. He also did lots of delta function gymnastics, but no mention of general orthogonal curvilinear coordinates and scale factors. I remember the delta stuff being painful to a beginner. Recall that Smythe refused to use deltas at all, while Jackson has them everywhere. Then we had the full Green's identities and global theorems, and finally we had lots of Smythian form examples.

His task is to teach the student who knows practically nothing! Smythe and M&F are more advanced books (electrostatic parts) but also have all the special function details. Stakgold is perhaps the most advanced mathematically, but does not stray into obscure coordinate systems.