The Agassi Problem PhL 12.29.13

He wants to know a Green's function which is constant on a torus in toroidal coordinates.

$(\nabla^2 + k^2)\, g(x|\xi) = b\, \delta(\text{ line source along torus})$

That seems to me to be a doable problem. He also wants g = constant on a specific torus, and g = 0 at infinity. I might do this for point source then integrate around the ring.

Question 1: Does this have anything to do with my various 1/R expansions?

That 1/R is the 3D free space Laplace Greens.
More appropriate here would be e^{-ikR}/R for Helmholtz.

Plan A: Suppose you just write

$g = e^{-ikR}/R$ where $R = |x-\xi|$

That IS the Helmholtz *free space* Green's function, but he needs it to be a constant on some surface, so that is not going to cut the mustard.

Let's review Stak's methods in chapter 6. hold on that

Question 2: Does the fact that toroidals are only R-separable affect the eigenfunction expansion form for g?

Maybe now is a good time to review the bowl paper. // OK, I did so and it was a VERY long haul. There is certainly a lot of related stuff in there, but I don't know where to begin.

Maybe I will collected a few facts of possible use here.

Toroidal Atoms for Helmholtz?

My bowl paper does not deal with Helmholtz equation anywhere. It does have Mehler integrals and such things. The Stackel paper is entirely about Helmholtz, however. At least I do know that Helmholtz is R-separable in toroidals.

Fact: It also turns out that with R ≠ constant the Helmholtz PDE is not separable unless $K_1^2 = 0$, in which case it is the Laplace PDE. // In other words, no Helmholtz equation with $K_1^2 \neq 0$ is R-separable unless R = constant.

This seems to say that Helmholtz is not separable in toroidals! So we are dead meat at the git go. Toroidals are not one of the 11 classical systems. Laplace is R-separable, Helm is NOT separable. This is confirmed on page 101 if M&S.

1. The Helmholtz equation is not separable in toroidal coordinates. This is confirmed on page 99 of Moon and Spencer. It is only separable if $k^2 = 0$ in which case it is really the Laplace equation.

2. The Laplace equation is R-separable in toroidal coordinates with the "atom choices" shown in my bowl paper, section 2 (b)

ξ in $(0,\infty)$ u in $(0,2\pi)$ φ in $(0,2\pi)$

		expo	osc	osc
(1)	$\sqrt{ch\xi - cosu}$	$[P_{n-1/2}^{m}(ch\xi), Q_{n-1/2}^{m}(ch\xi)]$	$[\sin(nu),\cos(nu)]$	$[\sin(m\varphi),\cos(m\varphi)]$

		osc	expo	osc
(2)	$\sqrt{ch\xi - cosu}$	$[P_{i\tau-1/2}^{m}(ch\xi), Q_{i\tau-1/2}^{m}(ch\xi)]$	$[\exp(\tau u), \exp(-\tau u)]$	$[\sin(m\varphi),\cos(m\varphi)]$

The "R-separable" means that there is as non-separable overall function sitting in all solutions, here that function is $\sqrt{ch\xi - cosu}$.

But of course this is not what you want.

3. I think you are interested in the solution of the Helmholtz equation driven by a ring source, which I guess could be written as

$$(\nabla^2 + k^2)g(u,\xi|u_0,\xi_0) = b\,\delta(a\,\coth\xi_0 - \rho_0)$$

$$[\text{lap } f](\mathbf{x}) = 1/(h_1h_2h_3)\,\{\,\partial_1\,[\,(h_2h_3/h_1)\,\partial_1 f\,]\ + \text{cyclic}\ \}$$

My coordinates are ξ = torus label, u = bowl label.

