Choice B Bowl Attempt PhL 1.16.10

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Overview (9.28.10 1/2 page)

This is a very short doc that comes after the Choice A doc.

In **Section 0** I review again the **two bowl/disk inversion options** and this time I select Choice B which means we will assume we know the on-axis disk Green's function and use that to get the charged bowl result.

In **Section 1** I **quote my own on-axis disk Green's function** that I derived elsewhere starting with an oblate spheroid and taking the disk and on-axis limits. This potential is a single n sum of products of Legendre functions which is not very nice looking. I also quote the corresponding σ on the disk. Though simpler, it is still a bit ugly, involving an n sum of $P_n(\pm\sqrt{1 - (\rho/b)^2}$ and other factors. I then comment that I know ahead of time that σ on the charged bowl has a relatively simple form involving only trig functions. I quote this result from three sources: Canonical, Smythe Problem 42, and Kelvin. The three results have slightly different form and symbology, but they are certainly all the same. I then make the observation that this is probably the "wrong way" to solve the charged bowl problem because you start in R' space with a σ which is very complicated, then you do the inversion which adds more complexity, and you are supposed to end up with a simple trig result. What you will really end up with is some obscure "sum rule" involving Legendre functions.

In **Section 2** ("does **Smythe** do this problem?") I am looking for a solution write-up for the charged bowl problem that I can understand. Kelvin and Canonical were too hard for me. I search in Smythe and find the σ result stated in Problem 42, but to do that problem I have to do Problems 39,40 and 41 first, none of which I can do. [But eventually I did all these problems!]

In **Section 3** I search **M&F** and find no charged bowl solution (called there a spherical cap).

In **Section 4** I consider the **dual-series approach** (but I did not know to call it that). I now know that the problem is solved using this method in Sneddon, and appears also in Canonical.

So at this point I still have no understandable way to solve the charged bowl problem. Kelvin is too obscure, Canonical uses methods I don't understand, and Smythe requires me to do problems I cannot do.

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0. Introduction.

With the inversion method you get two choices for the bowl-disk situation-pair you can study:

Choice A:
R-space: metal bowl at V = 0 with Green's charge at the opposite pole
R'-space: charged metal disk in isolation with a cancelling constant -A added so disk is at V = 0

Choice B:
R-space: charged metal bowl in isolation with a cancelling constant -A added so bowl is at V = 0
R'-space: metal disk at V=0 with Green's charge at the origin.

I tried hard with Choice A, even parsing a lot of Kelvin's painful detailed chatter on this subject. I now think Kelvin solved it without using inversion, using some other fancy geometry method. Kelvin solved something by inversion, but I think it was the Choice A Green's situation, not the free bowl situation.

So here I will try once again with the Choice B.

1. What is the Green's Function for the Disk?

From the end of my Jackson method of inversion META document, I quote:

$$V_o(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_{n,even} (2n+1) [(2j/\pi) Q_n(j\zeta_0) - P_n(j\zeta_0)] Q_n(j\zeta) P_n(\xi) - q/[4\pi j c_1 \varepsilon] \Sigma_{n,odd} (2n+1) P_n(j\zeta_0) Q_n(j\zeta) P_n(\xi)$$

$$V_i(\zeta,\xi) = q/[4\pi j c_1 \varepsilon] \Sigma_{n,even} (2n+1) [(2j/\pi) Q_n(j\zeta) - P_n(j\zeta)] Q_n(j\zeta_0) P_n(\xi) - q/[4\pi j c_1 \varepsilon] \Sigma_{n,odd} (2n+1) P_n(j\zeta) Q_n(j\zeta_0) P_n(\xi)$$

where

q = Green's point charge
c_1 = radius of the disk
ζ_0 = "radial" location of the Green's point charge relative to the center of the disk
ζ = oblate radial coordinate
ξ = oblate angular coordinate

This IS the on-axis Green's function for the disk. I am sure of it. I agree it is ugly.

One hope is that we can ignore this ugly potential and just deal with the charge distributions on the two sides of the disk which the above potential creates. Those are: (I set disk radius now to b)

$$\sigma_{disk} = - q/[4\pi b^2](1/\sqrt{1 - (\rho/b)^2}) \Sigma_n (2n+1) [Q_n(j\zeta_0)/Q_n(0+)] P_n(\pm\sqrt{1 - (\rho/b)^2})$$

