Green's Functions using the Smythe Method META PhL 6.24.10

This HUGE 52 page doc is a grab-bag of many important items. It was written while I was doing my main Smythe readings circa Jan 3, 2010. My meta notes here are 12 pages, so 4:1 compression. While writing here, I added some stuff to the raw doc.

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0. Overview Summary (1 page, updated 12.4.10)

I was reading Smythe's 3D potential theory stuff and took notes here. In doing so, I learned very important ideas, one being that of using a **Smythian Form** for Green's, Neumann and Dirichlet problems, and the other being the neat little **pillbox condition** used when computing Green's Functions.

In **Section 1** I compare the pillbox condition in 3D to the string jump condition in 1D.

Section 2 contains notes on Smythe's first example: doing the **Green's Function for a cone**, which presents some novel features in terms of the θ SL problem. We work here in sphericals.

In **Section 2a** we add a ceiling and floor to this cone and get the **Green's Function for a conical box**.

In **Section 3** I (solo) compute the **Green's function for a sphere** using this method, and I show (using two 1/R expansions) that my result matches the usual image method result.

In **Section 4** I comment that **Stakgold** used, but did not really emphasize, the Smythian Form Method for computing Green's Functions. His famous list does not include it.

In **Section 5** I comment that his method can be applied in any coordinate system. Since Smythe is next going to do some examples in **oblate spheroidal coordinates**, I go off in **Section 6** and review these coordinates a bit.

In **Section 7** Smythe does the **Neumann problem for an oblate spheroid**. Soup to nuts. A special case of this is the expansion of 1/R in oblate coordinates which I do in Appendix D.

In **Section 8** I (solo) do the **Dirichlet problem for an oblate spheroid**. A special case is V = constant on a metal spheroid, and in **Section 9** I present all the **metal spheroid** results of this great case including capacitance. I then examine a special case of this special case which is the **metal disk**, and I state all those results.

The last four sections are all "comments":

In **Section 10** I just comment on how you might do a Smythian Form in **ellipsoidal coordinates**.

Section 11 is a somewhat obscure PhL comment about how dV does not factor in general, so 3D orthogonality is in general not the simple product of three 1D **orthogonalities**.

Section 12 comments on the fact that there must be some **group theory** method to understand the analog of $\nabla^2 = \nabla_r^2 + (1/r^2) L^2_{\theta,\varphi}$ in oblate coordinates. I have a pdf on this subject in fact.

Section 13 wonders what the **analog** for $J_{n+1/2}(\beta^{(n+1/2)}_k r)$ might be in the 3D eigenfunctions of an oblate spheroid, and notes how we don't need to know the answer to do certain oblate problems. [I added a note 12.4.10 since Moon and Spencer answer this question. Legendre Wave Functions are involved in two of the three coordinates, see below.]

The first three **Appendices A,B,C** are details of doing orthogonality in oblate coordinates and using the special way Smythe presents his φ dependence in his forms, namely, as $\cos[n(\varphi-\varphi_m)]$.

Appendix D treats the special case of Section 7 above to get a 1/R expansion in oblates.

Appendix E shows that $Q_n^m(jx)$ $P_n^m(jy)$ is imaginary for integer m and n (and real x and y). This then relieves the observer of the 1/R expansion which has an overall factor of j.

1. Introduction.

Here I show that the 1D string Green's "pillbox condition" can be written $\partial_x V(\xi+\varepsilon) - \partial_x V(\xi-\varepsilon) = -1$, and that we will generalizing this to 3D in this doc. A typical form is $\partial_r V_i - \partial_r V_e = (1/\varepsilon a^2)\,\delta(z-z_\beta)\delta(\varphi-\varphi_\beta)$, but the exact form depends on your coordinate system and coordinate. (It is just Gauss's Law)

2. Smythe Green's Function for a Cone.

(a) Setting things up to match the BC's

Here I accept his Smythian Form (see below), and explain (in an aside) why negative m values are not needed. The n quantization condition for the cone is $P_n^m(\cos\alpha) = 0$ and for m = 3 and $\cos\alpha = 0.7$ I plot $P_n^m(\cos\alpha) = f(n)$ versus n and show that indeed, there are an infinite number of zeros. I then quote Bateman's large-n formula for P and write a large-n formula for these zeros which is this (step N_{odd} through large odd integers and solve for n. Of course the n values won't be integers.)

$$[\,(n+1/2)\theta - \pi/4 + m\pi/2\,] = N_{odd}\,(\pi/2)$$

(b) Understanding things as distributions.

