The Pillbox Condition PhL 5.3.10

The time has come to isolate this thing so I can find it when I look for it. Right now it is inside the doc "On-axis Green's Function for an Oblate Spheroid.doc". I copy that here and will perhaps edit it. The main results that we always want is this, shown at the end of (a),

$\sigma = q\,(1/h_2h_3) * \delta(q_2-q_2')\,\delta(q_3-q_3')$ // σ and q both in same units

$\partial_{q1}V_i - \partial_{q1}V_o = (q/\varepsilon)\,(h_1/h_2h_3)\,\delta(q_2-q_2')\,\delta(q_3-q_3')$ // $(q/\varepsilon) \rightarrow 4\pi q$ in Jackson q/r units

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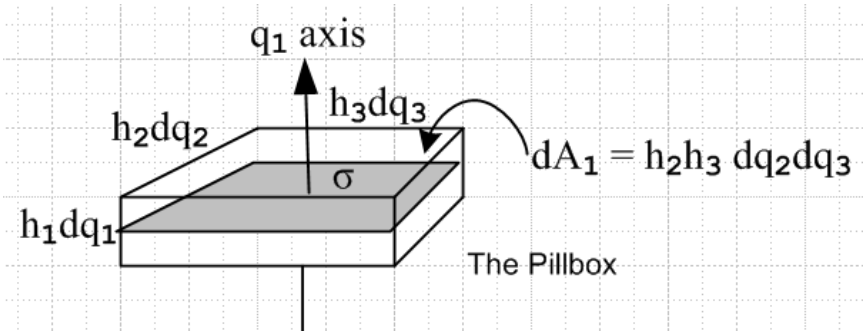
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(a) Finding the surface charge density σ in the pillbox for a point charge.

We are going to show here how to write ρ and σ for a point charge.

Let's do this for general curvilinear coordinates I will call q_1, q_2, q_3 having scale factors h_1 h_2 h_3. I should be putting the q indices up, but I will put them down for convenience. Assume that the coordinate 1 is perpendicular to our thin pillbox which lies surrounding a piece of surface.



Here are some facts we know:

$dA_1 = h_2h_3\,dq_2dq_3$ = area of either end of the pillbox (or of its center layer where σ lies)

$dV = h_1h_2h_3\,dq_1dq_2dq_3$ = volume of the pillbox. h_3dq_3

The perpendicular dimension of our pillbox must therefore be $h_1 dq_1$. Our pillbox here is really a curvilinear parallelepiped, very small and very flat, compressed down onto the surface it straddles. Since the coordinates are orthogonal, in the limit it becomes a right-angles rectangular solid.

Assume we have a point charge q inside the pill box at location **r**'. We then know that

$$\rho = q\,\delta(\mathbf{r}-\mathbf{r}') = q/(h_1h_2h_3) * \delta(q_1-q_1')\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

This is the key fact! We know that $\delta(\mathbf{r}-\mathbf{r}')$ works the opposite of dV, so that

$$q = \int dV\,\rho = \int h_1h_2h_3\,dq_1dq_2dq_3\; q/(h_1h_2h_3) * \delta(q_1-q_1')\,\delta(q_2-q_2')\,\delta(q_3-q_3') = q$$

$$= \int \rho\, h_1h_2h_3\,dq_1dq_2dq_3$$

[The charge happens to have the name q which is the same as the name of the coordinates. I hope this won't confuse me later.]

Now we also know that

$$q = \int dA_1\,\sigma \qquad dA_1 = h_2h_3\,dq_2dq_3 \qquad \Rightarrow \qquad q = \int \sigma\, h_2h_3\,dq_2dq_3$$

where here we are integrating over a layer-cake internal frosting-layer which contains σ, which lies along the center "plane" (gray in picture), let us say, of our pillbox. Perhaps σ varies, so we have an integral.