$$Q_n(0+) = (-1)^{(n+1)/2} (n-1)!! / n!! \qquad n \text{ odd}$$

$$Q_n(0+) = (-1)^{n/2}\,(-j\pi/2)\;(n-1)!!\,/\,n!! \qquad n \text{ even}$$

Comment: I know the answer to my problem because I have seen it in various places. One good place is the Canonical Structures PDF. I have detailed notes on that in Stakgold/ "potential of half-spherical shell.doc" (there are v2 and v3 docs, ignore them for now). The problem is treated as a mixed-BC problem. The bowl has $\varphi = 0$ from angle $(0,\theta_0)$ [origin at center of bowl]. So this is the Dirichlet part of the BC. Then the bowl has $\partial_r\varphi^+ - \partial_r\varphi^- = 0$ on the rest of the angle range. This is the Neumann part where it is just saying $\sigma = 0$ on this part of the sphere. The authors set this derivative to $g(\theta)$ and they claim this IS the σ on the cap. Here is their result then for $g(\theta)$

$$g(\vartheta) = \frac{\sqrt{2}}{\pi}\left\{\frac{2\cos\frac{1}{2}\theta_0}{\sqrt{\cos\vartheta - \cos\theta_0}} + \frac{\pi}{2} - \arcsin\left(\frac{\cos\frac{1}{2}\theta_0}{\cos\frac{1}{2}\theta}\right)\right\}. \qquad (1.\ 111)$$

My point is this: since this is a relatively simply result, and since this must be related back to the on-axis Green's function for the disk by the inversion method, at least the charge distributions on the disk must have some simpler formulas that are not infinite n sums!

By the way, using the Geometrical Kinematics of "Kelvin Bowl attempt.doc" I know that $\rho/b = \tan(\theta/2)/\tan(\theta_0/2)$. So the argument of the Legendre function above is

$$P_n\left(\pm\sqrt{1 - \tan^2(\theta/2)/\tan^2(\theta_0/2)}\right)$$

This just shows how impossibly complicated my σ_{disk} form above is. It is the wrong path!

2. Does Smythe do this problem?

I don't see it listed in his huge Chapter 5 contents detail list. But "bowl" does appear in the problems. And here is our problem of interest:

42. Invert the result of problem 39 and find the charge density induced on a spherical earthed conducting bowl by a uniform charge distribution $-\delta_1$ over the remainder of the sphere. By superimposing a uniform spherical surface charge of density σ_1 show that the charge density at P on the inside or outside, respectively, of a freely charged conducting spherical bowl at potential V_0 is

$$\sigma_i = \frac{\epsilon V_0}{\pi a}\left[\frac{\sin\alpha}{(\sin^2\theta - \sin^2\alpha)^{\frac{1}{2}}} - \sin^{-1}\frac{\sin\alpha}{\sin\theta}\right], \qquad \sigma_o = \frac{\epsilon V_0}{a} + \sigma_i$$

where the angles are defined as in problem 40.

Notice the same idea we saw in Kelvin that the outer charge is the inner charge plus a constant! Although Kelvin's answer is different, I am sure it can be show to match the above

$$\rho = \frac{V}{2\pi^2 f}\left\{\sqrt{\frac{\cos\alpha + 1}{\cos\eta - \cos\alpha}} - \tan^{-1}\sqrt{\frac{\cos\alpha + 1}{\cos\eta - \cos\alpha}}\right\} \ldots (18),$$

For now, my point is that we have that same "answer" we saw in the Canonical document. So I have two sources on this now. In order to follow Smythe on this you have to do three problems in order:

42 requires 39
39 requires 38

None of these problem is even clear to me, so it would take me many days to travel this path. So let's not do that. Smythe search on "spherical cap" turns up nothing relevant.

3. Do MF do this problem?

Neither volume has the word "bowl". But "spherical cap" turns up a section in Volume 2. It is page 1269 in Section 10.3. But for some reason they start off with a "dipole layer of surface density I" on the cap. But this is harder than the problem I want! They have some in-problem references that mention inversion, but nothing that attracts me.

So, I have looked at Canonical, Smythe, MF, and Kelvin, and nobody is giving me a way to solve this problem that I like.

4. What about the form fit method?

If you do the "form fit" method, you start off like the Canonical Russian guys: (p 39)

$$\sum_{n=0}^{\infty} a_n P_n(\cos\theta) = 1,\ \theta \in (0, \theta_0), \qquad (1.\ 100)$$

$$\sum_{n=0}^{\infty} (2n+1) a_n P_n(\cos\theta) = 0,\ \theta \in (\theta_0, \pi), \qquad (1.\ 101)$$

I show in my notes that this second equation says the jump in $\partial_{\mathbf{r}}\varphi$ across the dotted boundary is 0.
This is the "mixed BC" approach in spades. Since the $P_{\mathbf{n}}$ are not complete on the partial interval, you can't just invert the first equation for the $a_{\mathbf{n}}$. You will need the information in both equations!