The Mehler-Fock Transform PhL 12.29.15

This plays a crucial role in bowl doc, but I seem never to have derived it. It would be a good addition to bowl doc it not too hard. Here is the full thing

$$g(y) = \int_0^\infty d\tau\, P^m_{-1/2+i\tau}(y)\, f(\tau) \qquad \text{// expansion}$$

$$f(\tau) = (\tau/\pi)\, \mathrm{sh}(\pi\tau)\, \Gamma(1/2-m+i\tau)\, \Gamma(1/2-m-i\tau) \int_1^\infty dy\, P^m_{-1/2+i\tau}(y)\, g(y) \qquad \text{// projection} \qquad (2.4.1)$$

So I presume the interval for y is $(1,\infty)$ and for τ is $(0,\infty)$.

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The differential operator must be the Legendre one somehow. Here is the ODE from GR7

8.700 An *associated Legendre function* is a solution of the differential equation

$$1. \qquad (1-z^2)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right] u = 0,$$

So in my theory I would say (don't know which sign)

$$L = (1-z^2)\partial_z^2 - 2z\partial_z + [\nu(\nu+1)-\mu^2/(1-z^2)]$$

or

$$L = -(z^2-1)\partial_z^2 - 2z\partial_z + [\nu(\nu+1)+\mu^2/(z^2-1)]$$

The general form is

$$L = -\partial_z[p(z)\partial_z] + q(z) = -p(z)\partial_z^2 - (\partial_z p)\partial_z + q$$

So let's try

$$p(z) = (z^2-1) \qquad \partial_z p = 2z \qquad q = [\nu(\nu+1)+\mu^2/(z^2-1)]$$

Then the general form says

$$L = -p(z)\partial_z^2 - (\partial_z p)\partial_z + q$$

$$= -(z^2-1)\partial_z^2 - 2z\partial_z + [\nu(\nu+1)+\mu^2/(z^2-1)]$$

and this matches the Legendre, so we have accomplished the first step. What's next? Let's rewrite using τ so that

$$\nu(\nu+1) = (i\tau-1/2)(i\tau+1/2) = -\tau^2 - 1/4 = -(\tau^2+1/4)$$

Then we have

$$L = -(z^2-1)\partial_z^2 - 2z\partial_z + [\mu^2/(z^2-1) - (\tau^2+1/4)]$$

Think of μ as a bystander variable and τ is the item of interest, like the discrete n.

How do you find the weight function s(x) ? The eigenvalue problem is this

$$L\varphi_n(x) = \lambda_n s(x)\varphi_n(x)$$

which becomes

$$L\,\varphi(\tau,x) = \lambda(\tau)\, s(x)\, \varphi(\tau,x)$$

Aha! Start over now that I recognize the eigenvalue. In this restart we write

$$L = -(z^2-1)\partial_z^2 - 2z\partial_z + \mu^2/(z^2-1)$$

$$p(z) = (z^2-1) \qquad \partial_z p = 2z \qquad q = \mu^2/(z^2-1)$$

Now the EV equation is this

$$L\varphi(\nu,z) = -\nu(\nu+1)s(z)\varphi(\nu,z)$$

But I know that

$$LP_\nu^\mu(z) = -\nu(\nu+1)P_\nu^\mu(z)$$

and then

$$LP^\mu_{i\tau-1/2}(z) = +(\tau^2+1/4)\, P^\mu_{i\tau-1/2}(z) \qquad \text{same for } Q^\mu_{i\tau-1/2}(z)$$

So luckily s(x) = 1 and we can forget about that. so we know that

$$\lambda(\tau) = (\tau^2+1/4) \text{ = the eigenvalue!}$$

Question: are both P and Q in the eigenfunction set? I know that Q diverges at z = 1 so am tempted to thing it is not allowed. So for now, assume only the P are eigenfunctions.

Now we come to orthogonality

$$\int_a^b dx\, s(x)\, \bar{\varphi}_n(x)\, \varphi_m(x) = \delta_{n,m}$$

which becomes

$$\int_1^\infty dz\, [P^\mu_{i\tau-1/2}(z)]^*\ P^\mu_{i\tau'-1/2}(z) = K\, \delta(\tau-\tau')$$

I think $P_{-\nu-1} = P_\nu$ says we can remove the * for real μ (z is real) so looking for

$$\int_1^\infty dz\, P^\mu_{i\tau-1/2}(z)\ P^\mu_{i\tau'-1/2}(z) = K^\mu(\tau)\, \delta(\tau-\tau') \qquad \text{// orthogonality}$$

Plans to determine the normalizing factor

Plan A: integrate orthog over dτ'

Setup: Suppose the above were true. Then integrate both sides on dτ' from 0 to infinity

$$\int_0^\infty d\tau' \int_1^\infty dz\, P^\mu_{i\tau-1/2}(z)\ P^\mu_{i\tau'-1/2}(z) = \int_0^\infty d\tau'\, K^\mu(\tau)\, \delta(\tau-\tau')$$

or

$$\int_1^\infty dz\, P^{\mu}_{i\tau-1/2}(z) \int_0^\infty d\tau'\, P^{\mu}_{i\tau'-1/2}(z) = K^{\mu}(\tau)$$

Now I have the dτ' integral here from PBM

$$\int_0^\infty P^{\mu}_{ix-1/2}(c)\, dx = \sqrt{\frac{\pi}{2}}\, \frac{(c-1)^{-(\mu+1)/2}}{\Gamma(1/2-\mu)}\, (c+1)^{\mu/2} \qquad [\mathrm{Re}\,\mu < 1/2].$$

Therefore,

$$\int_0^\infty d\tau'\, P^{\mu}_{i\tau'-1/2}(z) = \sqrt{\pi/2}\ (z\text{-}1)^{-(\mu+1)/2}\ (1/\Gamma(1/2\text{-}\mu))\ (z\text{+}1)^{\mu/2}$$

I then have

$$K^{\mu}(\tau) = \int_1^\infty dz\, P^{\mu}_{i\tau-1/2}(z)\ [\sqrt{\pi/2}\ (z\text{-}1)^{-(\mu+1)/2}\ (1/\Gamma(1/2\text{-}\mu))\ (z\text{+}1)^{\mu/2}\ ]$$

$$= \sqrt{\pi/2}\ (1/\Gamma(1/2\text{-}\mu))\ \int_1^\infty dz\, P^{\mu}_{i\tau-1/2}(z)(z\text{-}1)^{-(\mu+1)/2}(z\text{+}1)^{\mu/2}$$

Search: Does this integral exist somewhere?

$$(z\text{-}1)^{-(\mu+1)/2}(z\text{+}1)^{\mu/2} = (z\text{-}1)^{-(\mu+1)/2}(z\text{+}1)^{\mu/2} * (z\text{-}1)^{\mu/2}(z\text{-}1)^{-\mu/2}$$

$$= (z^2\text{-}1)^{\mu/2}\ (z\text{-}1)^{-\mu-1/2}$$

so then I am looking for

$$\int_1^\infty dz\, P^{\mu}_{i\tau-1/2}(z)\ (z^2\text{-}1)^{\mu/2}\ (z\text{-}1)^{-\mu-1/2}$$

