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Although I have fiddled with toroidal coordinates as you have seen (for the Laplace equation), I certainly am not an expert in this subject. However, I am quite sure of item 1 below, which is not what you want to hear! Below are some notes which probably are not of much value. It is an interesting problem and I presume it has a reasonable numerical solution.

1. The (scalar) Helmholtz equation is not separable in toroidal coordinates. This is confirmed on page 101 of Moon and Spencer (see next page). It is only separable if $k^2 = 0$ in which case it is really the Laplace equation and then all the stuff of my bowl paper applies. I suppose one could do perturbation theory for small k^2 but that sounds pretty horrible. (The *vector* Helmholtz equation is then even *more* non-separable!) Here is a shot of the Moon and Spencer page where S means Simple-separable ($R=1$), R means R-separable ($R \neq 1$), and **X** means *not* separable: ($g_{11} = h_1^2$) [toroidal 4th up from the bottom]

Separability				One-dimensional Solution	
HELMHOLTZ		LAPLACE		HELMHOLTZ	LAPLACE
$\psi(u^1, u^2, u^3)$	$\psi(u^1, u^2)$	$\psi(u^1, u^2, u^3)$	$\psi(u^1, u^2)$		

Table 4.02. Continuation

No.	Name of Coordinate System	Fig. No.	Equations	Metric Coefficients	Separability				One-dimensional Solution		α_i
					HELMHOLTZ	LAPLACE	HELMHOLTZ	LAPLACE	HELMHOLTZ	LAPLACE	
P4R	Hyperbolic	2.04 Rot. about y axis 59	$x = (\varrho + \mu)^{1/2} \cos \psi$ $y = (\varrho + \mu)^{1/2} \sin \psi$ $z = (\varrho - \mu)^{1/2}$ where $\varrho = +(\mu^2 + r^2)^{1/2}$	$E_{11} = E_{22} = \frac{1}{2}(\mu^2 + r^2)^{-1}$ $E_{33} = \varrho + \mu$	×	×	×	×	×	×	-
E1R	Spherical	1.05 Rot. about x axis 61	$x = e^\psi \sin \theta \cos \varphi$ $y = e^\psi \sin \theta \sin \varphi$ $z = e^\psi \cos \theta$	$E_{11} = E_{22} = e^{2\psi}$ $E_{33} = e^{2\psi} \sin^2 \theta$	S	S	S	S	•	•	0
E4Rx	Bispherical	4.03 Rot. about x axis 64	$x = \frac{a \sin \theta \cos \varphi}{\cosh \eta - \cos \theta}$ $y = \frac{a \sin \theta \sin \varphi}{\cosh \eta - \cos \theta}$ $z = \frac{a \sinh \eta}{\cosh \eta - \cos \theta}$	$E_{11} = E_{22} = \frac{a^2}{(\cosh \eta - \cos \theta)^2}$ $E_{33} = \frac{a^2 \sinh^2 \eta}{(\cosh \eta - \cos \theta)^2}$	×	×	R	R	×	×	$-\frac{1}{2}$
E4Ry	Toroidal	4.04 Rot. about y axis 64	$x = \frac{a \sinh \eta \cos \varphi}{\cosh \eta - \cos \theta}$ $y = \frac{a \sinh \eta \sin \varphi}{\cosh \eta - \cos \theta}$ $z = \frac{a \sin \theta}{\cosh \eta - \cos \theta}$	$E_{11} = E_{22} = \frac{a^2}{(\cosh \eta - \cos \theta)^2}$ $E_{33} = \frac{a^2 \sinh^2 \eta}{(\cosh \eta - \cos \theta)^2}$	×	×	R	R	×	×	$\frac{1}{2}$
H1Rx	Prolate spheroidal	1.06 Rot. about x axis 65	$x = a \sinh \eta \sin \theta \cos \varphi$ $y = a \sinh \eta \sin \theta \sin \varphi$ $z = a \cosh \eta \cos \theta$	$E_{11} = E_{22} = a^2(\cosh^2 \eta - \cos^2 \theta)$ $= a^2(\sinh^2 \eta + \sin^2 \theta)$ $E_{33} = a^2 \sinh^2 \eta \sin^2 \theta$	S	S	S	S	×	•	0
H1Ry	Oblate spheroidal	1.07 Rot. about y axis* 65	$x = a \cosh \eta \sin \theta \cos \varphi$ $y = a \cosh \eta \sin \theta \sin \varphi$ $z = a \sinh \eta \cos \theta$	$E_{11} = E_{22} = a^2(\cosh^2 \eta - \sin^2 \theta)$ $E_{33} = a^2 \cosh^2 \eta \sin^2 \theta$	S	S	S	S	×	•	0
2Rx	Inversion of prolate spheroidal	4.05 Rot. about x axis 70	$x = \frac{a \sinh \eta \sin \theta \cos \varphi}{\cosh^2 \eta - \sin^2 \theta}$ $y = \frac{a \sinh \eta \sin \theta \sin \varphi}{\cosh^2 \eta - \sin^2 \theta}$ $z = \frac{a \cosh \eta \cos \theta}{\cosh^2 \eta - \sin^2 \theta}$	$E_{11} = E_{22} = \frac{a^2(\sinh^2 \eta + \sin^2 \theta)}{(\cosh^2 \eta - \sin^2 \theta)^2}$ $E_{33} = \frac{a^2 \sinh^2 \eta \sin^2 \theta}{(\cosh^2 \eta - \sin^2 \theta)^2}$	×	×	R	R	×	×	0