This is my first crude attempt to derive "the pillbox condition" noted above. It is done much better in a later document (see XXXXX) but the result I get here is correct and I already quoted it in the intro above. Smythe avoided using δ functions and has lots of words about regions being small.

(c) The eigenfunctions of the θ ODE BV problem on (0,α) are complete and orthogonal.

In order to pick off our Smythian form coefficients, we need P orthogonality for our strange values of n, and this is of course part of the SL problem in θ. I show that only the positive part of the n spectrum should be kept. The orthogonality and completeness is this: ($z = \cos\theta$)

$$\int_0^\alpha d\theta\, \sin\theta\, P_n^m(\cos\theta)\, P_{n'}^m(\cos\theta) = K_n^m(\alpha)\, \delta_{n,n'} \qquad \text{// orthogonality}$$

$$\text{where } K_n^m(\alpha) = -\sin^2\alpha/(2n+1) \;*\; \partial_z P_n^m(\cos\alpha)\, \partial_n P_n^m(\cos\alpha)$$

$$\Sigma_n\, P_n^m(z')\, P_n^m(z)/\, K_n^m(\alpha) = \delta(z'-z) \qquad \text{// completeness}$$

I comment that every self-adjoint L and interval (a,b) will have an orthog set of eigenfunctions, and that you won't find all possible such functions in any book, and I was unable to find the above anywhere else.

(d) Computation of the A_{mn} coefficients.

Here I use orthogonality in φ and θ to "pick off" the coefficients in the Smythian Form. Here is that Form and those coefficients. The Green's point charge is located at (a,β,φ=0), cone angle is α : (β < α)

$$r<a \qquad V_i = \Sigma_{m=0}^{\infty} \Sigma_n\, A_{mn}\, (r/a)^n \quad P_n^m(z)\cos(m\varphi)$$

$$r<a \qquad V_e = \Sigma_{m=0}^{\infty} \Sigma_n\, A_{mn}\, (r/a)^{-n-1}\, P_n^m(z)\cos(m\varphi)$$

$$A_{mn} = -(2-\delta_{m,0})\,(2\pi\varepsilon a)^{-1}\,P_n^m(\cos\beta)\,\{\sin^2\alpha\;\partial_z P_n^m(\cos\alpha)\,\partial_n P_n^m(\cos\alpha)\}^{-1}$$

This was my first real exposure to using the Smythian Form Method in 3D.

(e) Comments on Smythe's Method.

I summarize his excellent method. We are now at page 11 in the raw doc.

2a. Adding two walls to make a conical box.

This is another Smythe example which I went through in full detail. You have a Form, there are 5 coefficients, and you find them. All we are doing is adding a spherically shaped ceiling (r = d) and floor (r=c) to our conical region, and we put our same Green's charge (a,β,0) inside this conical box. Here is the Smythian form:

r<a	$V_i = \Sigma_{m=0}^{\infty}\,\Sigma_n\,A_{mn}\,[\,C_n(r/a)^n + D_n\,(r/a)^{-n-1}\,]\,P_n^m(z)\,\cos(m\varphi)$	c
r>a	$V_e = \Sigma_{m=0}^{\infty}\,\Sigma_n\,A_{mn}\,[\,C'_n(r/a)^n + D'_n\,(r/a)^{-n-1}]\,P_n^m(z)\,\cos(m\varphi)$	d

We have 3 conditions on our 5 coefficients,

$C_n(c/a)^n + D_n\,(c/a)^{-n-1} = 0$	V= 0 on floor
$C_n'(d/a)^n + D_n'\,(d/a)^{-n-1} = 0$	V = 0 on ceiling
$C_n + D_n = C_n' + D_n'$	V continuous at r = a

This is 3 conditions on 4 unknowns. Following the lead of Smythe, we make an arbitrary scale choice and obtain these solutions for the 4 unknowns. (ie, these solve the 3 equations above)

$$C_n = (a^{2n+1} - d^{2n+1}) / (c^{2n+1} - d^{2n+1})$$
$$C'_n = (a^{2n+1} - c^{2n+1}) / (c^{2n+1} - d^{2n+1})$$
$$D_n = (-c^{2n+1})\,(1 - (d/a)^{2n+1}) / (c^{2n+1} - d^{2n+1})$$
$$D'_n = (-d^{2n+1})\,(1 - (c/a)^{2n+1}) / (c^{2n+1} - d^{2n+1})$$

The A_{mn} coefficient still remains. I solve for this using the usual pillbox method and find this result

$$A_{mn} = -(2-\delta_{m,0})\,(2\pi\varepsilon a)^{-1}\,P_n^m(\cos\beta)/\;\{\sin^2\alpha\;\partial_z P_n^m(\cos\alpha)\,\partial_n P_n^m(\cos\alpha)\}^{-1} \;\; * (2n+1)/\,[n+(n+1)\,E_n]$$

where - $E_n = D_n - D_n' = (\;(c/a)^{2n+1} - (d/a)^{2n+1}\;) / \;(1 - (d/c)^{2n+1})$

I then comment briefly on how things would change if you added side walls so the box becomes shaped a little like a central chunk of an orange slice.

