

1. Ring source in cylindrical coordinates

In a potential theory problem (Laplace equation) and in electrostatics language, a ring source of total charge q in cylindrical coordinates with radius ρ_c and at $z = 0$ would be written as

$$\text{source} = q \delta(z) \delta(\rho - \rho_c) / [2\pi\rho] .$$

The reason for the $2\pi\rho$ is that we need the 3D integral of the source to be q :

$$\int d\phi \int \rho d\rho \int dz \ q \delta(z) \delta(\rho - \rho_c) / [2\pi\rho]$$

$$= 2\pi \int \rho d\rho \int dz \ q \delta(z) \delta(\rho - \rho_c) / [2\pi\rho]$$

$$= q \int d\rho \int dz \ \delta(z) \delta(\rho - \rho_c) = q$$

The problem then is to express this ring source in toroidal coordinates.

2. Digression on the equations for the toroidal coordinate level curves

(I have derived these equations elsewhere and they probably appear somewhere in M&F or elsewhere.)

horizontal circles (tori cross sections, let ρ = horizontal coordinate, z = vertical coordinate)

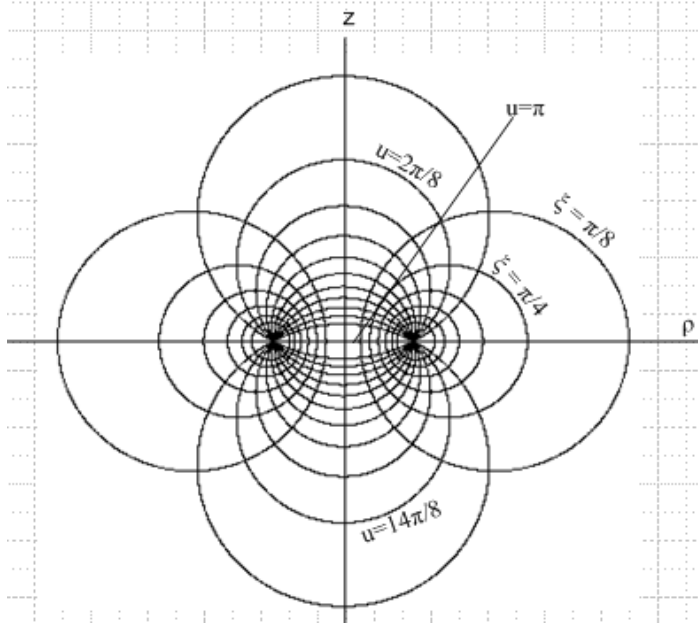
$$(\rho - a \coth \xi)^2 + z^2 = a^2 / \sinh^2 \xi \qquad \rho_c = a \coth \xi \qquad R = a / |\sinh \xi|$$

(Historically (200BC), the horizontal circles are called the "circles of Appollonius" and they have some very simple properties, notably, $|\mathbf{r} - \mathbf{a}| / |\mathbf{r} + \mathbf{a}| = e^{-\xi}$.)

vertical circles (when truncated at $z=0$, they give the bowls)

$$\rho^2 + (z - a \cot u)^2 = a^2 / \sin^2 u \qquad z_c = a \cot(u) \qquad R = a / |\sin u|$$

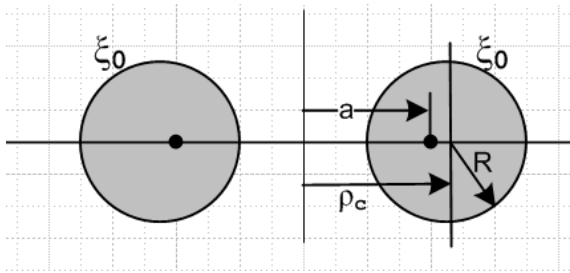
The picture going with these equations showing the various circles is this



Remember that you rotate this bipolar 2D system around the vertical z axis by ϕ to get toroidals.