So at this point we have these two equations:

$$q = \int dq_2 \int dq_3\,\sigma\, h_2h_3$$

$$q = \int dq_2 \int dq_3 \int dq_1\;\rho\, h_1h_2h_3$$

In order for these to be equal we should (can) have

$$\sigma\, h_2h_3 = \int dq_1\;\rho\, h_1h_2h_3 = \int dq_1\;\{\, q/(h_1h_2h_3) * \delta(q_1-q_1')\,\delta(q_2-q_2')\,\delta(q_3-q_3')\}\, h_1h_2h_3$$

$$= q \int dq_1\,\delta(q_1-q_1')\,\delta(q_2-q_2')\,\delta(q_3-q_3') = q\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

Therefore we learn through this tortuous exercise that

$$\sigma = q(1/h_2h_3) * \delta(q_2-q_2')\,\delta(q_3-q_3') = \text{the charge density in our pillbox for a point charge}$$

and of course we also know that

$\rho = q\ 1/(\ h_1h_2h_3) * \delta(q_1-q_1') \delta(q_2-q_2') \delta(q_3-q_3')$

Here is a faster way we could have found the result for σ. Imagine the 2D space spanned by the "tangent vectors" of the two coordinates q_2 and q_3 at the point charge location **r'**. See "tensors and curvilinear" on this. In this subspace, our volume charge density would be $\rho = \sigma$, and our volume element would be $dV = h_2h_3\ dq_2dq_3 = dA$ and we would describe our point charge as $\rho = \sigma = (1/h_2h_3)\ \delta(q_2-q_2')\ \delta(q_3-q_3')$ and we would have our answer.

NOTE: Our above result for σ may have to be modified at singular points of the coordinate system. We have assumed we are at some "general point" in space where things are not singular.

(b) Applying Gauss's Law to the pillbox

The reason we are so interested in "σ" above will be seen in a moment. Consider now some equations relating to Gauss's Law [we could set ε to different things in different systems of units. At least we have "something there" that lets us do this.]

$\nabla\bullet\mathbf{E} = \rho/\varepsilon \quad \mathbf{E} = -\nabla V \qquad \rho = q\ \delta(\mathbf{r}-\mathbf{r'}) \qquad$ // replace $1/\varepsilon$ by 4π in Jackson units

$\int_V dV \nabla\bullet\mathbf{E} = \int_S d\mathbf{S}\bullet\mathbf{E}$ // divergence theorem = Gauss's Law when the vector is **E**

$\text{LHS} = \int_V dV \nabla\bullet\mathbf{E} = \int_V dV\ \rho/\varepsilon = \int_V dr\ q\ \delta(\mathbf{r}-\mathbf{r'})/\varepsilon = q/\varepsilon$

$\text{RHS} = \int_S d\mathbf{S}\bullet\mathbf{E} = -\int_S d\mathbf{S}\bullet\nabla V = -\int_S dS\ \partial_n V$

We can express the LHS ("the charge enclosed") in the following manner, for the limiting situation:

$\text{LHS} = (\sigma\ dA)/\varepsilon$

since σdA is "the charge enclosed" which is of course just q. We can then write the RHS as

$\text{RHS} = -\int_S dS\ \partial_n V = -[\ (\partial_n V_o)dA - (\partial_n V_i)dA\ ] = [\ (\partial_n V_i) - (\partial_n V_o)]\ dA$

Here we have assumed that the normal (Cartesian) vector **n** is pointing "out", so $\hat{\mathbf{n}} = \hat{\mathbf{q}}_1$. The potential V has the functional form called V_o on the "outside" of the surface, and V_i on the "inside". So it is on the outside that $d\mathbf{S} = d\mathbf{A}$ lines up with $\hat{\mathbf{n}}$ and gives the positive contribution to $\int_S dS\ \partial_n V$. In general, both terms shown on the far right will be positive. Just inside the point charge, $(\partial_n V_i)$ is very positive because we are approaching an infinite (q>0) point charge. Just outside, $(\partial_n V_o)$ is very negative because we are moving away from this same point charge. So we do not have nearly equal and opposite terms here, both terms including their signs are very positive.

Now, if we set LHS = RHS above we get

$[\ (\partial_n V_i) - (\partial_n V_o)]\ dA = (\sigma\ dA)/\varepsilon$

or

$\partial_n V_i - \partial_n V_o = \sigma/\varepsilon$ (*) // write = $4\pi\sigma$ if V = a/r for a point charge!