How about this

7.134

1. $$\int_1^\infty (x-1)^{\lambda-1}\left(x^2-1\right)^{\frac{1}{2}\mu} P_\nu^\mu(x)\, dx = \frac{2^{\lambda+\mu}\,\Gamma(\lambda)\,\Gamma(-\lambda-\mu-\nu)\,\Gamma(1-\lambda-\mu+\nu)}{\Gamma(1-\mu+\nu)\,\Gamma(-\mu-\nu)\,\Gamma(1-\lambda-\mu)}$$
$$[\mathrm{Re}\,\lambda > 0, \quad \mathrm{Re}(\lambda+\mu+\nu) < 0, \quad \mathrm{Re}(\lambda+\mu-\nu) < 1] \quad \text{ET II 321(4)}$$

Set λ-1 = -μ-1/2 so λ = -μ+1/2 so need μ< 1/2 . Just try it.

λ = 1/2-μ λ+μ = 1/2 λ+μ+ν = 1-μ < 0 ouch cannot have μ = 0

-λ-μ-ν = -λ - 1/2 = μ-1/2 - 1/2 = μ-1

$1-\lambda - \mu+\nu = \mu+1/2 - \mu - \nu = 1/2-\nu$

$1-\mu+\nu$ = itself

$-\mu-\nu = -1/2$

$1-\lambda-\mu = \mu+1/2 - \mu = 1/2$

so integral from GR7 is then

$$\sqrt{2}\ \Gamma(1/2-\mu)\Gamma(\mu-1)\Gamma(1/2-\nu) / (\Gamma(1-\mu+\nu)\Gamma(-1/2)\ \Gamma(1/2))$$

So here is what I then have

$$K^{\mu}(\tau) = \sqrt{\pi/2}\ (1/\Gamma(1/2-\mu))\ \sqrt{2}\ \Gamma(1/2-\mu)\Gamma(\mu-1)\Gamma(1/2-\nu) / (\Gamma(1-\mu+\nu)\Gamma(-1/2)\ \Gamma(1/2))$$

now

```
GAMMA(-1/2);
```

$$-2\sqrt{\pi}$$

so rewrite the above as

$$K^{\mu}(\tau) = -\ \sqrt{\pi/2}\ (1/\Gamma(1/2-\mu))\ \sqrt{2}\ \Gamma(1/2-\mu)\Gamma(\mu-1)\Gamma(1/2-\nu) / (\Gamma(1-\mu+\nu)2\sqrt{\pi}\ \sqrt{\pi}\ )$$

$$= -\ \sqrt{\pi/2}\ (1/2\pi)\ \sqrt{2}\ \Gamma(\mu-1)\Gamma(1/2-\nu)/\Gamma(1-\mu+\nu)$$

$$= -\ (1/2\sqrt{\pi}\ )\ \Gamma(\mu-1)\Gamma(1/2-\nu)/\Gamma(1-\mu+\nu)$$ <u>ouch pole at $\mu = 0$</u>

Now set $\nu = i\tau-1/2$. Then

$1/2-\nu = 1/2 - (i\tau-1/2) = 1-i\tau$

$1-\mu+\nu = 1-\mu+(\ i\tau-1/2\ ) = i\tau + 1/2 - \mu$

so then

$$K^{\mu}(\tau) = -\ (1/2\sqrt{\pi}\ )\ \Gamma(\mu-1)\Gamma(1-i\tau)/\Gamma(\ i\tau + 1/2\ - \mu)$$

At least I have *something*! But I was hoping to get

$K^{\mu}(\tau) = 1/f(\tau,\mu)$ where $f(\tau,\mu) = (\tau/\pi)\ \mathrm{sh}(\pi\tau)\ \Gamma(1/2-\mu+i\tau)\ \Gamma(1/2-\mu-i\tau)$

If his were true, I would need

$K^{\mu}(\tau)f(\tau,\mu) = 1$

or

$$- (1/2\sqrt{\pi}\,)\,\Gamma(\mu-1)\Gamma(1-i\tau)/\Gamma(\,i\tau + 1/2\, - \mu) * (\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu+i\tau)\,\Gamma(1/2-\mu-i\tau) = 1$$

or

$$- (1/2\sqrt{\pi}\,)\,\Gamma(\mu-1)\Gamma(1-i\tau) * (\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu-i\tau) = 1$$

Stop. This is singular when $\mu = 0$ due to $\Gamma(\mu-1)$ and I now see that way back above.

Plan B: insert int rep #1 for both P's in orthog (μ=0 only)

But you have to prove this somehow. So we need an integral rep for P. How about

$$P_{-1/2+i\tau}(z) = (\sqrt{2}\,/\pi\,)\,\mathrm{ch}(\pi\tau)\ \int_0^\infty dx\,\cos(\tau x)\,/\sqrt{z+\mathrm{ch}x} \qquad \text{int rep \#1} \qquad \textit{(L.3.5)}$$

Insert this twice to get (doing $\mu = 0$ only here)

$$\int_1^\infty dz\, P_{i\tau-1/2}(z)\ P_{i\tau'-1/2}(z)$$

$$= \int_1^\infty dz[\,(\sqrt{2}\,/\pi\,)\,\mathrm{ch}(\pi\tau)\ \int_0^\infty dx\,\cos(\tau x)\,/\sqrt{z+\mathrm{ch}x}\,]\,[(\sqrt{2}\,/\pi\,)\,\mathrm{ch}(\pi\tau')\ \int_0^\infty dx'\,\cos(\tau' x')\,/\sqrt{z+\mathrm{ch}x'}\,]$$

$$= (\sqrt{2}\,/\pi\,)^2\ \int_0^\infty dx\,\cos(\tau x)\int_0^\infty dx'\,\cos(\tau' x')\int_1^\infty dz\ \ 1/[\sqrt{z+\mathrm{ch}x}\,\sqrt{z+\mathrm{ch}x'}\ ]$$

I suspect that dz integral gives $f(x)\,\delta(x-x')$. If I assume that then I can continue

$$= (\sqrt{2}\,/\pi\,)^2\int_0^\infty dx\,\cos(\tau x)\int_0^\infty dx'\,\cos(\tau' x')\,f(x)\delta(x-x')$$

$$= (\sqrt{2}\,/\pi\,)^2\int_0^\infty dx\,\cos(\tau x)\cos(\tau' x)\,f(x)$$

Now I suspect that $f(x) = f_0 =$ constant and then we get

$$= (\sqrt{2}\,/\pi\,)^2\,f_0\int_0^\infty\ \int_0^\infty dx\,\cos(\tau x)\cos(\tau' x)$$