Names of the coordinates:

My Stackel paper:	(ξ_1,ξ_2,ξ_3)	toroid, bowl, azimuthal plane	
Moon & Spencer	(η,θ,ψ)	toroid, bowl, azimuthal plane	
My bowl paper	(ξ,u,φ)	toroid, bowl, azimuthal plane	
M&F:	(μ,θ,φ)	toroid, bowl, azimuthal plane	Vol 1 p 666

The scale factors:

$$h_\xi = h_u = a/(ch\xi - cosu) \qquad h_\varphi = sh\xi\, h_\xi = a\, sh\xi\,/(ch\xi - cosu)$$

$$h_1 = h_2 = a/(ch\xi - cosu) \qquad h_3 = sh\xi\, h_1$$

$$\nabla^2 = 1/(h_1h_2h_3)\,\{\,\partial_1\,[\,(h_2h_3/h_1)\,\partial_1 f\,]\ + \text{cyclic}\ \}$$

$$\nabla^2 = 1/(h_1h_2h_3)\,\{\,\partial_1\,[\,(h_2h_3/h_1)\,\partial_1 f\,]\ + \partial_2\,[\,(h_3h_1/h_2)\,\partial_2 f\,] + \partial_3\,[\,(h_1h_2/h_3)\,\partial_3 f\,]\,\}$$

$$\nabla^2 = 1/(h_1^3 sh\xi) \{ \partial_1 [(sh\xi\, h_1) \partial_1 f] + \partial_2 [(sh\xi\, h_1) \partial_2 f] + \partial_3 [(h_1^2/sh\xi h_1) \partial_3 f] \}$$

$$\nabla^2 = 1/(h_1^3 sh\xi) \{ \partial_1 [(sh\xi\, h_1) \partial_1 f] + \partial_2 [(sh\xi\, h_1) \partial_2 f] + \partial_3 [(h_1/sh\xi) \partial_3 f] \}$$

$$\nabla^2 = 1/(h_1^3 sh\xi) \{ \partial_1 [(sh\xi\, a/(ch\xi - cosu)) \partial_1 f] + \partial_2 [(sh\xi\, a/(ch\xi - cosu)) \partial_2 f] + \partial_3 [(a/(ch\xi - cosu)/sh\xi) \partial_3 f] \}$$

$$a^2\nabla^2 f(\xi_1,\xi_2) = [ch(\xi_1)-cos(\xi_2)]^2 (\partial_{\xi 1}^2 + \partial_{\xi 2}^2) f$$

$$(\nabla^2 + k^2)\, f(\xi_1,\xi_2) = a^{-2} [ch(\xi_1)-cos(\xi_2)]^2 (\partial_{\xi 1}^2 + \partial_{\xi 2}^2) f + k^2 f = b\, \delta(a\, coth\xi_0 - \rho_0)$$

$$(\nabla^2 + k^2)\, g(\xi,u \mid \xi_0,u_0) = b\, \delta(a\, coth\xi_0 - \rho_0) \qquad g(\xi,u_1 \mid \xi_0,u_0) = \text{constant}$$

$$a^{-2} [ch(\xi)-cos(u)]^2 (\partial_\xi^2 + \partial_u^2)\, g(\xi,u \mid \xi_0,u_0) + k^2\, g(\xi,u \mid \xi_0,u_0) = b\, \delta(a\, coth\xi_0 - \rho_0)$$

Restrospective.

I see now how I misled Agassi with my paper title which is "Stackel separation of the H equation" and I use toroidals as a constant example throughout, but toroidal is only Laplace separable. I think I should repair this error and add some clear statements at the beginning. I so say this right at the start

The subject at hand is separation of the Helmholtz equation (and its special case the Laplace equation) in various 3D curvilinear coordinate systems whose coordinates we shall call ξ_1,ξ_2,ξ_3. The term

It turns out that only the 11 "classical" 3D Euclidean orthogonal coordinate systems are *simple* separable for the Helmholtz equation (and therefore also for the Laplace equation). These systems are discussed in the first chapter (called Section I) of Moon & Spencer (1961). This separability is why they are the classical systems.

At this point the reader might assume that toroidal is R separable and might be H separable. I do say that

Problem A = simple sep
Problem B = R sep

I continue then with

Section 7 discusses the solution of Problem B as recast at the end of Section 5. Section 7 (a) specifies a set of Conditions which must be met for a solution to exist, and Section 7 (b) gives a sequence of Steps one can follow to find the solution. These Steps are then applied to two examples. The first is a Problem B problem: R-separation of toroidal coordinates. The second is a Problem A problem: simple-separation of circular cylindrical coordinates. The examples are just exercises in turning a crank. In each case the solution functions are stated, these being the toroidal and cylindrical harmonics.