3. Green's Function for a Sphere using this Smythe Method.

I am sailing on my own in this section, just trying out my new Smythian toy. I basically take the limit of the previous conical box problem where $c \to 0$ (no floor) and $\alpha \to \pi$ (cone is entire sphere), to get a sphere or radius d. This might leave a tiny radial fiber of V = 0 at the south pole, but I don't think so, maybe in 3D that thread does not have enough "measure". Using the same Form as above, I have the same C and D equations now setting c = 0. But then to get A_{mn} I use the conventional P orthogonality, and my final result is this: (Green's charge (a,β,φ=0))

$$r<a \qquad V_i = \Sigma_{m=0}^{\infty} \Sigma_{n=m}^{\infty} A_{mn} [1 - (a/d)^{2n+1}] (r/a)^n P_n^m(z) \cos(m\varphi)$$

$$r>a \qquad V_e = \Sigma_{m=0}^{\infty} \Sigma_{n=m}^{\infty} A_{mn} [- (a/d)^{2n+1} (r/a)^n + (r/a)^{-n-1}] P_n^m(z) \cos(m\varphi)$$

$$A_{mn} = (2-\delta_{m,0}) /[4\pi \varepsilon a] P_n^m(\cos\beta) (n-m)! / (n+m)! = \varepsilon_m/[4\pi \varepsilon a] P_n^m(\cos\beta) f(n,-m)$$

I show in the raw notes (in a note added today) that this result does in fact agree exactly with $V_{green's} + V_{image}$, and to show this I got to use my recently derived 1/R expansions in spherical coordinates.

4. How does Smythe compare to Stak's list of ways to compute Green's Functions?

Here I give Stak's list of ways and that list does not include what I called way #6 which is the Smythian Form Method. This method is of course implied by Stak, he just doesn't explicitly put it in his list. He uses the Smythian Form method in very many examples! I then have a little comment that the spherical Green's function shown above can be written as a *triple* sum using eigenfunctions of the sphere. I think the exact triple sum formula is this

$$V(r,\theta,\varphi| a,\beta,0) = (1/\sqrt{ar}\,) \Sigma_{\ell m} \Sigma_k \, j_\ell(k_{\ell m} r) Y_{\ell m}(\theta,\varphi) \, j_\ell(k_{\ell m} a) Y_{\ell m}(\beta,0)/\lambda_{\ell mk} \qquad \lambda_{\ell mk} = k_{\ell m}^2$$

This is an example of a Green's function *full* eigenfunction expansion, $g(\mathbf{x}|\xi) = \Sigma_n \varphi_{\lambda n}(\mathbf{x}) \bar{\varphi}_{\lambda n}(\xi) / \lambda_n$.

5. Smythe Method for other orthogonal coordinate systems.

[verbatim] I can see it will work just fine. They key idea will be to "shape" the point charge in a manner which matches the coordinate system, and then come up with some appropriate form for the jump condition which will have delta functions. Something will then replace $\partial_r V_i - \partial_r V_e = (1/\varepsilon a^2) \delta(z-z_\beta)\delta(\varphi-\varphi_\beta)$. Then if we can find orthogonal functions, we can carry things through.

6. Review of Smythe Oblate Spheroidal Coordinates Work (Previously Read)

In (a) I review the form of the "oblate spheroidal atoms" in ξ,ζ,φ coordinates. In (b) I note that I used such an atomic Smythe form to solve for hole in plate with distant E field (I did this not the way Smythe did), and I quote the result. I then note that I went on to solve the charged iris problem in a similar fashion, and I show plots of both. I think I added this section just to get back "into" the oblate coordinates, because Smythe proceeds in them in his text.

7. Smythe Oblate Spheroidal Neumann Problem [165]

We are given some $\sigma(\xi,\zeta_o,\varphi)$ on a spheroid ζ_o and we are supposed to find the potential inside and outside. This is "the Neumann problem", it is NOT a Green's Function problem. Doing this problem consumes about 11 pages of the raw notes!

(a) Setting up the problem.