Now I need to quote the Fourier Cosine Series Transform

$$f(z) = \sqrt{2/\pi}\ \int_0^\infty dk\,\cos(kz)\,f^{k} \qquad \text{// expansion}$$

$$f^{k} = \sqrt{2/\pi}\ \int_0^\infty dz\,\cos(kz)\,f(z) \qquad \text{// projection}$$

$\int_0^\infty$ dz cos(kz) cos(k'z) = (π/2)δ(k-k') // orthogonality

$\int_0^\infty$ dk cos(kz) cos(kz') = (π/2)δ(z-z') // completeness

Using the last two lines, I can continue the above

$= (\sqrt{2}/\pi)^2 f_0(\pi/2)\,\delta(\tau-\tau')$

$= (2/\pi^2)(\pi/2) f_0\,\delta(\tau-\tau')$

$= (1/\pi)\, f_0\,\delta(\tau-\tau')$

So now then do I do this integral

$\int_1^\infty$ dz $1/[\sqrt{z+chx}\,\sqrt{z+chx'}\,]$

Oops, it diverges at the high end, so this path is a no go, wrong integral rep for P! Also, this is an integral of a positive-always integrand, so cannot possibly give 0 when x ≠ x'.

Plan C: insert int rep #2 for both P's in orthog

$\int_1^\infty$ dz $P^{\mu}_{i\tau-1/2}(z)\;P^{\mu}_{i\tau'-1/2}(z) = K\,\delta(\tau-\tau')$ // orthogonality

Need an integral rep for P. Try this one

8.715

1. $$P_\nu^\mu(\cosh\alpha) = \frac{\sqrt{2}\sinh^\mu\alpha}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\mu\right)}\int_0^\alpha \frac{\cosh\left(\nu+\frac{1}{2}\right)t\,dt}{(\cosh\alpha-\cosh t)^{\mu+\frac{1}{2}}}$$

Set ν = iτ-1/2 so then ν+1/2 = iτ. Then we have

$$P^{\mu}_{i\tau-1/2}(ch\alpha) = \frac{\sqrt{2}\,sh^{\mu}\alpha}{\sqrt{\pi}\,\Gamma(1/2-\mu)}\int_0^{\alpha} dt\; ch(i\tau t)\,/\,(ch\alpha-cht)^{\mu+1/2} \qquad \text{int rep \#2}$$

$$= \frac{\sqrt{2}\,sh^{\mu}\alpha}{\sqrt{\pi}\,\Gamma(1/2-\mu)}\int_0^{\alpha} dt\; \cos(\tau t)\,/\,(ch\alpha-cht)^{\mu+1/2}$$

If I try this rep, I get

$\int_1^\infty$ dz $P^{\mu}_{i\tau-1/2}(z)\;P^{\mu}_{i\tau'-1/2}(z)$

$$= \int_1^\infty d(ch\alpha)\ \left(\frac{\sqrt{2}\ sh^\mu\alpha}{\sqrt{\pi}\,\Gamma(1/2-\mu)}\right)^2 \int_0^\alpha dt\ \cos(\tau t) / (ch\alpha - cht)^{\mu+1/2} \int_0^\alpha dt'\ \cos(\tau' t') / (ch\alpha - cht')^{\mu+1/2}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}\,\Gamma(1/2-\mu)} \int_1^\infty dz\ (z^2-1)^{\mu/2}$$
$$\int_0^{\text{arccosh } z} dt\ \cos(\tau t) / (z - cht)^{\mu+1/2} \int_0^{\text{arccosh } z} dt'\ \cos(\tau' t') / (z - cht')^{\mu+1/2}$$

This seems a huge mess, so no go.

Plan D: try int rep #3 (μ=0 only)

I am in over my head, it is not so simple as I hoped. I look on the web. Here is one idea I see

$$P_{-\frac{1}{2}+i\tau}(\cosh x) = \frac{\sqrt{2}}{\pi}\cosh(\pi\tau)\int_0^\infty (\cosh x + \cosh\xi)^{-\frac{1}{2}}\cos(\tau\xi)d\xi$$

and integrating (2.2) by parts, one obtains

$$P_{-\frac{1}{2}+i\tau}(\cosh x) = \frac{\cosh(\pi\tau)}{\sqrt{2}\pi\tau}\int_0^\infty (\cosh x + \cosh\xi)^{-\frac{3}{2}}(\sin(\tau\xi))(\sinh\xi)d\xi.$$

By doing parts, he fixes my convergence problem! Let's assume this last line is valid. Then

$$P_{i\tau-1/2}(z) = ch(\pi\tau)/(\sqrt{2}\,\pi\tau) \int_0^\infty d\xi\ sh\xi\ \sin(\tau\xi)\ (z + ch\xi)^{-3/2} \qquad \text{int rep \#3}$$

where z is still nicely isolated. Then I would need to find this integral

$$\int_1^\infty dz\ (z + ch\xi)^{-3/2}\ (z + ch\xi')^{-3/2}$$

$$= \int_1^\infty dz\ \frac{1}{((z+a)(z+b))^{3/2}}$$

This is a doable integral according to Maple

```
Int(1/((x+a)*(x+b))^(3/2),x=1..infinity);
```

$$\int_1^\infty \frac{1}{((x+a)(x+b))^{\left(\frac{3}{2}\right)}}\,dx$$

```
value(%):simplify(%);
```

$$2\,\frac{-2\sqrt{(b+1)(1+a)}+b+a+2}{\sqrt{(b+1)(1+a)}\,(b^2-2ab+a^2)}$$

So just for fun, let's pursue this a bit

$$\int_1^\infty dz\, P_{i\tau-1/2}(z)\; P_{i\tau'-1/2}(z) =$$

$$= \int_1^\infty dz\, [\, ch(\pi\tau)/(\sqrt{2}\,\pi\tau)\;\; \int_0^\infty d\xi\, sh\xi\, \sin(\tau\xi)\, (z + ch\xi)^{-3/2}]\, [\, ch(\pi\tau')/(\sqrt{2}\,\pi\tau')\;\; \int_0^\infty d\xi'\, sh\xi'\, \sin(\tau'\xi')\, (z + ch\xi')^{-3/2}]$$

$$[ch(\pi\tau)/(\sqrt{2}\,\pi\tau)][\, ch(\pi\tau')/(\sqrt{2}\,\pi\tau')] *$$

$$\int_0^\infty d\xi\, sh\xi\, \sin(\tau\xi) \int_0^\infty d\xi'\, sh\xi'\, \sin(\tau'\xi')\;\; \int_1^\infty dz (z + ch\xi)^{-3/2} (z + ch\xi')^{-3/2}$$

But then I am stuck with three integrals to worry about, and the dz one is a bit ugly. So this seems a hopeless path. Once again, the dz integral cannot give $\delta(\xi-\xi')$ because it is the integral of an all positive integrand, so when $\xi \neq \xi'$ you won't get 0. I think I am doing non-uniform convergence somehow.