I don't make clear that it is R-separation of toroidal for Laplace, not Helmholtz. I then dive into details of hacking away at the H equation, grinding it down. I do say this during the processing:

Therefore, the toroidal system at least has a chance of being R-separable. It is in fact R-separable (for Laplace), but we don't know that yet since there might be other restrictions that will need checking.

So this seems a clear hint that it is not separable for Helmholtz. Then later

Problem C: (R-separation of the *Helmholtz* Equation with $K_1^2 \neq 0$).

Here one has the full bore (4.9) to contend with

$$- \Sigma_n(Q/h_n^2) [\kappa_1^2\Phi_{n1}(n) + k_2^2\Phi_{n2}(n) + k_3^2\Phi_{n3}(n)] + k_1^2 + QK_1^2 = 0 . \qquad (4.9C)$$

Remember that Q = Q(123) is (in general) a function of all three coordinates. It is not clear how to decouple that last two terms in this equation. What should κ_1^2 be set to? The method just outlined above for solving Problem B *does not work* for Problem C. Moon and Spencer state (p 96, 1961) that no curvilinear system has ever been found in which the Helmholtz equation with $K_1^2 \neq 0$ separates via R-separation (by which they mean with R ≠ constant). So having at least stated it, we shall give up on Problem C and continue now to solve Problem B.

So here I say Agassi cannot succeed.

R ≠ constant R = constant (=1 without loss of generality)

OK, I made a new chart and added new text to the opening section of Staekel. I think it is better and I will probably change the title, though that will mess up web search links.

Question 3. For an R separable Laplace system, what do the "basic analysis" equations look like?

Maybe I should start with spherical coordinates. They separate and we get three little ODE's, two of which are oscillatory. // Put a hold on this. Instead.

Question 4: What is the toroidal ∇^2 Dan references from M&F? Recall that MF use some alternate coordinates as I told Dan in my last email to him.

Answer: I resolved this matter completely on 12.30.13 and result is in an email to Dan. I also now have a doc which looks at all MF toroidal stuff, basically one chunk in each volume. Nothing new here.

Question 5: What are my Stackel paper's toroidal ODE's and solutions?

$$L_1X_1 = \partial_1^2X_1 + \coth(\xi_1)\,\partial_1X_1\} + [(1/4) - k_2^2 - k_3^2/\mathrm{sh}^2(\xi_1)]X_1 = 0$$
$$L_2X_2 = \partial_2^2X_2 + k_2^2X_2 = 0$$
$$L_3X_3 = \partial_3^2X_3 + k_3^2X_3 = 0 \quad .$$

$$\overset{\text{expo}}{[P^{q}_{p-1/2}(ch\xi_1)}, \overset{\text{osc}}{Q^{q}_{p-1/2}(ch\xi_1)\,[\,\sin(p\xi_2), \cos(p\xi_2)\,]}\,\overset{\text{osc}}{[\,\sin(q\xi_3), \cos(q\xi_3)\,]} * \,[ch(\xi_1)-\cos(\xi_2)]^{1/2}\,.$$

showing toroidal or ring functions. Here the separation constants are $k_2^2 = p^2$ and $k_3^2 = q^2$ with q an integer in the full 2π range of ξ_3 is present in a problem of interest. If $k_2^2 = p^2 < 0$, one writes $p = i\tau$ to get

$$\overset{\text{osc}}{[P^{q}_{i\tau-1/2}(ch\xi_1)}, \overset{\text{exp}}{Q^{q}_{i\tau-1/2}(ch\xi_1)\,[\,sh(\tau\xi_2), ch(\tau\xi_2)\,]}\,\overset{\text{osc}}{[\,\sin(q\xi_3), \cos(q\xi_3)\,]} * \,[ch(\xi_1)-\cos(\xi_2)]^{1/2}$$

and Legendre functions of this type are called Mehler functions.

Question 6: If we want a solution to vanish "far away", which form is appropriate?