2. The *Laplace* equation is "R-separable" in toroidal coordinates with the "atom choices" shown in my bowl paper, section 2 (b) :

$$\xi \text{ in } (0, \infty) \quad u \text{ in } (0, 2\pi) \quad \phi \text{ in } (0, 2\pi)$$

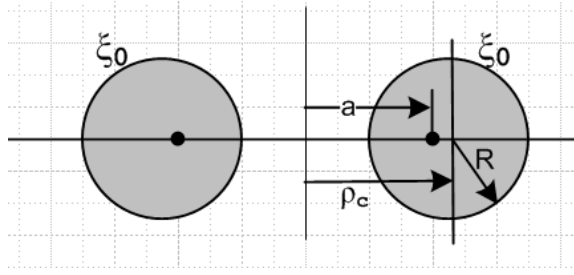
$$(1) \quad \sqrt{\text{ch}\xi - \text{cos}u} \quad \begin{matrix} \text{expo} \\ [P_{n-1/2}^m(\text{ch}\xi), Q_{n-1/2}^m(\text{ch}\xi)] \end{matrix} \quad \begin{matrix} \text{osc} \\ [\sin(nu), \cos(nu)] \end{matrix} \quad \begin{matrix} \text{osc} \\ [\sin(m\phi), \cos(m\phi)] \end{matrix}$$

$$(2) \quad \sqrt{\text{ch}\xi - \text{cos}u} \quad \begin{matrix} \text{osc} \\ [P_{i\tau-1/2}^m(\text{ch}\xi), Q_{i\tau-1/2}^m(\text{ch}\xi)] \end{matrix} \quad \begin{matrix} \text{expo} \\ [\exp(\tau u), \exp(-\tau u)] \end{matrix} \quad \begin{matrix} \text{osc} \\ [\sin(m\phi), \cos(m\phi)] \end{matrix}$$

The "R-separable" means that there is a non-separable overall function sitting in all solutions, and here that function is $\sqrt{\text{ch}\xi - \text{cos}u} = 1/R$. This is a Moon and Spencer term.

But of course this is not what you want.

3. Here is a picture showing a torus cross section taken from my bowl paper (this is a different R)



$$\rho_c = a \coth \xi_0 \quad R = a / \text{sh} \xi_0 \quad \Rightarrow \quad \sqrt{\rho_c^2 - R^2} = a, \quad \rho_c / R = \text{ch} \xi_0 \quad (10.9)$$

Based on this, I think your desired "ring source" could be written

$$\text{source} = b' \delta(a \coth \xi_0 - \rho_0) \quad // \delta(\rho - \rho_c) \text{ in cylindricals}$$

and then your desired Green's Function problem is something like this (I think $u' = 0$ for your ring source location)

$$(\nabla^2 + k^2) g(\xi, u | \xi_0, u') = b' \delta(a \coth \xi_0 - \rho_0) \quad g(\xi, u | \xi_0, u') = \text{constant} \quad g(\infty) = 0$$

where ∇^2 needs to be expressed in toroidal coordinates. I am not sure one would call this a Green's Function, it is more the Helmholtz equation with a ring source (I am used to point sources for Green). I suppose then

$$\begin{aligned} \delta(a \coth \xi' - \rho_0) &= b' \delta(\xi' - \xi_0) / |\partial_{\xi'}(a \coth \xi')| = b' \delta(\xi' - \xi_0) / [a \text{csch}^2 \xi'] = b' \text{sh}^2(\xi_0) \delta(\xi' - \xi_0) / a \\ &\equiv b \delta(\xi' - \xi_0) \end{aligned}$$

so that

$$(\nabla^2 + k^2) g(\xi, u | \xi', u') = b \delta(\xi' - \xi_0) \quad // u' = 0$$

which gets things closer to your emailed form,

Specifically, solving for G where $(\text{Laplacian} - k^2)(G) = b \cdot \text{Delta}(\text{torus coordinate} - \text{axis of torus})$, where b, k^2 are positive, real, known constants, and G is the unknown Green's function.

Here ξ_0 is a label for the torus having the axis of interest. Also I have a different sign for k^2 , just a convention.

4. Since your ring source lies inside the torus ξ' and you want $g = \text{constant}$ on the torus (inner) surface, you are doing an "interior problem" inside the torus. I am not sure that you are then allowed to require also that $g(\infty) = 0$ which seems part of a problem exterior to the torus. That is, you may be over-specifying the problem.