Plan E: find out what the normalization factor must be

OK, I give up on this effort for now. Let's go back to the claimed transform itself,

$$g(y) = \int_0^\infty d\tau\, P^{\mu}_{-1/2+i\tau}(y)\, f(\tau) \qquad \text{// expansion}$$

$$f(\tau) = (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-\mu+i\tau)\, \Gamma(1/2-\mu-i\tau) \int_1^\infty dy\, P^{\mu}_{-1/2+i\tau}(y)\, g(y) \qquad \text{// projection} \qquad (2.4.1)$$

My theory above says no s(z) weight functions What happens if I use these in the above transform. OK do the "two ways" thing:

$$g(y) = \int_0^\infty d\tau\, P^{\mu}_{-1/2+i\tau}(y)\, f(\tau)$$

$$= \int_0^\infty d\tau\, P^{\mu}_{-1/2+i\tau}(y)[\, (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-\mu+i\tau)\, \Gamma(1/2-\mu-i\tau) \int_1^\infty dy'\, P^{\mu}_{-1/2+i\tau}(y')\, g(y')]$$

$$= \int_1^\infty dy'\, g(y') \{ \int_0^\infty d\tau\, P^{\mu}_{-1/2+i\tau}(y) P^{\mu}_{-1/2+i\tau}(y')\, (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-\mu+i\tau)\, \Gamma(1/2-\mu-i\tau) \}$$

So the implication here is that

$$\int_0^\infty d\tau\, P^{\mu}_{-1/2+i\tau}(y) P^{\mu}_{-1/2+i\tau}(y')\, (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-\mu+i\tau)\, \Gamma(1/2-\mu-i\tau) \;\; = \delta(y-y')$$

Now go the other way

$$f(\tau) = (\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu+i\tau)\,\Gamma(1/2-\mu-i\tau)\int_1^\infty dy\, P^{\mu}_{-1/2+i\tau}(y)\, g(y)$$

$$= (\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu+i\tau)\,\Gamma(1/2-\mu-i\tau)\int_1^\infty dy\, P^{\mu}_{-1/2+i\tau}(y)\int_0^\infty d\tau'\, P^{\mu}_{-1/2+i\tau'}(y)\, f(\tau')$$

$$= (\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu+i\tau)\,\Gamma(1/2-\mu-i\tau)\int_0^\infty d\tau'\, f(\tau')\,\{\int_1^\infty dy\, P^{\mu}_{-1/2+i\tau}(y)\, P^{\mu}_{-1/2+i\tau'}(y)\}$$

with the implication that

$$(\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu+i\tau)\,\Gamma(1/2-\mu-i\tau)\int_1^\infty dy\, P^{\mu}_{-1/2+i\tau}(y)\, P^{\mu}_{-1/2+i\tau'}(y) = \delta(\tau-\tau')$$

So rewrite this implications:

$$\int_0^\infty d\tau\,[(\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu+i\tau)\,\Gamma(1/2-\mu-i\tau)\,]\, P^{\mu}_{-1/2+i\tau}(y) P^{\mu}_{-1/2+i\tau}(y') = \delta(y-y') \quad \text{// comp}$$

$$\int_1^\infty dy\,[(\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu+i\tau)\,\Gamma(1/2-\mu-i\tau)\,]\, P^{\mu}_{-1/2+i\tau}(y)\, P^{\mu}_{-1/2+i\tau'}(y) = \delta(\tau-\tau') \quad \text{// orthog}$$

So why is there a kernel in there when I thought not from my simple theory? OK, not function of z.

Now define

$$f(\tau,\mu) = (\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu+i\tau)\,\Gamma(1/2-\mu-i\tau)$$

Then here is what things must look like:

$$\int_0^\infty d\tau\, f(\tau,\mu)\, P^{\mu}_{-1/2+i\tau}(y) P^{\mu}_{-1/2+i\tau}(y') = \delta(y-y') \quad \text{// completeness}$$

$$\int_1^\infty dy\, f(\tau,\mu)\, P^{\mu}_{-1/2+i\tau}(y)\, P^{\mu}_{-1/2+i\tau'}(y) = \delta(\tau-\tau') \quad \text{// orthogonality}$$

Now you can remove $f(\tau,\mu)$ from the second line to write

$$\int_0^\infty d\tau\, f(\tau,\mu)\, P^{\mu}_{-1/2+i\tau}(y) P^{\mu}_{-1/2+i\tau}(y') = \delta(y-y') \quad \text{// completeness}$$

$$\int_1^\infty dy P^{\mu}_{-1/2+i\tau}(y)\, P^{\mu}_{-1/2+i\tau'}(y) = \delta(\tau-\tau')/\, f(\tau,\mu) \quad \text{// orthogonality}$$

Comment: Notice that ANY $f(\tau,\mu)$ makes "both directions work", and so "both directions" does not determine $f(\tau,\mu)$

Plan F: Try completeness with int rep #1 and known f(τ,μ)

OK. I have now written up the above in Section J.5 of bowl doc. But now I would like to use

$$P^{\mu}_{-1/2+i\tau}(z) = (\sqrt{2}/\pi)\, ch(\pi\tau) \int_0^{\infty} dx\, \cos(\tau x)/\sqrt{z+chx} \qquad \text{int rep \#1} \qquad (L.3.5)$$

to show that

$$\int_0^{\infty} d\tau\, f(\tau,\mu)\, P^{\mu}_{i\tau-1/2}(z)\, P^{\mu}_{i\tau-1/2}(z') = \delta(z-z')$$

with

$$f(\tau,\mu) = (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-\mu+i\tau)\, \Gamma(1/2-\mu-i\tau)\ .$$

I had trouble in the other direction, maybe I can do this direction?

$$\int_0^{\infty} d\tau\, f(\tau,\mu)\, P^{\mu}_{i\tau-1/2}(z)\, P^{\mu}_{i\tau-1/2}(z')$$

$$= \int_0^{\infty} d\tau\, f(\tau,\mu)$$

$$[(\sqrt{2}/\pi)\, ch(\pi\tau) \int_0^{\infty} dx\, \cos(\tau x)/\sqrt{z+chx}\,]\ [(\sqrt{2}/\pi)\, ch(\pi\tau) \int_0^{\infty} dx'\, \cos(\tau x')/\sqrt{z'+chx'}\,]$$

$$= (\sqrt{2}/\pi)^2 \int_0^{\infty} dx\, (1/\sqrt{z+chx}) \int_0^{\infty} dx'\, (1/\sqrt{z'+chx'}) \int_0^{\infty} d\tau\, f(\tau,\mu)\, \cos(\tau x)\, \cos(\tau x')$$

so again I am faced with a painful dτ integral, namely

$$\int_0^{\infty} d\tau\, (\tau/\pi)\, sh(\pi\tau)\, \Gamma(1/2-\mu+i\tau)\, \Gamma(1/2-\mu-i\tau)\, \cos(\tau x)\, \cos(\tau x')$$