THIS is why we were so interested in an expression for σ in the previous section! Since the dA's cancel, we can avoid worrying about the exact form of dA.

Before taking the final step, we note that

$dn = h_1 dq_1 = ds_1$ = the "radial" distance scale factor situation.

$\Rightarrow \partial_n = (1/h_1)\,\partial_{q1}$

We can thus restate our result (*) above as

$\partial_{q1} V_i - \partial_{q1} V_o = h_1\sigma/\varepsilon$

Now recall from above that, for our point charge situation, we had

$\sigma = q(1/h_2h_3) * \delta(q_2-q_2')\,\delta(q_3-q_3')$

Therefore, we conclude that

$\partial_{q1} V_i - \partial_{q1} V_o = (q/\varepsilon)\,(h_1/h_2h_3)\,\delta(q_2-q_2')\,\delta(q_3-q_3')$ // $(q/\varepsilon) \rightarrow 4\pi\sigma$ in Jackson q/r units

This is our **Big Result** that I have been having so much trouble with! Again, we take as a warning that this may need repair at singular points of the coordinate system.

Comments: (just repeating)

(1) notice that we never have to worry about the details of what dA actually looks like in curvilinear coordinates. It cancels out on both sides of our equation to give (*).

(2) the "units system" is entirely contained in (q/ε). Everything else is units-independent.

Application #1: Pillbox on spherical surface

closing off our cone in our cone problem of another document, where I first started pondering this important pillbox component of "Smythe's Method" .

Our coordinates are

$q_1, q_2, q_3 = r,z,\varphi$

$h_1, h_2, h_3 = 1,r,r$ // see page 178 of M&M, but using z instead of θ

[I will try to remember to confirm this h_i claim in an Appendix at the end of this doc.] Our desired surface is in fact perp to coordinate $q_1 = r$ in this application example. Our end result is then this

$$\partial_{q1} V_i - \partial_{q1} V_o = (q/\varepsilon)\,(h_1/h_2h_3)\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

$$\partial_r V_i - \partial_r V_o = (q/\varepsilon)\,(1/a^2)\,\delta(z-z')\,\delta(\varphi-\varphi') \qquad \text{since } r = a \text{ at our pillbox}$$

This agrees with the result I got earlier in a different doc (by less clear means) where I said

$$[\,\partial_r V_i - \partial_r V_e\,] = (1/\varepsilon a^2)\,(1/\Delta\Omega_\beta) \rightarrow (1/\varepsilon a^2)\,\delta(z-z_\beta)\delta(\varphi-\varphi_\beta)$$

Application #2: Pillbox on surface of an oblate spheroid.

I am going to carefully make use of Smythe's slightly illogical 1,2,3 definitions here so we will have

$q_1, q_2, q_3 = \xi, \zeta, \varphi$ $\qquad\qquad$ ζ = "the radial coordinate"

In our previous examples we always had "1" as the perpendicular coordinate, but here we have "2". Therefore we can do a cyclic rotation on our result:

$$\partial_{q1} V_i - \partial_{q1} V_o = (q/\varepsilon)\,(h_1/h_2h_3)\,\delta(q_2-q_2')\,\delta(q_3-q_3') \qquad \text{// pre cyclic}$$

$$\partial_{q2} V_i - \partial_{q2} V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(q_3-q_3')\,\delta(q_1-q_1') \qquad \text{// post cyclic}$$

[Everything is positive, so there is no confusion really about this cyclic order business.] So

$$\partial_\zeta V_i - \partial_\zeta V_o = (q/\varepsilon)\,(h_2/h_3h_1)\,\delta(\varphi-\varphi')\,\delta(\xi-\xi') \qquad \text{// post cyclic}$$

and from Smythe we have for the three h_i scale factors,

$$h_1 = c_1^2\xi\left[\frac{x^2}{(-c_1^2\xi^2)^2} + \frac{\rho^2}{c_1^4(1-\xi^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1-\xi^2}\right)^{\frac{1}{2}} \qquad (6)$$

$$h_2 = c_1^2\zeta\left[\frac{x^2}{c_1^4\zeta^4} + \frac{\rho^2}{c_1^4(1+\zeta^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1+\zeta^2}\right)^{\frac{1}{2}} \qquad (7)$$

$$h_3 = \frac{\rho d\phi}{d\phi} = \rho = c_1[(1+\zeta^2)(1-\xi^2)]^{\frac{1}{2}} \qquad (8)$$

By the way, the area of our pill box cross section for this problem is

$$dA_2 = h_3h_1dq_3dq_1 = h_3h_1d\varphi\,d\xi$$

(c) Question: how do we take our pillbox "to the north pole" ? Finding G.