5. Here are four different notations for the toroidal coordinates that I know of:

My Stackel paper:	(ξ_1, ξ_2, ξ_3)	toroid, bowl, azimuthal plane	
Moon & Spencer	(η, θ, ψ)	toroid, bowl, azimuthal plane	
My bowl paper	(ξ, u, ϕ)	toroid, bowl, azimuthal plane	p 101
Morse and Feshbach:	(μ, θ, ϕ)	toroid, bowl, azimuthal plane	Vol 1 p 666

On p 666 M&F complicate the issue by using $\xi_1 = \cosh(\mu)$, $\xi_2 = \cos\theta$ and $\xi_3 = \cos\phi$. These are different ξ_i from my Stackel document shown above. I think they regard toroidal coordinates as some kind of limit of the ellipsoidal ones.

6. I tried to get Maple to compute scalar ∇^2 using the (ξ_1, ξ_2, ξ_3) toroidal notation and

$$\nabla^2 f(\xi_1, \xi_2) = 1/(h_1 h_2 h_3) \{ \partial_1 [(h_2 h_3 / h_1) \partial_1 f(\xi_1, \xi_2)] + \text{cyclic} \}$$

where I assume azimuthal symmetry so no ξ_3 dependence. The scale factors are

$$h_1 = h_2 = a / (\cosh \xi_1 - \cos \xi_2)$$

$$h_3 = \sinh \xi_1 \sin \xi_2$$

Here is my result,

T1 := Diff((h2*h3/h1)*Diff(g(xi1,xi2),xi1),xi1);

$$T1 := \frac{\partial}{\partial \xi_1} \frac{h_2 h_3 \left(\frac{\partial}{\partial \xi_1} g(\xi_1, \xi_2) \right)}{h_1}$$

T2 := Diff((h3*h1/h2)*Diff(g(xi1,xi2),xi2),xi2);

$$T2 := \frac{\partial}{\partial \xi_2} \frac{h_3 h_1 \left(\frac{\partial}{\partial \xi_2} g(\xi_1, \xi_2) \right)}{h_2}$$

T3 := Diff((h1*h2/h3)*Diff(g(xi1,xi2),xi3),xi3);

$$T3 := \frac{\partial}{\partial \xi_3} \frac{h_1 h_2 \left(\frac{\partial}{\partial \xi_3} g(\xi_1, \xi_2) \right)}{h_3}$$

del2g := (h1*h2*h3)^(-1)*(T1+T2+T3): simplify(%);

$$\frac{\left(\frac{\partial}{\partial \xi_1} \frac{h_2 h_3 \left(\frac{\partial}{\partial \xi_1} g(\xi_1, \xi_2) \right)}{h_1} \right) + \left(\frac{\partial}{\partial \xi_2} \frac{h_3 h_1 \left(\frac{\partial}{\partial \xi_2} g(\xi_1, \xi_2) \right)}{h_2} \right) + \left(\frac{\partial}{\partial \xi_3} \frac{h_1 h_2 \left(\frac{\partial}{\partial \xi_3} g(\xi_1, \xi_2) \right)}{h_3} \right)}{h_1 h_2 h_3}$$

> h2 := h1:

> h3 := sinh(xi1)*h1;

$$h_3 := \sinh(\xi_1) h_1$$

> h1 := a/(cosh(xi1)-cos(xi2));

$$h_1 := \frac{a}{\cosh(\xi_1) - \cos(\xi_2)}$$

> d := collect(value(del2g),diff):

> del2 := subs({xi1 = xi,xi2 = u},d);

$$\begin{aligned} del2 := & - \frac{(\cosh(\xi) - \cos(u)) \sin(u) \left(\frac{\partial}{\partial u} g(\xi, u) \right)}{a^2} + \frac{(\cosh(\xi) - \cos(u))^2 \left(\frac{\partial^2}{\partial u^2} g(\xi, u) \right)}{a^2} \\ & + \frac{(\cosh(\xi) - \cos(u))^3 \left(- \frac{a \sinh(\xi)^2}{(\cosh(\xi) - \cos(u))^2} + \frac{a \cosh(\xi)}{\cosh(\xi) - \cos(u)} \right) \left(\frac{\partial}{\partial \xi} g(\xi, u) \right)}{a^3 \sinh(\xi)} \\ & + \frac{(\cosh(\xi) - \cos(u))^2 \left(\frac{\partial^2}{\partial \xi^2} g(\xi, u) \right)}{a^2} \end{aligned}$$

So this expression is $\nabla^2 g$ (unless I screwed up, which I often do) and your equation is then,

$$(\nabla^2 + k^2) g(\xi, u | \xi', u') = b \delta(\xi' - \xi_0) \quad g(\xi, u | \xi_0, u') = \text{constant} \quad g(\infty) = 0$$

which I agree is not very nice. It really makes me think this is not such a good way to approach the problem. Maybe that small k^2 perturbation theory is a better path, where one then starts with the known Laplace toroidal harmonics and can perhaps work up a solution for k^2 not so small. Or maybe model the torus as a polygonal structure in *cylindrical* coordinates (if nothing else, a thick annular ring).

So this is all nothing new to you I am sure. Sorry that I have no magic bullet!

Maybe those Tokamak people know about this stuff. Or smoke-ring researchers ?

Best regards,
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12/29/13