Plan G: Integrate both dies of completeness over z

Assume that

$$\int_0^{\infty} d\tau\, f(\tau,\mu)\, P^{\mu}_{i\tau-1/2}(z)\, P^{\mu}_{i\tau-1/2}(z') = \delta(z-z')$$

Integral both sides 1 to infinity to get

$$\int_1^{\infty} dz' \int_0^{\infty} d\tau\, f(\tau,\mu)\, P^{\mu}_{i\tau-1/2}(z)\, P^{\mu}_{i\tau-1/2}(z') = \int_1^{\infty} dz'\, \delta(z-z')$$

or

$$\int_0^\infty d\tau\, f(\tau,\mu)\, P^{\mu}_{i\tau-1/2}(z) \int_1^\infty dz'\, P^{\mu}_{i\tau-1/2}(z') = 1$$

Now I know that integral (I think). It is not a Mehler integral. But this integral only converges for the Mehler value if ν, so no one has this integral. But it certainly looks like one that could be done.

Comment: The real problem here is that the functions $P^{\mu}_{i\tau-1/2}(z)$ are just completely different from the normal Legendre functions and all the usual methods just don't apply! The integrals in the references are not geared to this conical function.

Plan H: Rosenthal thesis int rep #4

I just looked at Rosenthal's thesis, and he seems interested in this integral rep,

$$3. \qquad P_\nu^{-\mu}(z) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\mu+\frac{1}{2}\right)\left(z^2-1\right)^{\frac{\mu}{2}}}{\Gamma(\nu+\mu+1)\,\Gamma(\mu-\nu)} \int_0^\infty \frac{\cosh\left(\nu+\frac{1}{2}\right)t\,dt}{(z+\cosh t)^{\mu+\frac{1}{2}}}$$
$$[\mathrm{Re}\, z > -1, \quad |\arg(z\pm 1)| < \pi, \quad \mathrm{Re}(\nu+\mu) > -1, \quad \mathrm{Re}(\mu-\nu) > 0] \quad \text{MO 89}$$

though he quotes it from page 156 of HTF 1. Here is his quote

$$(0.2) \quad P^{\mu}_{-\frac{1}{2}+ix}(y) = \sqrt{\frac{2}{\pi}}\,\Gamma(\tfrac{1}{2}-\mu)\left[\Gamma(\tfrac{1}{2}-\mu+ix)\,\Gamma(\tfrac{1}{2}-\mu-ix)\right]^{-1} \cdot (y^2-1)^{\frac{1}{2}\mu} \int_0^\infty (y+\cosh t)^{\mu-\frac{1}{2}} \cos(xt)\,dt, \quad R_e\,\mu < \tfrac{1}{2},\ y > 1$$

Can I connect these two? Well, the first step is to negate μ everywhere in my quote above,

MO Magnus, W. and Oberhettinger, F., *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*, Springer–Verlag, Berlin, 1948.

Here is my negation of the GR7 quote above

$$P_\nu^{\mu}(z) = \sqrt{2/\pi}\ \ \Gamma(1/2-\mu)\,(z^2-1)^{-\mu/2} / [\,\Gamma(\nu-\mu+1)\Gamma(-\mu-\nu)\,] * \int_0^\infty dt\, \cosh[(\nu+1/2)t] / (z+\cht)^{-\mu+1/2}$$

Now set $\nu = i\tau - 1/2$ so that

$$-\mu+\nu+1 = -\mu + (i\tau-1/2)+1 = -\mu+i\tau + 1/2 = 1/2-\mu+i\tau$$

$$-\mu-\nu = -\mu - i\tau + 1/2 = 1/2 - \mu - i\tau$$

$\cosh[(\nu+1/2)t] = \cosh[(i\tau)t] = \cos(\tau t)$

Then my translation of the HTF 1 result is

$$P_{i\tau-1/2}^{\mu}(y) = \sqrt{2/\pi}\ \ \Gamma(1/2-\mu)\,(y^2-1)^{-\mu/2} / [\ \Gamma(1/2-\mu+i\tau)\Gamma(\ 1/2 - \mu -i\tau\ ]$$
$$* \int_0^{\infty} dt\, \cos(\tau t) * (y+\cht)^{\mu-1/2} \qquad \text{int rep \# 4}$$

Comments:
(1) this agrees with Rosenthal except I think he is missing a minus sign I show in red, let's assume this is just a local typo in his thesis.

(2) Now recall that

$$f(\tau,\mu) = (\tau/\pi)\ \sh(\pi\tau)\ \Gamma(1/2-\mu+i\tau)\ \Gamma(1/2-\mu-i\tau)$$

So the good news is that you see exactly that product of gamma functions showing up in his integral rep!! That I think is very significant. Let's now look at the adjusted conditions:

$\mathrm{Re}(z) > -1$	just fine		$\nu = i\tau - 1/2$				
$\mathrm{Re}(\nu-\mu) > -1$	$\Rightarrow$	$-1/2 - \mu > -1$	$\Rightarrow$	$-\mu > -1/2$	$\Rightarrow$	$\mu < 1/2$	ok
$\mathrm{Re}(-\mu-\nu) > 0$	$\Rightarrow$	$-\mu + 1/2 > 0$	$\Rightarrow$	$\mu < 1/2$	same condition!		

So I like these conditions very much. He even writes $\mathrm{Re}(\mu) < 1/2$ and adds $y > 1$.

OK, so we are sitting on this interesting integral representation,

$$P_{i\tau-1/2}^{\mu}(z) = \sqrt{2/\pi}\ \ (z^2-1)^{-\mu/2}\ \frac{\Gamma(1/2-\mu)}{\Gamma(1/2-\mu+i\tau)\Gamma(\ 1/2 - \mu -i\tau\ )}\ \ \int_0^{\infty} dt\, \cos(\tau t) * (z+\cht)^{\mu-1/2} \qquad \texttt{ir \#4}$$

Can this somehow help me in either of these problems:

$$\int_1^{\infty} dz\, P^{\mu}_{i\tau-1/2}(z)\, P^{\mu}_{i\tau'-1/2}(z) = \delta(\tau-\tau') / f(\tau,\mu) \qquad \text{// orthogonality}$$

$$\int_0^{\infty} d\tau\, f(\tau,\mu)\, P^{\mu}_{i\tau-1/2}(z)\, P^{\mu}_{i\tau-1/2}(z') = \delta(z-z') \qquad \text{// completeness}$$

where $f(\tau,\mu) = (\tau/\pi)\ \sh(\pi\tau)\ \Gamma(1/2-\mu+i\tau)\ \Gamma(1/2-\mu-i\tau)$ (J.5.15)