In the problem we are now working on, our point charge is "on axis" so we have to get things to the north pole. For our general oblate spheroidal pillbox we have

$\sigma = q\,(1/h_3h_1) * \delta(q_3-q_3')\,\delta(q_1-q_1')$ // general curvilinear coordinates

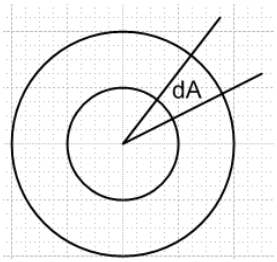
so

$\sigma(\xi,\varphi,\zeta) = q\,(1/h_3h_1) * \delta(\varphi-\varphi')\,\delta(\xi - \xi')$ // oblate spheroidal coordinates

so

$\partial_n V_i - \partial_n V_o = \sigma/\varepsilon = (q/\varepsilon)\,(1/h_3h_1) * \delta(\varphi-\varphi')\,\delta(\xi - \xi')$ // pillbox condition

Here is a picture of the situation at the north pole



Our point charge is inside dA and as $\xi' \rightarrow 1$, this approaches the north pole and becomes a little wedge of area there. One idea I have now is to "diffuse" the charge density in dA over the entire little annular ring. Then in the limit of "north pole", it does not matter whether we use dA or the ring.

Reduction of a delta function product to a product of *fewer* delta functions at singular points.

Well, in the end, we are looking for something like this:

$$\lim_{\xi'\rightarrow 1} \sigma(\xi,\varphi,\zeta) = \lim_{\xi'\rightarrow 1} [\, q(1/h_3h_1) * \delta(\varphi-\varphi')\,\delta(\xi - \xi')] = q\, f(\zeta)\,\delta(\xi - 1)$$

and so the subject here is how a "two coordinate delta function" can be a "one coordinate" delta function in the limit. Luckily for me, I have seen this discussed in Stakgold, and I will now scan for it. One example we say in Stak in Section p 14 was this

$$\lim_{\alpha\rightarrow 0} [\, \pi^{-3/2} \exp(-r^2/\alpha^2) / \alpha^3\,] = \delta(x_1)\delta(x_2)\delta(x_3)$$

But this is not what we want, but I think we are in the right ball park. I think the Example on Stak page 21 might be relevant. He is talking spherical coordinates and notes that things are singular on the z axis and we have to ponder what to do. Aha? I have spotted what I want on page 17 of my raw Stak Ch 5 notes:

$$\delta^3(\mathbf{x}) = 1/[r^2\sin\theta] * \delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')$$

$$\delta(\mathbf{x} - \mathbf{x}') = \delta(r - r')\,\delta(\theta) / [2\pi r^2\sin\theta] \qquad \text{when } x' \text{ has } \theta' = 0 \text{ (on z' axis)}$$

$$\delta^3(\mathbf{x}) = \delta(r)/[4\pi r^2]$$

The first line is the general case. The second line is what happens if $\mathbf{x}'$ = on z' axis. In this case, a triplet of deltas becomes a doublet of them! And finally, if $\mathbf{x}' = 0$, we have the last line where the triplet has been

reduced to a single delta function! So yes, this is definitely the right ball park, and I have luckily been "trained" in this area, though I now forget the details. Unfortunately, Stak in his discussion on p 21-22 uses the obscure names u_1, u_2, u_3 for r,θ,φ, so you have to "translate things" to get what I state above.

So I am now going to study Stak's example again very carefully. I did this quite well in my notes where I "verify" the equalities above. Very good, it all comes back and makes sense. [See page 17 of my raw Chap 5 notes!]