Using Smythe only for verification, I write my own Smythian form for this problem:

$$V_i(\zeta,\xi,\varphi) = (j/\varepsilon)\,\Sigma_{n,m}\,(-1)^m\,C_{n,m}\,f(n,-m)\,(1+\zeta_0^2)\,Q_n^m(j\zeta_0)\,P_n^m(j\zeta)\,P_n^m(\xi)\cos(m[\varphi-\varphi_m])$$
$$V_o(\zeta,\xi,\varphi) = (j/\varepsilon)\,\Sigma_{n,m}\,(-1)^m\,C_{n,m}\,f(n,-m)\,(1+\zeta_0^2)\,P_n^m(j\zeta_0)\,Q_n^m(j\zeta)\,P_n^m(\xi)\cos(m[\varphi-\varphi_m])$$

and I note that

$$= -\partial_n V_o + \partial_n V_i = E_o - E_i = \sigma(\zeta_0,\xi,\varphi)/\varepsilon \qquad \varepsilon = 4\pi \text{ for Jackson} \qquad n = \text{normal}$$

So this is going to replace the "pillbox condition" we have in a Green's Function problem. Instead of having a point charge somewhere, we have charge on a whole surface. Smythe has allowed for an arbitrary mix of sin and cos φ functions in each m wave, this is the purpose of the constants φ_m. We don't know in this problem that things are even in φ.

(b) Review of the scale factors for oblates. (comparing Smythe and MF)

The main point result of interest here is

$$dn = ds = h_2^{SMY}\,d\zeta = h_2 d\zeta \qquad \text{so} \qquad \partial_n = 1/h_2\,\partial_\zeta \qquad h_2 = c_1\sqrt{\xi^2+\zeta^2}\,/\sqrt{1+\zeta^2}$$

where c_1 is the focal distance of our spheroid, so we know now how to do $\partial_n V_o$ etc above. I also compare the Smythe and MF notations, $(\eta,\xi)_{MF} = (\xi,\zeta)_{Smythe}$, so not very pleasant! In any event, we now have

$$\partial_\zeta V_i - \partial_\zeta V_o = h_2\sigma(\zeta_0,\xi,\varphi)/\varepsilon \qquad (*)$$

so our task is to compute the LHS.

(c) Condition at the surface where the charge is

Here I compute the LHS OF (*) and find that a Wronskian appears. Perhaps this is the first place I ever saw this happen, but now I know it happens all the time. We find that

$$\partial_\zeta V_i - \partial_\zeta V_o = (j/\varepsilon)\,\Sigma_{n,m}\,(-1)^m\,C_{n,m}\,f(n,-m)(1+\zeta_0^2)\,P_n^m(\xi)\cos(m[\varphi-\varphi_m])\,(-j)\,W[P_n^m(j\zeta_0), Q_n^m(j\zeta_0)]$$

(d) Digression on the Wronskian.

We are talking off-the-cut P and Q functions here, and I convert Bateman's Wronskian to a simpler form which is this:

$$W[P_n^m(z), Q_n^m(z)] = (-1)^m f(n,m) /(1-z^2)$$

In this digression I notice that MF do a very bad job on the P and Q function stuff because they are enthralled with their T functions which no one else uses.

(e) Resuming now that we have the right Wronskian.

I install the Wronskian into the $\partial_\zeta V_i - \partial_\zeta V_o$ RHS and things then simplify a lot and we end up with

$$h_2(\zeta_o,\xi)\, \sigma(\zeta_0,\xi,\varphi)/\varepsilon \;=\; (1/\varepsilon)\, \Sigma_{n,m}\, C_{n,m}\;\; P_n^m(\xi)\, \cos(m[\varphi-\varphi_m])$$

We can then get the $C_{n,m}$ coefficients using double orthogonality. I leave the details to Appendix C and just quote the result which is this

$$C_{n,m} = (2n+1)\, (2-\delta_{m,0})\, /4\pi * f(n,-m) * \int d\xi\;\; P_n^m(\xi)\;\; \int d\varphi\, \cos(m[\varphi-\varphi_m])\, h_2(\zeta_o,\xi)\, \sigma(\zeta_0,\xi,\varphi)$$

and after a small fiddle $(1+\zeta^2)\, c_1\, h_2 \;=\; h_1 h_3$ I show that this agrees with Smythe's result. To have a complete solution, you must know the φ_m constants, and in Appendix A I show that you find these by doing sin(mφ) and cos(mφ) integrals of $\sigma(\zeta_0,\xi,\varphi)$, you "scan" the function to see, a separate little problem. In particular, if σ is even, you find that $\varphi_m = 0$ for all m.