Plan I: Use Rosen int rep #4 to replace just one of the P functions in orthog

It seems that I only want to replace ONE of the P functions with this integral rep. Let's try orthogonality and see what happens:

$$\int_1^\infty dz\, P^{\mu}_{i\tau-1/2}(z)\, P^{\mu}_{i\tau'-1/2}(z)$$

$$= \int_1^\infty dz \left[\sqrt{2/\pi}\ (z^2-1)^{-\mu/2}\ \frac{\Gamma(1/2-\mu)}{\Gamma(1/2-\mu+i\tau)\Gamma(1/2-\mu-i\tau)}\ \int_0^\infty dt\, \cos(\tau t) * (z+cht)^{\mu-1/2}\right] P^{\mu}_{i\tau'-1/2}(z)$$

$$= \sqrt{2/\pi}\ \frac{\Gamma(1/2-\mu)}{\Gamma(1/2-\mu+i\tau)\Gamma(1/2-\mu-i\tau)}\ \int_1^\infty dz\, (z^2-1)^{-\mu/2} \int_0^\infty dt\, \cos(\tau t) * (z+cht)^{\mu-1/2}\right] P^{\mu}_{i\tau'-1/2}(z)$$

$$= \sqrt{2/\pi}\ \frac{\Gamma(1/2-\mu)}{\Gamma(1/2-\mu+i\tau)\Gamma(1/2-\mu-i\tau)}\ \int_0^\infty dt\, \cos(\tau t)\ \int_1^\infty dz\, (z^2-1)^{-\mu/2}\ (z+cht)^{\mu-1/2}\ P^{\mu}_{i\tau'-1/2}(z)$$

Do I have any chance on this integral? I don't see it in GR7. If it came out being cos(τ't) then I would be in fat city!

Plan J: Use Rosen int rep #4 to replace both of the P functions in orthog

Suppose I use the expansion twice? Then we get

$$\int_1^\infty dz\, P^{\mu}_{i\tau-1/2}(z)\, P^{\mu}_{i\tau'-1/2}(z)$$

$$= \int_1^\infty dz\ *$$

$$\sqrt{2/\pi}\ (z^2-1)^{-\mu/2}\ \frac{\Gamma(1/2-\mu)}{\Gamma(1/2-\mu+i\tau)\Gamma(1/2-\mu-i\tau)}\ \int_0^\infty dt\, \cos(\tau t) * (z+cht)^{\mu-1/2}$$

$$\sqrt{2/\pi}\ (z^2-1)^{-\mu/2}\ \frac{\Gamma(1/2-\mu)}{\Gamma(1/2-\mu+i\tau')\Gamma(1/2-\mu-i\tau')}\ \int_0^\infty dt'\, \cos(\tau' t') * (z+cht')^{\mu-1/2}$$

Now I am faced with this integral

$$\int_1^\infty dz\ (z^2-1)^{-\mu}\ (z+cht)^{\mu-1/2}\ (z+cht')^{\mu-1/2}$$

which has this form

$$\int_1^\infty dz\, (z^2-1)^{-\mu}\ [(z+a)(z+b)]^{\mu-1/2}$$

Do I have any chance on this integral? How about for μ = 0. Then it is this

$$\int_1^\infty dz\, [(z+a)(z+b)]^{-1/2}$$

and we have that same convergence issue for μ = 0.

Plan K: Use Rosen int rep #4 to replace one of the P functions in completeness

OK, lets switch over to the completeness condition instead:

$$\int_0^\infty d\tau\, f(\tau,\mu)\, P^{\mu}_{i\tau-1/2}(z)\, P^{\mu}_{i\tau-1/2}(z') = \delta(z-z') \qquad \text{// completeness}$$

and insert his expansion for the first P function, so we get

$$\int_0^\infty d\tau\, f(\tau,\mu)\, P^{\mu}_{i\tau-1/2}(z)\, P^{\mu}_{i\tau-1/2}(z')$$

$$= \int_0^\infty d\tau\, f(\tau,\mu)\left[\sqrt{2/\pi}\ (z^2-1)^{-\mu/2}\,\frac{\Gamma(1/2-\mu)}{\Gamma(1/2-\mu+i\tau)\Gamma(1/2-\mu-i\tau)}\ \int_0^\infty dt\, \cos(\tau t) * (z+\mathrm{cht})^{\mu-1/2}\right] P^{\mu}_{i\tau-1/2}(z')$$

Now

$$f(\tau,\mu)\frac{\Gamma(1/2-\mu)}{\Gamma(1/2-\mu+i\tau)\Gamma(1/2-\mu-i\tau)}$$

$$= (\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu+i\tau)\,\Gamma(1/2-\mu-i\tau)\frac{\Gamma(1/2-\mu)}{\Gamma(1/2-\mu+i\tau)\Gamma(1/2-\mu-i\tau)}$$

$$= (\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu)$$

Then we continue

$$= \sqrt{2/\pi}\ \int_0^\infty d\tau\, (\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu)\left[\,(z^2-1)^{-\mu/2}\ \int_0^\infty dt\, \cos(\tau t) * (z+\mathrm{cht})^{\mu-1/2}\right] P^{\mu}_{i\tau-1/2}(z')$$

$$= \sqrt{2/\pi}\,\Gamma(1/2-\mu)(z^2-1)^{-\mu/2}\ \int_0^\infty d\tau\, \tau\,\mathrm{sh}(\pi\tau)\ \int_0^\infty dt\, \cos(\tau t) * (z+\mathrm{cht})^{\mu-1/2}\Big] P^{\mu}_{i\tau-1/2}(z')$$

$$= \sqrt{2/\pi}\,\Gamma(1/2-\mu)(z^2-1)^{-\mu/2}\ \int_0^\infty dt\, (z+\mathrm{cht})^{\mu-1/2} \int_0^\infty d\tau\, \tau\,\mathrm{sh}(\pi\tau)\cos(\tau t)\, P^{\mu}_{i\tau-1/2}(z')$$

and I am then facing a Mehler integral. It is not one that I have done. It is not in the Boeing book either.

