Pondering the Green's function of ellipsoid PhL 12.3.09

Review 4.3.10. For a sphere, we know the Green's function is simply the potential of the point charge plus that of the image charge, since these make V = 0 on the sphere. We could nevertheless write an expression for the Green's function of a sphere in terms of the eigenfunctions of the sphere, using Stakgold's "full eigenfunction method", $g(\mathbf{x}|\xi) = \Sigma_{\mathbf{n}}\, u_{\mathbf{n}}(\mathbf{x})\bar{u}_{\mathbf{n}}(\xi)/\lambda_{\mathbf{n}}$ (6.108), and $u_{\mathbf{n}}(\mathbf{x})$ would involve the spherical Bessel functions and $\Sigma_{\mathbf{n}}$ would be over I think three indices, one of them labeling the zeros of the Bessel function. Somehow, in similar fashion, it must be possible to write the Green's function for an ellipsoid in this manner, ie, in terms of the eigenfunctions of the ellipsoid. The simple image method is obviously not going to work in this case. I eventually solved for the Green's function of a *oblate spheroidal* ellipsoid, but I never got it as above in terms of the eigenfunctions of a spheroid. I still have no idea what those eigenfunctions might be. In this early 2-page document, I am just pondering the entitled problem, I am in no way attempting to solve it.

This problem may be "too hard". Maybe just do it for oblates, or just on axis, or just in the disk limit where I eventually want it.

I notice that MF talk about "Green's Function for Prolate Spheroids" in their Chapter 10 which sounds maybe close enough for me. Let's go take a look. I think MF are going to have it all here.

Tour of MF Section 10.3 on "solutions for 3 dimensions" (Laplace and Poisson)

(1252) MF are a bit sad that we don't have the simple complex variable approach in 3D, and the solutions always end up in 3D being infinite sums or integrals

(1253) Of our separable systems, certain ones are extruded (which they call cylindrical in nature), while others are obtained by doing rotation of some 2D plan (which they call rotational in nature). Sphericals eg are polar 2D rotated about the z axis to add azimuth, and same idea for toroidal. They put parabolic and ellipsoidals into a less pleasant third class, one that is "less symmetric".

Second item here is this idea:

We shall cling to our useful complex variables as long as possible. For instance, any function of a complex variable X, corresponding to a plane parallel to the z axis and at an angle u with respect to the x axis,

$$X = z + i(x \cos u + y \sin u) \qquad (10.3.1)$$

is a solution of Laplace's equation, and any combination of such solutions

$$\psi(x,y,z) = \int_0^{2\pi} F(X)f(u)\,du \qquad (10.3.2)$$

is also a solution of Laplace's equation. We could, of course, try differ-

In 2D, we would just say f(X) was a Laplace solution where f is any analytic function. Here he has a strange version of this statement where we superpose analytic functions which have a parameterized imaginary part (variable u). We shall later see what this does, he says.

(1254) Integral form of Green's Function.

Conic ≡ a conic section, like a parabola, ellipse, or hyperbola

Conicoid ≡ what you get when you rotate one of the above about an aligned axis.

Appendix A. What are the eigenfunctions for an ellipsoid $\xi = \xi_1$?

In the ξ direction, these are "like" some sine functions that you make vanish on the boundary of the ellipsoid, and which you require to vanish at infinity. Our candidate functions are the $E_m^p(\xi)$ and the second kind $F_m^p(\xi)$. [see comment below] You might have to lincom these to get vanishing at ξ_1. I have not seen this issue brought up anywhere yet. We have u=0 on the metal surface, so ξ is the "radial coordinate". Then we will have angular harmonics of the same E form for the other coordinates. I do wonder how this looks.

If we worked inside the ellipsoid, we might expect something like $J_n(\beta^{(n)}_m) = 0$ so we might get involved with the zeros of the Lame functions.

Note added Jan 11, 2010 regarding the above paragraph. First think of the 3D sphere "eigenfunctions". These are not in fact the "solution terms" or "atoms" you get from separating Laplace in sphericals. Those terms have the form $r^{n,-n-1}\, P_n^m(z)\, e^{im\varphi}$. When you talk "eigenfunctions of a sphere", you add in another quantum number which is the actual eigenvalue λ, as in $\nabla^2\psi = \lambda\psi$, as opposed to Laplace $\nabla^2\varphi = 0$. This extra eigenvalue is like the k^2 term that turns Laplace into Helmholtz. When you do this with the sphere, you find that the radial eigenfunctions are $\sim j_n(kr)$ where $k(\lambda)$ are the zeros of j_n (unit sphere). Obviously this j_n is not r^n or r^{-n-1} which is what you get when $\lambda = 0$.

Similarly, for the ellipsoid we have "solution terms" which are like $E_m^p(\xi_1)\, E_m^p(\xi_2)\, E_m^p(\xi_3)$ (we could throw in the second kind F term in each factor). The factor $E_m^p(\xi_1)$ here is like $r^{n,-n-1}$. If you wanted the eigenfunctions for the ellipsoid, you would add that Helmholtz constant, and then the ODE would be even more complicated than the Lamé equation and the solution would doubtless be something more complicated than $E_m^p(\xi_1)$. This new radial function would be to $E_m^p(\xi_1)$ as $j_n(kr)$ is to r^n . As of this date, I have not seen this problem addressed. MF talk about the Laplace equation only, not the Helmholtz or the eigenfunctions of an ellipsoid. Even Smythe stays away from this problem. And I see nothing on the web in a quick look. You might first try to find the eigenfunctions of, say, an oblate spheroid. There the radial functions are $Q_n(j\zeta)$ which would then be like the r^{-n-1} for spherical or the $F_m^p(\xi_1)$ for ellipsoidal. This would be "bad enough" I think.