Answer: In my notes sent to Dan, I claim that large r goes with both $\xi_1 = \xi$ and $\xi_2 = u$ being small. Why is this? Small ξ means the fat torus whose surface is large and in some sense far away. Small u means we approach the iris limit of the bowl, also far away. I claim various things in this limit: First

Fact 1: $(ch\xi - \cos u) \approx (1/2)(\xi^2+u^2)$

Proof: $ch\xi \approx 1 + \xi^2/2$ $\cos u \approx 1 - u^2/2$

$(ch\xi - \cos u) \approx 1 + \xi^2/2 - (1 - u^2/2) = \xi^2/2 + u^2/2$ QED.

Fact 2: $\sqrt{\xi^2+u^2} \approx \sqrt{2}\,\sqrt{ch\xi - \cos u}$ proof: just solve Fact 1

Fact 3: I presume the first form here is exact from the x,y,z equations:

$$r^2 = a^2\,(sh^2\xi + \sin^2 u)/(ch\xi - \cos u)^2 \approx a^2\,(\xi^2+u^2) / [\,(1/2)(\xi^2+u^2)]^2 = (2a)^2/\,(\xi^2+u^2)$$

Proof: $r^2 \approx a^2\,(\xi^2 + u^2) / [(1/2)(\xi^2+u^2)]^2 = 4a^2 / (\xi^2 + u^2)$ QED.

So, as ξ and u get small, you see that r gets large.

Question 7: What do my two atomic forms do for large r ?

$$P^{q}_{p-1/2}(ch\xi_1) \approx P^{q}_{p-1/2}(1 + \xi_1^2/2) \approx P^{q}_{p-1/2}(1)$$

My careful Legendre property notes say that if q is in integer and p-1/2 is "anything" then

$$P^{q}_{p-1/2}(1) = \delta_0{}^{q}$$

So this vanishes for all integer $q \neq 0$, and equals 1 for q = 0. Similarly,

$$P^{q}_{i\tau-1/2}(ch\xi_1) \approx P^{q}_{i\tau-1/2}(1) = \delta_0{}^{q}$$

So in both cases, the P function vanishes as $r \to \infty$, causing any P-containing atom to be small, and this is what Dan wants for large r. But we have to treat q = 0 separately! We don't need to look at the ξ_2 coordinate to get a small limit for the atom, I don't think it is a problem.

I would guess that the Q function blows up at z = 1 and must be excluded. let's check on that.

$\mathrm{Re}(q) < 0$ and $q \neq$ integer or half integer $\Rightarrow$ $Q_\nu^q(\pm 1) = 0$ for $\nu = q - J$ only, ELSE $Q_\nu^q(\pm 1) = \infty$

In our case q = integer, so $Q_\nu^q(\pm 1) = \infty$ and yes, we can discard the Q atomic form!

Question 8: How can you specify the ring source that Dan wants?

I took a shot at this here, wrote it up, then deleted it here. I was surprised to get $\xi_1 = 2\xi_0$, See that email to Dan for details. So at least I know HOW to compute the ring source, even if I might have done it wrong.

Question 9: How do you express $\delta(\xi - \xi')$ in a useful way?

Maybe if comes from the transform I recently sent to Dan involving the Q function. If I use the first atom form above, then things are periodic in u and φ and we know how completeness works for these atoms, and that then leaves only the P function to worry about. I think I have done this somewhere already, it just rings a bell, though a very distant one.

Let's just back up a bit. My problem is going to be something like this:

$$\nabla^2 g = \text{constant}\ \delta(\xi - \xi_1)\delta(u - u_1)$$

If I use the form where u and φ are oscillatory, then the P function is going to be expo, and then we don't have a Sturm Liouville problem in ξ, and then I cannot come up with a completeness for $\delta(\xi - \xi_1)$. So maybe I need to instead use a form where ξ is oscillatory, such as

$$\begin{matrix} & \text{osc} & & \text{exp} & & \text{exp} & \\ [P^q_{i\tau-1/2}(ch\xi_1), & Q^q_{i\tau-1/2}(ch\xi_1) & [\ sh(\tau\xi_2), & ch(\tau\xi_2)\ ] & [\ \sin(q\xi_3), & \cos(q\xi_3)\ ] * & [ch(\xi_1) - \cos(\xi_2)]^{1/2} \end{matrix}$$

$$\begin{matrix} & \text{osc} & & \text{exp} & & \text{osc} & \\ [P^m_{i\tau-1/2}(ch\xi), & Q^m_{i\tau-1/2}(ch\xi) & [\ sh(\tau u), & ch(\tau u)\ ] & [\ \sin(m\varphi), & \cos(m\varphi)\ ] * & [ch\xi - \cos u]^{1/2} \end{matrix}$$

where m = integer. What is the transform here? It is the generalized Mehler-Fock which appears in the bowl paper:

The generalized Mehler-Fock (Mehler-Fok) transform (Oberhettinger and Higgins page 2) is this,

$$g(y) = \int_0^\infty d\tau\ P^m_{-1/2+i\tau}(y)\ f(\tau) \qquad \text{// expansion}$$

$$f(\tau) = (\tau/\pi)\ sh(\pi\tau)\ \Gamma(1/2 - m + i\tau)\ \Gamma(1/2 - m - i\tau) \int_1^\infty dy\ P^m_{-1/2+i\tau}(y)\ g(y) \qquad \text{// projection}$$

I bet I have more detail on this somewhere! I would like to get this transform "normalized" so I have eigenfunctions and completeness and orthogonality and all that good stuff.

Maybe first guess, then look things up.

$\varphi_{m\tau}(y) = P^{m}_{-1/2+i\tau}(y)$ = eigenfunction $\qquad y = ch\xi$

I then expect something like this for orthogonality

$$\int_0^\infty dy\ \varphi_{m\tau}(y)\ \varphi_{m'\tau'}(y)^* = \delta_{m,m'}\ \delta(\tau-\tau') \qquad \text{// orthog?}$$

and something like this for completeness

$$\Sigma_m \int_0^\infty d\tau\ \varphi_{m\tau}(y)\ \varphi_{m\tau}(y')^* = \delta(y-y') \qquad \text{// completeness?}$$

So I am looking for an integral of this form

$$\Sigma_m \int_0^\infty d\tau\ P^{m}_{-1/2+i\tau}(y)\ P^{m}_{-1/2-i\tau}(y') = \delta(y-y')$$

where there is likely some other function in the integrand. This is some kind of generalized Mehler integral.

My bowl paper has a waypoint gap between (2.4) and (2.5). I don't follow it! I must have done this in some other document, and that doc will have what I want! Now I have to start searching my Alta database.

1. Search math for docs containing Fock. I get some hits, the search is slow. It finds these:

Name	Date Created	Size	In Folder
Sturm Liouville atoms in various coord systems.doc	7/15/2010 8:01 AM	393 KB	D:\Work\My Interests\Math\Curvilinear Systems
doing Mehler transform integrals.doc	1/27/2011 8:03 AM	688 KB	D:\Work\My Interests\Math\Integrals series sums+ GR
Auto Comp of Zeros of Bessel.doc	10/16/2004 10:54 AM	31 KB	D:\Work\My Interests\Math\Jim Ball GQ Binder
Dan DE handbook errata Fock.doc	3/6/2011 5:16 PM	254 KB	D:\Work\My Interests\Math\Curvilinear Systems\Toroidals
Toroidal coordinates.doc	2/5/2011 2:59 PM	1,348 KB	D:\Work\My Interests\Math\Curvilinear Systems\Toroidals

I now repeat the search with Witzend: I get the exact same hits. I scan each of these:

- Doing mehler -- lots of stuff, but no completeness or orthog. This is where I did all the integrals that I report out in bowl doc.

- SL -- not much

- Auto comp -- Jim's thing, nothing here

- Dam DE errata -- not here

• Tor coords -- not here.

Now search physics instead. Got 11 hits.

