Brief Comments on Methods for the charged bowl problem PhL 7.9.10

As a first method, I think this problem can be fully solved in **toroidal** coordinates. I think this because the surfaces of constant potential in that system are in fact toroids and spherical bowls! There is good web action on this subject, and I have the problem in Lebedev's book. So this is a first possible "method".

A second method is to use inversion.

Start with an offset charged metal disk + neutralizing constant potential and end up with a grounded metal spherical bowl with a point charge at an arbitrary point on the cap. This is of course a special case of the **Green's function for the bowl**. I did this problem (both Φ and σ), and went on to rotate the bowl and invert a second time to get the Green's function for an iris with point charge in the hole. (And I later found out how to do this iris problem more directly, and I also did the Green's function for the disk.) The main point here is that this entire effort produces only a special case of the bowl Green's function, which is a very different problem from that of the charged bowl (but see next paragraph).

Then if you integrate the solution of the above problem over the entire cap (perhaps first doing a ring), you can obtain results for a grounded metal bowl in the presence of a uniformly charged cap. I have done this for σ but not for Φ. At first, this situation seems of little interest until you realize that you can superpose on it a uniformly charged sphere which cancels the cap charge and raises the bowl potential from 0, and you end up with the **charged bowl situation**! In this way I have derived expressions for the charge density on the inner and outer surface of the bowl (they differ by a constant). This is a famous result, and Kelvin did it in 1847 and even plotted the charge distribution for various bowl cases. Perhaps he just gave numbers, I forget. Smythe uses this same method embedded in a sequence of problems.

A third method is the Sneddon approach using dual series equations, which involves assuming a reasonable Smythian form for the potential.

Comment on the first method.

Realizing that the "bowl" is a constant parameter surface in toroidal coordinates, I am motivated to find the potential of a charged bowl in these coordinates. Even in messy *ellipsoidal* coordinate, we know the potential of an ellipsoid whose surface has the constant value $\xi = \xi_1$, to wit

$V(\xi_1) = V_0 \, F[\sin^{-1}(g/\xi_1), f/g] \; / \, F[\sin^{-1}(g/a), f/g]$ // restatement of MF above

$V(\xi_1) = V_0 \, F[\sin^{-1}(f/\xi_1), g/f] \; / \, F[\sin^{-1}(f/a), g/f]$ // this line is "the Kelvin solution"

semis: $\xi_1 = a \geq b \geq c$ focals: $g^2 \equiv a^2 - c^2$ and $f^2 \equiv a^2 - b^2$ $g > f$

The way this simple result arises (ie, it is not some messy double sum) is that one conjectures a single term separated form as the product of three Lamé functions two of which are 1, and the third is the above (which is the lowest second-kind Lamé). This result has constant $V=V_0$ on the ellipsoid $\xi = \xi_1$, has the right large distance decay, and solves Laplace, so it must be the one and only solution.

As of 9.26.10, I have not carried out this solution method, but I still intend to.