3. Geometry associated with the ring source

Now consider the following picture, where on the right is one of our "horizontal circles" being a cross section of a gray toroid,



We know at once from the previous section that the center of the circle is at $\rho_c = a \coth(\xi_0)$. We also know that $R = a/|\text{sh}\xi_0|$. Then

$$\rho_c^2 - R^2 = a^2 \coth^2(\xi_0) - a^2/(\text{sh}\xi_0)^2 = a^2[\coth^2(\xi_0) - \text{csch}^2(\xi_0)] = a^2$$

$$\rho_c/R = a \coth(\xi_0) * \text{sh}\xi_0/a = \text{ch}(\xi_0) \quad // \ 0 < \xi_0 < \infty \text{ in normal usage}$$

We have now derived this set of equations which appeared in my earlier email and which appear in the bowl paper,

$$\rho_c = a \coth \xi_0 \quad R = a / \text{sh} \xi_0 \quad \Rightarrow \quad \sqrt{\rho_c^2 - R^2} = a, \quad \rho_c/R = \text{ch} \xi_0 \quad (10.9)$$

Also, we can get an expression for cylindrical ρ expressed in toroidals as follows (these x,y,z equations are those which in fact define toroidal coordinates (ξ,u,φ))

$$\begin{aligned} x &= a \cos\varphi \operatorname{sh}\xi / (\operatorname{ch}\xi - \cos u) &= \cos\varphi h_\varphi \\ y &= a \sin\varphi \operatorname{sh}\xi / (\operatorname{ch}\xi - \cos u) &= \sin\varphi h_\varphi \\ z &= a \sin u / (\operatorname{ch}\xi - \cos u) &= \sin u h_\xi \end{aligned}$$

we find that

$$\rho^2 = x^2 + y^2 = a^2 \operatorname{sh}^2 \xi / (\operatorname{ch}\xi - \cos u)^2 \quad \Rightarrow \quad \rho = a \operatorname{sh}\xi / (\operatorname{ch}\xi - \cos u) .$$

The inverse equations are given by

$$\begin{aligned} \xi &= \tanh^{-1} [2ap / (a^2 + \rho^2 + z^2)] \\ \tan(u) &= [-2az / (a^2 - \rho^2 - z^2)] & // \text{ note that } z = 0 \Rightarrow u = 0 \text{ as one possibility} \\ \tan\varphi &= y/x \end{aligned}$$

However, one really needs to execute the following algorithm to find compute ξ, u, φ from x, y, z :

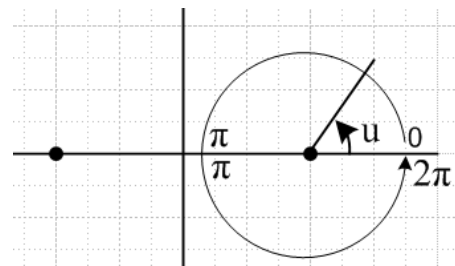
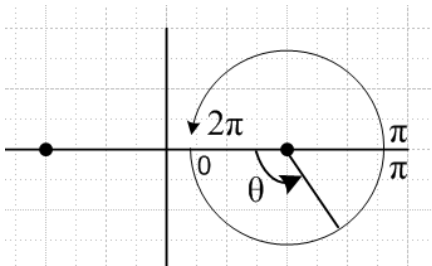
1. Compute φ from the last equation above, in range $(0, 2\pi)$ say. Then $\sin\varphi$ and $\cos\varphi$ are known.
2. Compute ξ from the unambiguous formula shown above involving $\tanh^{-1}$.
3. Compute $\cos u$ from the " $x =$ " formula above.
3. Compute $\sin u$ from the " $z =$ " formula above.
4. Then compute u manually from these values.

The scale factors are given by

$$h_\xi = h_u = a / [\operatorname{ch}\xi - \cos u] \quad h_\varphi = a \operatorname{sh}\xi / [\operatorname{ch}\xi - \cos u] = \operatorname{sh}\xi h_u$$

$$h_\xi h_u h_\varphi = \operatorname{sh}\xi h_\xi^3 = a^3 \operatorname{sh}\xi / [\operatorname{ch}\xi - \cos u]^3 \text{ which will be used below.}$$

Note: M&F deal with this stuff on page 1210. Instead of using our angle u , they use an angle θ . Here is the connection between their θ and our u :