D:\Work\My Interests\Physics\E&M\Electrostatics\bowl\bowl in toroidals\Bowl Green's Function Attempt using Toroidals.doc
D:\Work\My Interests\Physics\E&M\Electrostatics\bowl\bowl in toroidals\charged bowl in toroidals (INCOMPLETE).doc
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D:\Work\My Interests\Physics\E&M\Electrostatics\bowl\bowl in toroidals\The Charged Bowl in Toroidal Coordinates.doc
D:\Work\My Interests\Physics\E&M\Electrostatics\bowl\bowl in toroidals\release of 11_30_11\The Charged Bowl in Toroidal Coordinates.doc
D:\Work\My Interests\Physics\E&M\Electrostatics\Canonical\Canonical-Structures Ch 6 notes.doc
D:\Work\My Interests\Physics\Geoff Chew\Cosmological Hilbert Space 3.doc
D:\Work\My Interests\Physics\Geoff Chew\geoff paper.doc
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D:\Work\My Interests\Physics\Quantum Mechanics\Schiff\Schiff Chap 7 Symmetry.doc
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I eliminate obvious ones to shorten the list

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Two of these are copies of the bowl paper, leaving a still shorter list

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• Not in the canonical paper.

• bowl Green's attempt -- not there

• INCOMPLETE -- here I actually have a web quote for the m = 0 case

$$\int_1^\infty P_{-1/2+i\tau_1}(x)\,P_{-1/2+i\tau_2}(x)\,dx = \frac{\delta(\tau_1-\tau_2)}{\tau_1\tanh(\pi\tau_1)}. \qquad \text{(A3)}$$

but I just found this on the web and I don't have the more general case.

• the 1_11 doc has my same waypoint situation, but more words. OK, there is the answer. You just do a "read off" trick.

$$(V_0 / \sqrt{\mathrm{ch}\xi - \cos u_0}) = \int_0^\infty d\tau\, P_{i\tau-1/2}(\mathrm{ch}\xi)[\, A(\tau)\mathrm{ch}(u_0\tau) + B(\tau)\mathrm{sh}(u_0\tau)\,]$$

$$= \int_0^\infty d\tau\, \tau\, th(\pi\tau) P_{i\tau-1/2}(ch\xi)\, \{ [\, A(\tau)ch(u_0\tau) + B(\tau)sh(u_0\tau)\,]/[\, \tau\, th(\pi\tau)] \}$$

Then the "read off" says,

$$g(\xi) = \int_0^\infty d\tau\, \tau\, th(\pi\tau)\, P_{i\tau-1/2}(ch\xi)\, G(\tau) \qquad \text{// expansion}$$

so if $g(\xi) = (V_0 / \sqrt{ch\xi - cosu_0}\,)$, then $G(\tau) = \{ [\, A(\tau)ch(u_0\tau) + B(\tau)sh(u_0\tau)\,]/[\, \tau\, th(\pi\tau)] \}$

OK, I am now ready to take the plunge. Start with the transform this way

$$g(y) = \int_0^\infty d\tau\, P^m_{-1/2+i\tau}(y)\, f(\tau) \qquad \text{// expansion}$$

$$f(\tau) = (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-m+i\tau)\, \Gamma(1/2-m-i\tau) \int_1^\infty dy\, P^m_{-1/2+i\tau}(y)\, g(y) \qquad \text{// projection}$$

Insert the first into the second

$$f(\tau) = (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-m+i\tau)\, \Gamma(1/2-m-i\tau) \int_1^\infty dy\, P^m_{-1/2+i\tau}(y)\, g(y)$$

$$= (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-m+i\tau)\, \Gamma(1/2-m-i\tau) \int_1^\infty dy\, P^m_{-1/2+i\tau}(y) \int_0^\infty d\tau'\, P^m_{-1/2+i\tau'}(y)\, f(\tau')$$

$$= (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-m+i\tau)\, \Gamma(1/2-m-i\tau) \int_0^\infty d\tau' \{ \int_1^\infty dy\, P^m_{-1/2+i\tau}(y)\, P^m_{-1/2+i\tau'}(y) \}\, f(\tau')$$

Now let's assume that (note * applied to second factor)

$$\int_1^\infty dy\, P^m_{-1/2+i\tau}(y)\, P^m_{-1/2-i\tau'}(y) = \delta(\tau-\tau')F(\tau)$$