Plan L: Use Rosen int rep #4 to replace one of the P functions in dz integration of completeness

Go back to R's integral rep,

$$P^{\mu}_{i\tau-1/2}(z) = \sqrt{2/\pi}\ \ \Gamma(1/2-\mu)\,(z^2-1)^{-\mu/2} / [\,\Gamma(1/2-\mu+i\tau)\Gamma(1/2-\mu-i\tau\,]$$
$$* \int_0^\infty dt\, \cos(\tau t) * (z+\mathrm{cht})^{\mu-1/2} \qquad \text{int rep \#4}$$

Go back to // $f(\tau,\mu) = (\tau/\pi)\,\mathrm{sh}(\pi\tau)\,\Gamma(1/2-\mu+i\tau)\,\Gamma(1/2-\mu-i\tau)$

$$\int_0^\infty d\tau\ f(\tau,\mu)\ P^{\mu}_{i\tau-1/2}(z)\ P^{\mu}_{i\tau-1/2}(z')\ = \delta(z-z')$$

Now integrate both sides over z' as I did earlier in this doc.,

$$\int_1^\infty dz\ \ \int_0^\infty d\tau\ f(\tau,\mu)\ P^{\mu}_{i\tau-1/2}(z)\ P^{\mu}_{i\tau-1/2}(z')\ = 1$$

or

$$\int_0^\infty d\tau\ f(\tau,\mu)\ P^{\mu}_{i\tau-1/2}(z')\ \ \int_1^\infty dz\ \ P^{\mu}_{i\tau-1/2}(z) = 1$$

or

$$\int_0^\infty d\tau\ P^{\mu}_{i\tau-1/2}(z')\ \left[\ f(\tau,\mu)\ \int_1^\infty dz\ \ P^{\mu}_{i\tau-1/2}(z)\ \right] = 1$$

So does int rep #4 help with this dz integral I could not do above?

$$f(\tau,\mu)\ \int_1^\infty dz\ \ P^{\mu}_{i\tau-1/2}(z)\ =$$

$$= f(\tau,\mu)\ \ \int_1^\infty dz\ \{\ \sqrt{2/\pi}\ \ \Gamma(1/2-\mu)\ (z^2-1)^{-\mu/2}\ /\ [\ \Gamma(1/2-\mu+i\tau)\Gamma(\ 1/2\ -\ \mu\ -i\tau\ ]$$

$$*\ \ \int_0^\infty dt\ \cos(\tau t)\ *\ (z+\mathrm{cht})^{\mu-1/2}\}$$

$$= (\tau/\pi)\ \mathrm{sh}(\pi\tau)\ \ \int_1^\infty dz\ \{\ (z^2-1)^{-\mu/2}\ \ \int_0^\infty dt\ \cos(\tau t)\ \ (z+\mathrm{cht})^{\mu-1/2}\ \ \}$$

$$= (\tau/\pi)\ \mathrm{sh}(\pi\tau)\ \ \int_0^\infty dt\ \cos(\tau t)\ \ \int_1^\infty dz\ \{\ (z^2-1)^{-\mu/2}\ \ (z+\mathrm{cht})^{\mu-1/2}\ \ \}$$

Even if I could do this dz integral, I then have two more integrals dt and then dτ to deal with. I call this just too messy.

So every road I try has a blockade in the form of a messy integral. Go back to,

$$\int_0^\infty d\tau\ f(\tau,\mu)\ P^{\mu}_{i\tau-1/2}(z)\ P^{\mu}_{i\tau-1/2}(z')\ = \delta(z-z')$$

$$f(\tau,\mu) = (\tau/\pi)\ \mathrm{sh}(\pi\tau)\ \Gamma(1/2-\mu+i\tau)\ \Gamma(1/2-\mu-i\tau)$$

I wonder if Maple would get zero if I numerically integrate this for some $z \neq z'$? Let's just try $\mu = 0$

$$\int_0^\infty d\tau\ f(\tau,0)\ P_{i\tau-1/2}(z)\ P_{i\tau-1/2}(z')\ = \delta(z-z')$$

$f(\tau,0) = (\tau/\pi)\ \mathrm{sh}(\pi\tau)\ \Gamma(1/2+i\tau)\ \Gamma(1/2-i\tau) = (\tau/\pi)\ \mathrm{sh}(\pi\tau)\ \pi/\cosh(\pi\tau) = \tau \tanh(\pi\tau)$

since $\Gamma(1/2+i\tau)\Gamma(1/2-i\tau) = \pi/\cosh(\pi\tau)$. Then you want to show that

$$\int_0^\infty d\tau\ \tau \tanh(\pi\tau)\ P_{i\tau-1/2}(z)\ P_{i\tau-1/2}(z') = \delta(z-z')$$

Question: Does this integral converge? I don't have data on the P's asym. I did some Maple plots and I now grok how the above equation works out.

I am giving up on trying to prove Mehler-Fock but my notes in J.5 of bowl doc are OK. It is now 9 PM or so on 12/29/15.

Next day 12/30/15 I added headings to all the above efforts.

I just paged through all of Sneddon. His only integrals with ∞ upper endpoint involve Bessel functions. He does nothing with Mehler. I need a source that is focused on this transform.

Jstor article few clips. They use the following L operator,

In this paper, we study the Mehler-Fock transform by using the method of adjoints. For this purpose, it will be required to introduce two Fréchet spaces V and W satisfying the condition that the linear operator $\mathcal{L} : W \longrightarrow V$ given by

$$(\mathcal{L}\psi)(x) = \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x)\psi(\tau)d\tau, \quad x > 0, \tag{1.2}$$

be continuous.

which basically describes transforming a function $\psi(\tau)$, so I would call this a Mehler integral. He then has a Mahler integral operator, so to speak. So the above defines the $\mathcal{L}$ operator.

They also define this A_x operator

where A_x is the differential operator

$$A_x \equiv D_x^2 + (\coth x)D_x \equiv (\sinh x)^{-1} D_x (\sinh x) D_x,$$

and D_x denotes derivative with respect to the x-variable.

Then we get this fact

Proof. First observe that for all $\tau > 0$,

$$A_x P_{-\frac{1}{2}+i\tau}(\cosh x) = -\left(\frac{1}{4} + \tau^2\right) P_{-\frac{1}{2}+i\tau}(\cosh x).$$

which must be the eigenvalue equation somehow. A_x does not look Legendre to me. I have

$$L = -(z^2-1)\partial_z^2 - 2z\partial_z + \mu^2/(z^2-1)$$

but he is doing $\mu = 0$ so then I have $L = -(z^2-1)\partial_z^2 - 2z\partial_z$. Then

$$(z^2-1)^{-1}L = -\partial_z^2 - 2z(z^2-1)^{-1}\partial_z$$

Now change to $z = ch\xi$ so that $dz = sh\xi d\xi$ and then $\partial_z = (1/sh\xi)\partial_\xi$ then the above reads

$$(1/sh^2\xi)\, L = -((1/sh\xi)\partial_\xi)((1/sh\xi)\partial_\xi) - 2ch\xi\,(1/sh^2\xi)\,(1/sh\xi)\partial_\xi$$

$$= -(1/sh\xi)^2\partial_\xi^2 + (1/sh\xi)(1/sh^2\xi)\, ch\xi\partial_\xi - 2ch\xi\,(1/sh^2\xi)\,(1/sh\xi)\partial_\xi$$

$$= -(1/sh\xi)^2\partial_\xi^2 - (1/sh\xi)(1/sh^2\xi)\, ch\xi\partial_\xi$$

and then

$$L = -\partial_\xi^2 - (1/sh\xi)ch\xi\partial_\xi = -\partial_\xi^2 - coth\xi\,\partial_\xi$$

which is what he shows above. Thus, A_x is really $A_\xi = L$ and so the above really is the Legendre equation and those really are the eigenvalues. Fine.