(f) Comments on the mysterious σ_n

Smythe uses weird notation which is why I rolled my own solution. One could say $\sigma(\zeta_o,\xi,\varphi) \;=\; \Sigma_{n,m}\, \sigma_{nm}(\xi,\varphi)$, but Smythe refers to $\sigma_{nm}(\xi,\varphi)$ as just $\sigma_n(\xi,\varphi)$ which is very confusing.

(g) The M and N constants

Smythe wants to write the solution to this problem in this way

$$\zeta < \zeta_0 \qquad V_i = \sum_{n=0}^{\infty} \sum_{m=0}^{n} M_{mn} P_n^m(j\zeta) P_n^m(\xi)\, \cos\, m(\phi - \phi_m) \qquad (9)$$

$$\zeta > \zeta_0 \qquad V_o = \sum_{n=0}^{\infty} \sum_{m=0}^{n} N_{mn} Q_n^m(j\zeta) P_n^m(\xi)\, \cos\, m(\phi - \phi_m) \qquad (10)$$

That is fine, and we find that

$$M_{mn} \;=\; (j/\varepsilon)\, (-1)^m\, C_{n,m}\;\; f(n,-m)\;\; (1+\zeta_0^2)\;\; Q_n^m(j\zeta_0) \qquad\qquad M_{mn}/\, N_{mn} \;=\; Q_n^m(j\zeta_0)/\, P_n^m(j\zeta_0)$$

$$N_{mn} \;=\; (j/\varepsilon)\, (-1)^m\, C_{n,m}\;\; f(n,-m)\;\; (1+\zeta_0^2)\;\; P_n^m(j\zeta_0)$$

and my stuff agrees with his. So there is the solution to this pretty tough problem.

(h) Shape of the spheroid.

The main point here is that we can relate the oblate coordinates ξ,ζ to cylindrical coordinates this way:

$z = c_1 \zeta \xi$
$\rho^2 = c_1^2 (1+\zeta^2)(1-\xi^2)$

but you have to be careful how you define ranges of parameters, it is done various ways.

(i) Why do you only use $P_n^m(j\zeta)$ inside the spheroid? Why not $Q_n^m(j\zeta)$ as well?

The answer is NOT that Q blows up at the origin, but rather that the Smythian form forces this to be the case if you want continuity on the spheroid boundary. On the outside you DO need Q and not P, so the matching condition forces your hand for the inside.

8. Smythe Oblate Spheroidal Dirichlet Problem

Since we just did the Neumann problem for a spheroid, it seems a good thing to attempt the Dirichlet problem for a spheroid. This is done in fashion very similar to the above, but things are simpler because we don't have any $\partial_\zeta V_i - \partial_\zeta V_o$ type derivatives to worry about. We just set in a Smythian form and then make it match the boundary prescribed by f(ξ,φ) and use Appendix C again to invert. The form is this

$$V_i(\zeta,\xi,\varphi) = (j/\varepsilon) \Sigma_{n,m} (-1)^m D_{n,m} f(n,-m) (1+\zeta_0^2) Q_n^m(j\zeta_0) P_n^m(j\zeta) P_n^m(\xi) \cos(m[\varphi-\varphi_m])$$
$$V_o(\zeta,\xi,\varphi) = (j/\varepsilon) \Sigma_{n,m} (-1)^m D_{n,m} f(n,-m) (1+\zeta_0^2) P_n^m(j\zeta_0) Q_n^m(j\zeta) P_n^m(\xi) \cos(m[\varphi-\varphi_m])$$

and the coefficients are these

$$D_{n,m} = (\varepsilon/j)(2-\delta_{m,0}) (1+\zeta_0^2)^{-1} [P_n^m(j\zeta_0) Q_n^m(j\zeta_o)]^{-1} (2n+1)/4\pi * \int d\xi \int d\varphi\, f(\xi,\varphi) P_n^m(\xi) \cos(m[\varphi-\varphi_m])$$

with the same rule for finding the φ_m as noted earlier. I point out that the result is independent of c_1.

9. Application: Potential of conducting spheroid of potential V_0.

So this is a big payoff. We set $f(\xi,\varphi) = V_0$ in the above to get the potential of a metal spheroid, both inside and out. We find quickly that

$$D_{n,m} = \delta_{n,0}\ \delta_{m,0}\ V_0\ (\varepsilon/j)(1+\zeta_0^2)^{-1}/Q_0(j\zeta_o)$$

and we get then these results:

$$V_i(\zeta,\xi,\varphi) = V_0\ Q_0(j\zeta_0) P_0(j\zeta) P_0(\xi)/ Q_0(j\zeta_o) = V_0$$
$$V_o(\zeta,\xi,\varphi) = V_0\ P_0(j\zeta_0) Q_0(j\zeta) P_0(\xi)/ Q_0(j\zeta_o) = V_0 Q_0(j\zeta)/Q_0(j\zeta_o)$$