Let's back up to here $[q_1, q_2, q_3 = \xi, \zeta, \varphi\ ]$

$$\delta(\mathbf{r}\text{-}\mathbf{r}') = 1/(h_1h_2h_3) * \delta(q_1\text{-}q_1')\,\delta(q_2\text{-}q_2')\,\delta(q_3\text{-}q_3')$$

$$\delta(\mathbf{r}\text{-}\mathbf{r}') = 1/(h_1h_2h_3) * \delta(\xi\text{-}\xi')\,\delta(\zeta - \zeta')\,\delta(\varphi\text{-}\varphi')$$

Now we are interested in the case where r' lies on the ξ' axis. We want to see something like this":

$$\delta(\mathbf{r}\text{-}\mathbf{r}') = g\ \delta(\xi\text{-}1)\,\delta(\zeta - \zeta')$$

and we want to find what g is by our "verification method". We use f(**r**) as our "test function".

$$f(\mathbf{r}') = \int\int\int dxdydx\ \delta(\mathbf{r}\text{-}\mathbf{r}')f(\mathbf{r}) = \int\int\int h_1h_2h_3\ d\xi\,d\zeta\,d\varphi\ \ \delta(\mathbf{r}\text{-}\mathbf{r}')\,f(\mathbf{r})$$

$$= \int\int\int h_1h_2h_3\ d\xi\,d\zeta\,d\varphi\ \ g\ \ \delta(\xi\text{-}1)\,\delta(\zeta - \zeta')\,f(\xi,\zeta,\varphi)$$

$$= \int\int\int d\xi\,d\zeta\,d\varphi\ \ \delta(\xi\text{-}1)\,\delta(\zeta - \zeta')\,f(\xi,\zeta,\varphi)\,h_1h_2h_3\,g$$

Do the dξ integral first. Since the range is (-1,1) you might wonder whether we pick up all of the delta function or just half of it. Looking at Stak, I see in his case δ(θ) he had the same issue, and he picked up the entire delta function. So I will do that here. We then have

$$= \int\int d\xi\,d\varphi\,\delta(\zeta - \zeta')\,f(1,\zeta,\varphi,)\,h_1h_2h_3\,g$$

Now once again we recall

$$h_1 = c_1^2\xi\left[\frac{x^2}{(-c_1^2\xi^2)^2} + \frac{\rho^2}{c_1^4(1-\xi^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1-\xi^2}\right)^{\frac{1}{2}} \qquad (6)$$

$$h_2 = c_1^2\zeta\left[\frac{x^2}{c_1^4\zeta^4} + \frac{\rho^2}{c_1^4(1+\zeta^2)^2}\right]^{\frac{1}{2}} = c_1\left(\frac{\xi^2+\zeta^2}{1+\zeta^2}\right)^{\frac{1}{2}} \qquad (7)$$

$$h_3 = \frac{\rho d\phi}{d\phi} = \rho = c_1[(1+\zeta^2)(1-\xi^2)]^{\frac{1}{2}} \qquad (8)$$

which shows that none of the h_i includes the φ variable. And we know that f(1,ζ,φ,) = f(1,ζ,0,) because a function on the z axis is the same for all angles φ. So we continue the above. Let's assume for the moment that g is not a function of φ as well, which I think is a reasonable thing to assume. Then continue

$$= \int d\xi\, d\varphi\, \delta(\zeta - \zeta')\, f(1,\zeta,0)\, h_1 h_2 h_3\, g$$

$$= \int d\xi\, \delta(\zeta - \zeta')\, f(1,\zeta,0)\, h_1 h_2 h_3\, g \int d\varphi$$

$$= 2\pi \int d\xi\, \delta(\zeta - \zeta')\, f(1,\zeta,0)\, h_1 h_2 h_3\, g$$

$$= f(1,\zeta',0)\, [2\pi\, h_1 h_2 h_3\, g\,]$$

where now the h's and g are evaluated at ζ = ζ' as well as ξ = 1. But f(1,ζ',0) = f(**r'**) since this is exactly how you write a point on the positive symmetry axis. So we have then shown that

$$f(\mathbf{r'}) = f(\mathbf{r'})\, [2\pi\, h_1 h_2 h_3\, g\,]$$

and we conclude that we must have

$$g = 1/[2\pi\, h_1 h_2 h_3]$$

so our rule must be

$$\delta(\mathbf{r}-\mathbf{r'}) = 1/[2\pi\, h_1 h_2 h_3]\ \delta(\xi-1)\, \delta(\zeta - \zeta')$$

Digression on Sphericals: Let's see if this gives the right spherical coordinates answer.