The relationship between u and θ is this:

$$u = (\theta + \pi) \bmod (2\pi)$$

or

$$\begin{aligned} u &= \theta + \pi & \text{for } \theta \text{ in } (0, \pi) \\ u &= \theta - \pi & \text{for } \theta \text{ in } (\pi, 2\pi) \end{aligned}$$

or

$$u = \theta + \pi \operatorname{sign}(\pi - \theta)$$

For any θ in $(0, 2\pi)$ we know the following to be true:

$$\begin{aligned} \sin \theta &= -\sin u \\ \cos \theta &= -\cos u \\ \tan \theta &= +\tan u . \end{aligned}$$

4. Jacobians and final expression for the ring source

We know in spherical coordinates that the Jacobian tells how volume elements transform,

$$dx dy dz = J(r, \theta, \varphi) dr d\theta d\varphi = r^2 \sin \theta dr d\theta d\varphi \quad J(r, \theta, \varphi) = r^2 \sin \theta .$$

Here J is the Jacobian for these curvilinear coordinates and for orthogonal coordinates ξ_1, ξ_2, ξ_3 we have

$$J(\xi_1, \xi_2, \xi_3) = h_1 h_2 h_3 . \quad // \text{ product of the scale factors}$$

In cylindrical coordinates we have

$$dx dy dz = J(\rho, z, \varphi) d\rho dz d\varphi = \rho d\rho dz d\varphi \quad J(\rho, z, \varphi) = \rho$$

In toroidal coordinates we have instead

$$dx dy dz = J(\xi, u, \varphi) d\xi du d\varphi \quad J(\xi, u, \varphi) = h_1 h_2 h_3 = \operatorname{sh} \xi h_1^3 = \operatorname{sh} \xi a^3 / [\operatorname{ch} \xi - \cos u]^3$$

Thus,

$$dx dy dz = a^3 \operatorname{sh} \xi (\operatorname{ch} \xi - \cos u)^{-3} d\xi du d\varphi .$$

We can then get this connection between cylindrical and toroidal coordinates using $dx dy dz = dx dy dz$ "

$$\rho d\rho dz d\varphi = a^3 \operatorname{sh} \xi (\operatorname{ch} \xi - \cos u)^{-3} d\xi du d\varphi . \quad // \text{ note that dimensionally correct}$$

But above we had

$$\rho = a \operatorname{sh} \xi (\operatorname{ch} \xi - \cos u)^{-1}$$

so then

$$a \operatorname{sh} \xi (\operatorname{ch} \xi - \cos u)^{-1} d\rho dz d\varphi = a^3 \operatorname{sh} \xi (\operatorname{ch} \xi - \cos u)^{-3} d\xi du d\varphi$$

or

$$dpdzd\phi = a^2 (\text{ch}\xi - \cos u)^{-2} d\xi du d\phi$$

or

$$dpdz = a^2 (\text{ch}\xi - \cos u)^{-2} d\xi du .$$

Since delta functions work as the inverse of volume elements, we then know that

$$\delta(\rho - \rho_1) \delta(z - z_1) = a^{-2} (\text{ch}\xi - \cos u)^2 \delta(\xi - \xi_1) \delta(u - u_1)$$

where ξ_1 and u_1 are the toroidal coordinates matching ρ_1 and z_1 in cylindrical coordinates.

We are more interested in [from above we know that $1/\rho = (\text{ch}\xi - \cos u)/(\text{ash}\xi)$]

$$\begin{aligned} \delta(z - z_1) \delta(\rho - \rho_1) / [2\pi\rho] &= a^{-2} (\text{ch}\xi - \cos u)^2 \delta(\xi - \xi_1) \delta(u - u_1) * (\text{ch}\xi - \cos u) / (2\pi \text{ash}\xi) \\ &= (1/2\pi) a^{-3} (\text{ch}\xi - \cos u)^3 \delta(\xi - \xi_1) \delta(u - u_1) / \text{sh}\xi \end{aligned}$$

For the desired ring source at the cross hairs of the above drawing, we have

$$\begin{aligned} \rho_1 &= \rho_c & // \rho_c &= a \coth \xi_0 \\ z_1 &= 0 \end{aligned}$$

$$\begin{aligned} u_1 &= 0 & // \text{the ring source lies on the "iris" limit of the bowl} \\ \xi_1 &= 2\xi_0 \end{aligned} .$$