Then we have

$$f(\tau) = (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-m+i\tau)\, \Gamma(1/2-m-i\tau) \int_0^\infty d\tau'\, \delta(\tau-\tau')F(\tau)\, f(\tau')$$

$$= (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-m+i\tau)\, \Gamma(1/2-m-i\tau)\, F(\tau)\, f(\tau)$$

Then it must be that

$$F(\tau) = [\, (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-m+i\tau)\, \Gamma(1/2-m-i\tau)\,]^{-1}$$

and then

$$\int_1^\infty dy\ P^m_{-1/2+i\tau}(y)\ \ P^m_{-1/2-i\tau'}(y) = [\ (\tau/\pi)\ sh(\pi\tau)\ \Gamma(1/2-m+i\tau)\ \Gamma(1/2-m-i\tau)\ ]^{-1}\ \ \delta(\tau-\tau')$$

and this is then the orthogonality relation for this particular transform.

Now let's try the other direction: insert the second into the first,

$$g(y) = \int_0^\infty d\tau\ P^m_{-1/2+i\tau}(y)\ f(\tau)$$

$$= \int_0^\infty d\tau\ P^m_{-1/2+i\tau}(y)\ (\tau/\pi)\ sh(\pi\tau)\ \Gamma(1/2-m+i\tau)\ \Gamma(1/2-m-i\tau)\ \int_1^\infty dy'\ P^m_{-1/2+i\tau}(y')\ g(y')$$

$$= \int_1^\infty dy'\ g(y')\ \int_0^\infty d\tau\ P^m_{-1/2+i\tau}(y)\ (\tau/\pi)\ sh(\pi\tau)\ \Gamma(1/2-m+i\tau)\ \Gamma(1/2-m-i\tau)\ P^m_{-1/2+i\tau}(y')$$

Now to make this work, we need to have

$$\int_0^\infty d\tau\ P^m_{-1/2+i\tau}(y)\ (\tau/\pi)\ sh(\pi\tau)\ \Gamma(1/2-m+i\tau)\ \Gamma(1/2-m-i\tau)\ P^m_{-1/2+i\tau}(y')\ = \delta(y-y')$$

or

$$\int_0^\infty d\tau\ \ (\tau/\pi)\ sh(\pi\tau)\ \Gamma(1/2-m+i\tau)\ \Gamma(1/2-m-i\tau)\ P^m_{-1/2+i\tau}(y)\ P^m_{-1/2+i\tau}(y')\ = \delta(y-y')$$

and this then is the completeness. So here are my tentative results:

orthogonality:

$$\int_1^\infty dy\ P^m_{-1/2+i\tau}(y)\ \ P^m_{-1/2-i\tau'}(y) = [\ (\tau/\pi)\ sh(\pi\tau)\ \Gamma(1/2-m+i\tau)\ \Gamma(1/2-m-i\tau)\ ]^{-1}\ \ \delta(\tau-\tau')$$

completeness:

$$\int_0^\infty d\tau\ [\ (\tau/\pi)\ sh(\pi\tau)\ \Gamma(1/2-m+i\tau)\ \Gamma(1/2-m-i\tau)]\ \ P^m_{-1/2+i\tau}(y)\ P^m_{-1/2+i\tau}(y')\ = \delta(y-y')$$

Now for more compaction, define

$$s_m(\tau) \equiv (\tau/\pi)\ sh(\pi\tau)\ \Gamma(1/2-m+i\tau)\ \Gamma(1/2-m-i\tau)$$

Then we get

$$\int_1^\infty dy\ P^m_{-1/2+i\tau}(y)\ P^m_{-1/2-i\tau'}(y) = \delta(\tau-\tau') / s_m(\tau) \qquad \text{orthogonality}$$

$$\int_0^\infty d\tau\ s_m(\tau)\ P^m_{-1/2+i\tau}(y)\ P^m_{-1/2+i\tau}(y') = \delta(y-y') \qquad \text{completeness}$$

Now let's check for the case m = 0.

$$s_0(\tau) = (\tau/\pi)\ \mathrm{sh}(\pi\tau)\ \Gamma(1/2+i\tau)\ \Gamma(1/2-i\tau)$$