The inside solution is obvious and checks our result. The outside solution is our very famous result which later let's us solve the charged metal disk. Since $Q_0(j\zeta) = -j \cot^{-1}(\zeta)$, we have this nice result:

$$V_{\text{metal spheroid}}(\zeta,\xi,\varphi) = V_0\ \cot^{-1}(\zeta)/ \cot^{-1}(\zeta_0)$$

I then express this in cylindrical coordinates (z = symmetry axis) in this way

$$\zeta(\rho,z) = \sqrt{r^2 - c_1^2 + \sqrt{(r^2 - c_1^2)^2 + 4z^2}} / (\sqrt{2}\, c_1) \qquad r^2 = \rho^2 + z^2$$

and of course ζ_0 is the label for our spheroid.

(a) Capacitance.

Since $\zeta \to r/c_1$ when large (just stare at the above), and since $\cot^{-1}(x) = 1/x$ for large x, we find $V \to V_0(c_1/r)/ \cot^{-1}(\zeta_0) = Q/r$, so $C = Q/V_0$ and we get

$$C = V_0(c_1)/ \cot^{-1}(\zeta_0) * 1/V_0 = c_1/ \cot^{-1}(\zeta_0) \qquad \text{// Jackson notation}$$

Again, for large ζ_0 we have $\cot^{-1}(\zeta_0) \to 1/\zeta_0 = c_1/r_0$ so $C \to r_0$, the known result for a sphere.

(b) Other forms of the solution

If A_0 is the semi-major axis of the ellipsoid ζ_0 and A for ζ, then an alternate form is

$$V_{\text{metal spheroid}}(A) = V_0 \sin^{-1}(c_1/A) / \sin^{-1}(c_1/A_0)$$

(c) Circular plate limit.

This means $\zeta_0 = 0$ and we know $\cot^{-1}(0) = \pi/2$. We then get (Jackson notation)

$$V_{\text{metal spheroid}}(\zeta,\xi,\varphi) = V_0 (2/\pi) \cot^{-1}(\zeta) \qquad C = (2/\pi)c_1$$

So as you vary ζ_0, the shape of the spheroid varies from disk to sphere, and C from $(2/\pi)c_1$ to c_1.

[verbatim] I see that I solved these last two problems in " 2 Oblate Spheroidal Coordinates and the Metal Disk Problem.doc". I went on there to compute the electric field and surface charge density and to get the latter into the nice "Kelvin form" . I won't repeat all that here. Notice that $(\zeta,\xi)_{\text{Smythe}} = (\xi,\eta)_{\text{MF}}$ which is an unfortunate fact -- ξ having reverse meaning in the two systems! In each case, the first is "radial".

10. Application Comparison: Potential of conducting *ellipsoid* of potential V_0.

I show here that if we take our general potential result (which I derive elsewhere) for a metal ellipsoid and take the oblate limit, that potential becomes the oblate form shown in (b) above. I then comment on how you might apply the entire Smythian form method to problems in general ellipsoidal coordinates. Byerly provides some interesting hints as to what the double orthogonality statement might be in that case, but I did not pursue it. Remember that it's all Lamé functions in this case and a Smythian form would look like this:

$\Sigma_{m,p}[\ A_m^p\ E_m^p(\xi_1)\ +\ B_m^p\ F_m^p(\xi_1)]\ [\ A'^p_m\ E_m^p(\xi_2)\ +\ B'^p_m\ F_m^p(\xi_2)]\ [\ A''^p_m\ E_m^p(\xi_3)\ +\ B''^p_m\ F_m^p(\xi_3)]$

11. Comment on the non-factorization of the 3D volume element, and on orthogonality.

Best to consider at once an example. For oblates we know that

$$\text{"dx"} = dV = h_1h_2h_3\ d\xi\ d\zeta\ d\varphi = c_1^3\ (\xi^2 + \zeta^2)\ d\xi\ d\zeta\ d\varphi = dx\ dy\ dz$$

and this volume element does not "factor" in the sense that $dV = f(\xi)g(\zeta)h(\varphi)\ d\xi\ d\zeta\ d\varphi$, although we do get such factorization in spherical and cylindrical coordinates. I claim that, although Stakgold does not say much about it explicitly, we know that the eigenfunctions of some PDE self-adjoint operator L will be orthogonal, so in some general system that would be stated this way, where u are eigenfunctions:

$$\int h_1h_2h_3\ d\zeta\ d\xi\ d\varphi\ u_{m,n,k}(\zeta, \xi,\varphi)\ u_{m',n',k'}(\zeta, \xi,\varphi) = \delta_{m,m'}\ \delta_{n,n'}\ \delta_{k,k'}$$