To verify this last equation from our inversion formula above,

$$\begin{aligned} \xi_1 &= \tanh^{-1} [2ap / (a^2 + \rho_1^2 + z_1^2)] = \tanh^{-1} [2ap_c / (a^2 + \rho_c^2)] \\ &= \tanh^{-1} [2a a \coth \xi_0 / (a^2 + a^2 \coth^2 \xi_0)] = \tanh^{-1} [2 \coth \xi_0 / (1 + \coth^2 \xi_0)] \\ &= \tanh^{-1} [2 / (\tanh \xi_0 + \coth \xi_0)] = \tanh^{-1} [2 \tanh \xi_0 / (\tanh^2 \xi_0 + 1)] \\ &= \tanh^{-1} [\tanh(2\xi_0)] \\ &= 2\xi_0 . \end{aligned}$$

Is this really true? To check, compute ρ_1 using formula above:

$$\begin{aligned} \rho_1 &= a \text{sh}\xi_1 / (\text{ch}\xi_1 - \cos u_1) = a \text{sh}(2\xi_0) / (\text{ch}(2\xi_0) - 1) \\ &= a 2 \text{sh}(\xi_0) \text{ch}(\xi_0) / [2\text{sh}^2(\xi_0)] = a \text{ch}(\xi_0) / \text{sh}(\xi_0) = a \coth(\xi_0) \end{aligned}$$

which is correct. Then we have shown that

$$\begin{aligned}
\delta(z-z_1) \delta(\rho-\rho_1) / [2\pi\rho] &= (1/2\pi) a^{-3} (\text{ch}\xi - \cos u)^3 \delta(\xi-\xi_1) \delta(u-u_1) / \text{sh}\xi \\
&= (1/2\pi) a^{-3} (\text{ch}\xi_1 - \cos u_1)^3 \delta(\xi-\xi_1) \delta(u-u_1) / \text{sh}\xi_1 \\
&= (1/2\pi) a^{-3} (\text{ch}\xi_1 - 1)^3 \delta(\xi-\xi_1) \delta(u) / \text{sh}\xi_1 \quad // u_1 = 0 \\
&= (1/2\pi) a^{-3} (2\text{sh}^2(\xi_1/2))^3 \delta(\xi-\xi_1) \delta(u) / [2\text{sh}(\xi_1/2) \text{ch}(\xi_1/2)] \\
&= (1/2\pi) a^{-3} (8\text{sh}^6(\xi_1/2)) \delta(\xi-\xi_1) \delta(u) / [2\text{sh}(\xi_1/2) \text{ch}(\xi_1/2)] \\
&= (1/2\pi) a^{-3} (4\text{sh}^5(\xi_1/2)) \delta(\xi-\xi_1) \delta(u) / [\text{ch}(\xi_1/2)] \\
&= (1/2\pi) a^{-3} (4\text{sh}^5(\xi_0)) \delta(\xi-2\xi_0) \delta(u) / [\text{ch}(\xi_0)] \\
&= (1/2\pi) a^{-3} 4\text{sh}^4(\xi_0) \text{th}(\xi_0) \delta(\xi-2\xi_0) \delta(u) \\
&= (2/\pi) a^{-3} \text{sh}^4(\xi_0) \text{th}(\xi_0) \delta(\xi-2\xi_0) \delta(u) .
\end{aligned}$$

Then the ring source of total charge q expressed in toroidal coordinates is given by

$$\text{source} = q \delta(z) \delta(\rho-\rho_c) / [2\pi\rho]$$

or

$$\text{source} = q (2/\pi) a^{-3} \text{sh}^4(\xi_0) \text{th}(\xi_0) \delta(\xi-2\xi_0) \delta(u)$$

I guess the main fact is that

$$\text{source} = \text{constant} \delta(\xi-2\xi_0) \delta(u)$$

where the details are only needed for normalization with respect to q. I am a little surprised at the $2\xi_0$, maybe it is wrong. I originally thought it would just be $\delta(\xi-\xi_0)$.

The work above is just a "quick shot" and is not fully checked!!