

1. Since the ring source includes a factor  $\delta(\xi-\xi')$ , one wants the  $\xi$  variable to be "oscillatory" in the atomic form expansion. Thus, the following form seems appropriate :

$$\sqrt{\text{ch}\xi - \text{cosu}} \quad \begin{matrix} \text{osc} & & \text{expo} & & \text{osc} \\ \text{[P}^{\mathbf{m}}_{-1/2+i\tau}(\text{ch}\xi) \text{]} & \text{[exp}(\tau\text{u}), \text{exp}(-\tau\text{u}) \text{]} & \text{[ sin(m}\phi\text{), cos(m}\phi\text{)]} \end{matrix}$$

The Q function is rejected because it does not vanish as  $r \rightarrow \infty$  (which means  $\xi$  and  $u$  are very small, so one ends up with  $Q^{\mathbf{m}}_{i\tau-1/2}(\text{ch}\xi) = Q^{\mathbf{m}}_{i\tau-1/2}(1) = \infty$ .) The function  $P^{\mathbf{m}}_{-1/2+i\tau}(y)$  is called a Mehler function and is defined for  $y \geq 1$ . It is of course just a Legendre function.

One could also write  $[\text{sh}(\tau\text{u}), \text{ch}(\tau\text{u})]$  for the  $u$  atoms.

In a problem which has full azimuth  $\phi$ , the separation constant  $m$  is an integer.

Since  $P^{\mu}_{-\nu-1}(z) = P^{\mu}_{\nu}(z)$  for any  $\mu$  and  $\nu$ , one finds that

$$P^{\mathbf{m}}_{-1/2-i\tau}(\text{ch}\xi) = P^{\mathbf{m}}_{-1/2+i\tau}(\text{ch}\xi)$$

so the sign of  $i\tau$  does not matter. You can plot these Mehler functions and they are perfectly reasonable oscillatory functions.

2. There exists a transform called the generalized Mehler-Fock transform (reference in the bowl doc)

$$g(y) = \int_0^{\infty} d\tau P^{\mathbf{m}}_{-1/2+i\tau}(y) f(\tau) \quad // \text{ expansion}$$

$$f(\tau) = (\tau/\pi) \text{sh}(\pi\tau) \Gamma(1/2-m+i\tau) \Gamma(1/2-m-i\tau) \int_1^{\infty} dy P^{\mathbf{m}}_{-1/2+i\tau}(y) g(y) \quad // \text{ projection}$$

where  $m$  is fixed. It is convenient to define

$$s_{\mathbf{m}}(\tau) \equiv (\tau/\pi) \text{sh}(\pi\tau) \Gamma(1/2-m+i\tau) \Gamma(1/2-m-i\tau)$$

so the transform becomes

$$g(y) = \int_0^{\infty} d\tau P^{\mathbf{m}}_{-1/2+i\tau}(y) f(\tau) \quad // \text{ expansion}$$

$$f(\tau) = s_{\mathbf{m}}(\tau) \int_1^{\infty} dy P^{\mathbf{m}}_{-1/2+i\tau}(y) g(y) \quad // \text{ projection}$$

3. Associated with any Sturm-Liouville transform is an orthogonality and completeness relation. By inserting the first in the second, and then the second in the first (of the above two equations) one finds that

$$\int_1^{\infty} dy P^{\mathbf{m}}_{-1/2+i\tau}(y) P^{\mathbf{m}}_{-1/2-i\tau'}(y) = \delta(\tau-\tau') / s_{\mathbf{m}}(\tau) \quad // \text{ orthogonality}$$

$$\int_0^{\infty} d\tau s_{\mathbf{m}}(\tau) P^{\mathbf{m}}_{-1/2+i\tau}(y) P^{\mathbf{m}}_{-1/2+i\tau'}(y') = \delta(y-y') \quad // \text{ completeness}$$

Replacing  $y = \text{ch}\xi$  and  $y' = \text{ch}\xi'$  we can rewrite these as

$$\int_0^\infty d\xi \text{sh}\xi P_{-1/2+i\tau}^m(\text{ch}\xi) P_{-1/2+i\tau'}^m(\text{ch}\xi) = \delta(\tau-\tau') / s_m(\tau) \quad // \text{ orthogonality}$$

$$\int_0^\infty d\tau s_m(\tau) P_{-1/2+i\tau}^m(\text{ch}\xi) P_{-1/2+i\tau}^m(\text{ch}\xi') = \delta(\text{ch}\xi - \text{ch}\xi') = \delta(\xi - \xi') / \text{sh}\xi \quad // \text{ completeness}$$

One then has the required eigenfunction expansion to use in the Green's function source,

$$\delta(\xi - \xi') = \text{sh}\xi \int_0^\infty d\tau s_m(\tau) P_{-1/2+i\tau}^m(\text{ch}\xi) P_{-1/2+i\tau}^m(\text{ch}\xi')$$

4. This transform exists for any  $m$ . For  $m = 0$  the transform is called the (regular) Mehler-Fock transform. The function  $s_m(\tau)$  simplifies since

$$\Gamma(1/2+i\tau) \Gamma(1/2-i\tau) = \pi / \text{ch}(\pi\tau) \quad // \text{ Euler's reflection formula } \Gamma(z) \Gamma(1-z) = \pi / \sin(\pi z)$$

so that

$$s_0(\tau) \equiv (\tau/\pi) \text{sh}(\pi\tau) * \pi / \text{ch}(\pi\tau) = \tau \text{th}(\pi\tau) .$$

In this case we get

$$\int_0^\infty d\xi \text{sh}\xi P_{-1/2+i\tau}(\text{ch}\xi) P_{-1/2+i\tau'}(\text{ch}\xi) = \delta(\tau-\tau') / [\tau \text{th}(\pi\tau)] \quad \text{orthogonality}$$

$$\int_0^\infty d\tau \tau \text{th}(\pi\tau) P_{-1/2+i\tau}(\text{ch}\xi) P_{-1/2+i\tau}(\text{ch}\xi') = \delta(\text{ch}\xi - \text{ch}\xi') = \delta(\xi - \xi') / \text{sh}\xi \quad \text{completeness}$$

so that

$$\delta(\xi - \xi') = \text{sh}\xi \int_0^\infty d\tau \tau \text{th}(\pi\tau) P_{-1/2+i\tau}(\text{ch}\xi) P_{-1/2+i\tau}(\text{ch}\xi') .$$