My rather vague "comment" being made in this section is this: (1) we could imagine some kind of SL problem in each of the three coordinates, with implied eigenfunctions and orthogonality in each variable; (2) we could imagine our 3D eigenfunctions shown above factor so that

$$u_{m,n,k}(\zeta, \xi,\varphi) = a_{m,n,k}(\zeta)b_{m,n,k}(\xi)c_{m,n,k}(\varphi)$$

where a, b and c are eigenfunctions found in item (1); (3) Even in this case, because the volume element does not "factor", we cannot say that the 3D orthogonality is the simple *product* of the three 1D orthogonalities.

Since writing the above, I have become aware that what we always do is select two of the three variables to be oscillatory, and we carry out SL analyses in each of these variables, and treat the third variable as a constant, such as the ζ_0 in our spheroid problem above. In this case, the volume element does in fact "factor". For example, for oblates in our spheroid problem

$$dV = h_1h_2h_3\ d\xi\ d\zeta\ d\varphi = c_1^3\ (\xi^2 + \zeta_0^2)\ d\xi\ d\zeta\ d\varphi = f(\xi)\ d\xi\ d\zeta\ d\varphi$$

12. Comment on oblate "spheroidal harmonics"

My comment has several parts.

(1) For sphericals, we write $\nabla_r^2 + (1/r^2)\ L^2_{\theta,\varphi}$ and we have L^2 and L_3 providing our "spherical harmonics" in a 2D space of θ,φ which are orthogonal and complete, $P_n^m(z)e^{im\varphi}$ for fixed $r = r_o$.

(2) For oblates, we have a known set of "oblate harmonics" which are also orthogonal and complete, these would be $P_n^m(\xi)e^{im\varphi}$ for example, for fixed $\zeta = \zeta_o$. So some sort of L^2 and L_3 here too.

(3) For oblates, we can say

$$\nabla^2 = \nabla_\zeta^2 + \nabla_\xi^2 + f(\zeta,\xi)\ \nabla_\varphi^2 = \nabla_\zeta^2 + \nabla_\xi^2 + [\ 1/(1-\xi^2) - 1/(1+\zeta^2)]\ \nabla_\varphi^2$$

but we can't really write this as $\nabla_{\zeta}^2 + f(\zeta) L^2_{\xi,\varphi}$. So things are now really the same.

(4) my group theory paper shows how this lack of sameness is explained in terms of two commuting operators like L^2 and L_3 for sphericals, but for oblates it is some Q^2 and L_3 instead. I will pursue this sometime, but not right now.

13. Comment on Full 3D Fourier Expansion and Recovery in Curvilinear Coordinates

This is another obscure PhL comment. Consider the eigenfunctions of a sphere which have this form

$$u_{m,n,k}(r,\theta,\varphi) = (1/r)\, J_{n+1/2}(\beta^{(n+1/2)}_k\, r)\; Y_{n,m}(\theta,\varphi) \qquad \lambda_{k,n,m} = [\beta^{(n+1/2)}_k]^2$$

We could use these to do a 3D Fourier transform (think Fourier transform on a group)

$$V_{mnk} = \int dx\, f(r,\theta,\varphi)\, u_{m,n,k}(r,\theta,\varphi) \qquad \text{Fourier projection}$$

$$V(r,\theta,\varphi) = \Sigma_{m,n,k}\; f_{mnk}\; u_{m,n,k}(r,\theta,\varphi) \qquad \text{Fourier expansion}$$

The question is: what happens in oblate coordinates? There must be some function *like* the $J_{n+1/2}$ shown above which appears in the $u_{m,n,k}(\zeta,\xi,\varphi)$ eigenfunctions of an oblate spheroid. I don't know what these functions are and have never seen them anywhere as far as I know. But for problems where the surface of interest is a spheroid at constant ζ_o, we never had to know what these functions are because we always worked with orthogonality involving $Y_{n,m}(\xi,\varphi)$. Again, there must be some group interpretation of the functions $J_{n+1/2}$ and the analogs for the oblate case. In my thesis, I often talked about certain P and Q functions as being the eigenfunctions of certain group operators in certain coordinates, but that point of view does not appear in Stakgold or any other general source I have. Might be worth pursuing at some point.