$q_1, q_2, q_3 = r,z,\varphi$

$h_1, h_2, h_3 = 1,r,r$ // see page 178 of M&M, but using z instead of θ

$h_1 h_2 h_3 = r^2$

$\Rightarrow\ \delta(\mathbf{r}-\mathbf{r'}) = 1/[2\pi\, r^2]\ \delta(z-1)\, \delta(r - r')$

and from above (somewhere) we had

$\delta(\mathbf{x} - \mathbf{x'}) = \delta(r - r')\, \delta(\theta) / [2\pi r^2 \sin\theta]$ when x' has θ' = 0 (on z' axis)

so I think we are cooking with gas here.

Back to oblate spheroidals. So far then we have:

$$\delta(\mathbf{r}\text{-}\mathbf{r}') = 1/[2\pi\, h_1h_2h_3]\ \delta(\xi\text{-}1)\,\delta(\zeta - \zeta') \qquad // \mathbf{r}' \text{ on the } \xi = 1 \text{ axis}$$

$$\delta(\mathbf{r}\text{-}\mathbf{r}') = 1/(\, h_1h_2h_3) * \delta(\xi\text{-}\xi')\,\delta(\zeta - \zeta')\,\delta(\varphi\text{-}\varphi') \qquad // \text{general location}$$

If we set the two quantities on the right equal, we get

$$1/[2\pi\, h_1h_2h_3]\ \delta(\xi\text{-}1)\,\delta(\zeta - \zeta') = 1/(\, h_1h_2h_3) * \delta(\xi\text{-}\xi')\,\delta(\zeta - \zeta')\,\delta(\varphi\text{-}\varphi')$$

$$\lim_{\xi'\to 1}\ [\ \delta(\xi\text{-}\xi')\,\delta(\varphi\text{-}\varphi')\ ] = (1/2\pi)\,\delta(\xi\text{-}1)$$

From above we had

$$\sigma(\xi,\varphi,\zeta) = q\,(1/h_3h_1) * \delta(\varphi\text{-}\varphi')\,\delta(\xi - \xi') \qquad // \text{oblate spheroidal coordinates}$$

so

$$\partial_n V_i - \partial_n V_o = \sigma/\varepsilon = (q/\varepsilon)\,(1/h_3h_1) * \delta(\varphi\text{-}\varphi')\,\delta(\xi - \xi') \qquad // \text{pillbox condition}$$

Therefore, in our north pole limit we are finding that

$$\partial_n V_i - \partial_n V_o = \sigma/\varepsilon = (q/\varepsilon)\,(1/h_3h_1) * \delta(\varphi\text{-}\varphi')\,\delta(\xi - \xi')$$
$$= (q/\varepsilon)\,(1/h_3h_1)\,(1/2\pi)\,\delta(\xi\text{-}1)$$

In Section X above we conjectured that

$$\partial_\zeta V_i(\zeta_0,\xi) - \partial_\zeta V_o(\zeta_0,\xi) = G\,\delta(\xi\text{-}1)$$

so we conclude that

$$G = (q/2\pi\varepsilon)\,(h_2/h_3h_1) = (q/2\pi\varepsilon)\,[\ c_1\,(1+\zeta_0^2)]^{-1}$$

where the second factor I get from hand calculation and it agrees with a result from my previous document where I had $(1+\zeta^2)\,c_1\,h_2 = h_1h_3$. So

$$G = q/[2\pi c_1\varepsilon]\ /\ (1+\zeta_0^2)$$

(d) Generalization to N dimensions

$$dA_1 = (h_2dq_2)(h_3dq_3).....(h_Ndq_N)$$