Let x = 1/2+iτ. Then

$$\Gamma(1/2+i\tau)\ \Gamma(1/2-i\tau) = \pi/\mathrm{ch}(\pi\tau) \qquad \text{// use Euler's Reflection Formula}$$

and then

$$s_0(\tau) = (\tau/\pi)\ \mathrm{sh}(\pi\tau)\ \pi/\mathrm{ch}(\pi\tau) = \tau\ \mathrm{th}(\pi\tau)$$

and then we have

$$\int_1^\infty dy\ P_{-1/2+i\tau}(y)\ P_{-1/2-i\tau'}(y) = \delta(\tau-\tau') / s_0(\tau) \qquad \text{orthogonality}$$

$$\int_0^\infty d\tau\ s_0(\tau)\ P_{-1/2+i\tau}(y)\ P_{-1/2+i\tau}(y') = \delta(y-y') \qquad \text{completeness}$$

or

$$\int_1^\infty dy\ P_{-1/2+i\tau}(y)\ P_{-1/2-i\tau'}(y) = \delta(\tau-\tau') / [\tau\ \mathrm{th}(\pi\tau)] \qquad \text{orthogonality}$$

$$\int_0^\infty d\tau\ \tau\ \mathrm{th}(\pi\tau)\ P_{-1/2+i\tau}(y)\ P_{-1/2+i\tau}(y') = \delta(y-y') \qquad \text{completeness}$$

The first agrees with my web quote except for the sign of τ'. Maybe sign does not matter. I know that

$$P^{\mu}_{-\nu-1}(z) = P^{\mu}_{\nu}(z)$$

Now suppose ν = -1/2+iτ. Then

$$-\nu-1 = -(-1/2+i\tau) - 1 = -i\tau - 1/2$$

and then

$$P^{\mu}_{-i\tau-1/2}(z) = P^{\mu}_{-1/2+i\tau}(z)$$

and thus the sign on τ does not matter! Good. Then I have an agreement.

Now express using y = chξ. Start with

$$\int_1^\infty dy\ P^m_{-1/2+i\tau}(y)\ P^m_{-1/2-i\tau'}(y) = \delta(\tau-\tau') / s_m(\tau) \qquad \text{orthogonality}$$

$$\int_0^\infty d\tau\ s_m(\tau)\ P^m_{-1/2+i\tau}(y)\ P^m_{-1/2+i\tau}(y') = \delta(y-y') \qquad \text{completeness}$$

then dy = sh(ξ)dξ and ξ ranges from 0 to ∞ so

$$\int_0^\infty d\xi\ sh\xi\ P^m_{-1/2+i\tau}(ch\xi)\ P^m_{-1/2-i\tau'}(ch\xi) = \delta(\tau-\tau') / s_m(\tau) \qquad \text{orthogonality}$$

$$\int_0^\infty d\tau\ s_m(\tau)\ P^m_{-1/2+i\tau}(ch\xi)\ P^m_{-1/2+i\tau}(ch\xi') = \delta(ch\xi-ch\xi') \qquad \text{completeness}$$

Then how to rewrite that delta?

$$\delta[f(x)-f(x')] = \delta(g(x)) = |g'(x)|^{-1}\delta(x-x') = |f'(x)|^{-1}\delta(x-x')$$

Let f(x) = chx. then f'(x) = shx so then

$$\delta(ch\xi-ch\xi') = \delta(\xi-\xi')/sh\xi$$

and then we have

$$\int_0^\infty d\xi\ sh\xi\ P^m_{-1/2+i\tau}(ch\xi)\ P^m_{-1/2-i\tau'}(ch\xi) = \delta(\tau-\tau') / s_m(\tau) \qquad \text{orthogonality}$$

$$\int_0^\infty d\tau\ s_m(\tau)\ P^m_{-1/2+i\tau}(ch\xi)\ P^m_{-1/2+i\tau}(ch\xi') = \delta(\xi-\xi')/sh\xi \qquad \text{completeness}$$

OK, let's write this up in another Dan document. DONE/.

Dec 31, 2013 6 PM. I sent Dan a note saying I have run out of time on his stuff, and asking what he is working on. He likes to ask questions and does not seem to answer my questions, so that is another reason I am cooled off a bit. I need to get back to lines doc now, and also do an update of Stackel.