Note added 12.4.10. What I show above with the $J_{n+1/2}$ are the eigenfunctions of a metal unit sphere which in fact are the solutions of the Helmholtz k^2 equation in spherical coordinates and k gets quantized to $k_j = \beta^{(n+1/2)}_j$. Moon and Spencer show these solutions in mid page 27, including the $Y_{n,m}$. If we then look at the M&S discussion of oblate coordinates, we find the solution on page 33 of M&S, and it is pretty complicated. I will write it down here using the M&S coordinates η,θ,ψ (last is azimuth).

$$u_{m,n,j}(\eta,\theta,\psi) = P_n^m(ik_j a, i\sinh\eta) * P_n^m(ika, \cos\theta) * \sin/\cos(m\psi) \qquad // \; \sh\eta = \xi \;\; \cos\theta = \eta$$

where a is the usual oblate coordinates definition constant. Here there are eigenvalues of k called k_j enumerated by j=1,2,3.. which cause $P_n^m(ik_j a, i\sinh\eta_1) = 0$ which of course is specific to our η_1 choice. My main point here is that finally I know *what* those more general functions are! They are the "**Legendre wave functions**" and they appear in two places! That is, they don't just appear in the "radial" coordinate's function. There is very little information available on these wave functions!

Appendix A. Azimuthal Orthogonality (integral is 0 to 2π)

In our work above, instead of using $A_m\sin(m\varphi) + B_m\cos(m\varphi)$ Smythe used $f_m \cos(\varphi-\varphi_m)$, fine. I first show in this appendix the following fact: (I did not know the ε_m Neumann symbol at time of writing)

$$\int d\varphi \cos[m(\varphi-\varphi_m)] \cos[m'(\varphi-\varphi_{m'})] = \delta_{m,m'}\pi (1+\delta_{m,0}) = \delta_{m,m'}\pi \varepsilon_m$$

which is a little impressive since the phases φ_m and $\varphi_{m'}$ can be completely arbitrary. This is the φ orthogonality that is used in the various problems attacked in this doc. I show that the corresponding expansion theorem is this:

$$f(\varphi) = \Sigma_m f_m \cos[m(\varphi-\varphi_m)] \qquad \text{// expansion}$$

$$a_m = (1+\delta_{m0}) (1/\pi) \int d\varphi \cos(m\varphi) f(\varphi) \qquad \text{// projections}$$

$$b_m = (1/\pi) \int d\varphi \sin(m\varphi) f(\varphi)$$

$$\tan(m\varphi_m) = -b_m/a_m \qquad \text{// how to get } f_m \text{ and } \varphi_m \text{ from } a_m \text{ and } b_m$$

$$f_m = a_m/\cos(m\varphi_m) = -b_m/\sin(m\varphi_m)$$

If you have some $f(\varphi)$ that is getting integrated, like $\sigma(\zeta,\xi,\varphi)$, to get some coefficient like $C_{n,m}$ above in the Neumann problem for a spheroid, the above shows at once how to compute the φ_m.

Appendix B. P Function Orthogonality (integral is -1 to 1)

This is just a derivation of a famous fact, and I think I have better derivations elsewhere. This is in the world where n,m take their usual integral values, which means problems with full θ or ξ sweep.

$$\int_{-1}^{1} dz\, P_n^m(z) P_k^m(z) = 2\delta_{n,k} (2n+1)^{-1} f(n,m) \qquad n,k = |m|, |m|+1, |m|+2 \ldots\ldots \infty$$

Appendix C. Combined P Function and φ orthogonality.

Here I just combine the results of appendices A and B to get:

$$f(\xi,\varphi) = \Sigma_{n,m} f_{mn} P_n^m(\xi) \cos(m[\varphi-\varphi_m]) \qquad \text{// expansion}$$

$$f_{mn} = (2-\delta_{m,0}) (2n+1)/4\pi * f(n,-m) * \int d\xi \int d\varphi\, f(\xi,\varphi) P_n^m(\xi) \cos(m[\varphi-\varphi_m]) \qquad \text{// projection}$$

and I use this combined orthogonality several times in the work above in this doc.

Appendix D. Use the Neumann Result to find an oblate 1/R expansion

If we assume σ on our spheroid is that of a point charge, the potential we get is just that of a point charge, but it is expressed in oblate coordinates. Here is the result shown as a 1/R expansion

$$1/R = j \Sigma_{n,m} (-1)^m (\varepsilon_m/c_1) (2n+1) f(n,-m)^2 Q_n^m(j\zeta_>) P_n^m(j\zeta_<) P_n^m(\xi) P_n^m(\xi_0) \cos(m[\varphi-\varphi_0])$$

and this result is verified in Smythe.

Appendix E. Show that $j\, Q_n^m(jx)\, P_n^m(jy)$ is real for n,m integers and x,y real.

I show this in order to show that each term in the above 1/R term is in fact real.