Now if you apply the A_x operator k times (k = positive integer) they show that

$$(2.1) \qquad A_x^k\,(\mathcal{L}\psi)\,(x) = (-1)^k \int_0^\infty \left(\frac{1}{4}+\tau^2\right)^k P_{-\frac{1}{2}+i\tau}(\cosh x)\psi(\tau)d\tau.$$

which seems correct to me.

Next, he rolls out the same Rosenthal integral rep and parts it

Using the integral representation (cf. [1, (11), p. 156]), which is valid for all $x > 0$ and all $\tau > 0$,

(2.2)

$$P_{-\frac{1}{2}+i\tau}(\cosh x) = \frac{\sqrt{2}}{\pi}\cosh(\pi\tau)\int_0^\infty (\cosh x + \cosh\xi)^{-\frac{1}{2}}\cos(\tau\xi)d\xi$$

and integrating (2.2) by parts, one obtains

$$P_{-\frac{1}{2}+i\tau}(\cosh x) = \frac{\cosh(\pi\tau)}{\sqrt{2}\pi\tau}\int_0^\infty (\cosh x + \cosh\xi)^{-\frac{3}{2}}(\sin(\tau\xi))(\sinh\xi)d\xi.$$

Next step is to claim

Substituting this expression in (2.1) and reversing the order of integration, it follows that

$$(2.3) \quad A_x^k(\mathcal{L}\psi)(x) = \frac{(-1)^k}{\sqrt{2}\pi}\int_0^\infty (\cosh x + \cosh\xi)^{-\frac{3}{2}}(\sinh\xi) \times \left(\int_0^\infty \tau^{-1}\cosh(\pi\tau)\left(\frac{1}{4}+\tau^2\right)^k \psi(\tau)\sin(\tau\xi)d\tau\right)d\xi.$$

where he uses that second parts rep thing and applies A_x k times. They then say

Note that the integral into the brackets is the Fourier sine transform of an odd function of the τ-variable, and consequently it is the Fourier transform of this function.

Therefore,

$$(2.4)\quad A_x^k(\mathcal{L}\psi)(x) = \frac{(-1)^k}{\sqrt{2\pi}} \int_0^\infty (\cosh x + \cosh \xi)^{-\frac{3}{2}} (\sinh \xi) \times \left(\mathcal{F}\left[\tau^{-1} \cosh(\pi\tau) \left(\frac{1}{4} + \tau^2 \right)^k \psi(\tau) \right] \right)(\xi)\, d\xi.$$

so that $\mathcal{F}$ means a full fourier transform. I wonder where this is all leading! They are defining a very special thread! The have somehow then shown in all this that $\mathcal{L}$ is continuous and that is the end of their section 2. We now start into section 3:

Section 3. Define the following ψ_ϕ operator.

$$(3.1)\qquad \psi_\phi(\tau) = \tau \tanh(\pi\tau) \int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x)\phi(x)dx,$$

for any $\phi \in V$ and all $\tau > 0$.

So this is basically just this definition,

$\psi_\phi(x) = \pi\tanh(\pi\tau) \int_0^\infty d(chx) P_{i\tau-1/2}(chx)\, \varphi(x)$.

They now define A_x' as the "adjoint" of A_x I guess in the Stak sense, so

$$A_x' = D_x(\sinh x) D_x(\sinh x)^{-1},$$

Then using this they claim,

$$A_x'\left[(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \right] = -\left(\frac{1}{4} + \tau^2 \right) (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x),$$

and then

$$(3.2)\qquad (-1)^k \left(\frac{1}{4} + \tau^2 \right)^k \psi_\phi(\tau) = \tau \tanh(\pi\tau) \int_0^\infty (A_x')^k \left[(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \right] \phi(x)dx.$$

Note that the integral in the right-hand side of (3.2) is equal to

$$\int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) A_x^k \phi(x) dx.$$

I could probably verify this just based on what is shown above. The second claim is k parts integrations I think. We then end up with

$$-\left(\frac{1}{4}+\tau^2\right)\psi_\phi(\tau) = \tau \tanh(\pi\tau) \int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) A_x \phi(x) dx.$$

and then with more work of iteration they get

Iterating this process one arrives at

$$(-1)^k \left(\frac{1}{4}+\tau^2\right)^k \psi_\phi(\tau) = \tau \tanh(\pi\tau) \int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) A_x^k \phi(x) dx.$$

This work has all been to prove some Lemma 3.1 which is a technical issue.

Lemma 3.2 claims this, where the first = is just a definition from above

$$(\mathcal{L}\psi_\phi)(x) = \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x)\psi_\phi(\tau) d\tau$$

$$= \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x)\tau \tanh(\pi\tau) \int_0^\infty (\sinh t) P_{-\frac{1}{2}+i\tau}(\cosh t)\phi(t) dt\, d\tau,$$

Most of their work is "technical" checking limits and convergence and all that stuff. The final conclusion is this,

$$\int_0^\infty F(\tau)\psi(\tau) d\tau.$$

$$= \lim_{T\to\infty} \langle \int_0^T \tau \tanh(\pi\tau)(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) F(\tau) d\tau, \phi(x) \rangle.$$

and somehow this is the regular MF transform. Wow, what a pile of work!

They then repeat it all for the generalized MF.

Starting from the integral representation [1, (11), p. 156], which is valid for all $n \in \mathbb{N} \cup \{0\}$, all $\tau > 0$ and all $x > 0$,

$$P^n_{-\frac{1}{2}+i\tau}(\cosh x) = \sqrt{\frac{2}{\pi}} \frac{\Gamma(n+\frac{1}{2})(\sinh x)^n}{\Gamma(n+\frac{1}{2}+i\tau)\Gamma(n+\frac{1}{2}-i\tau)} \int_0^\infty (\cosh x + \cosh \xi)^{-n-\frac{1}{2}} \cos(\tau\xi) d\xi,$$

$$A_x \equiv (\sinh x)^{-n-1} D_x (\sinh x)^{2n+1} D_x (\sinh x)^{-n},$$

The operator $\mathcal{L} : W \to V$, given by

$$(\mathcal{L}\psi)(x) = \int_0^\infty P^n_{-\frac{1}{2}+i\tau}(\cosh x)\psi(\tau) d\tau,$$

Also, for any $\phi \in V$ and all $\tau > 0$ set

$$\psi_\phi(\tau) = \frac{1}{\pi}\tau \sinh(\pi\tau)\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right) \times \int_0^\infty (\sinh x) P^n_{-\frac{1}{2}+i\tau}(\cosh x)\phi(x) dx.$$

OK, they just summarize the generalized part. I could quote this reference for a good proof! I said it was hard, and here they show it is hard.

They do NOT ever talk about orthogonality or completeness.

Mehler-Fock Transforms of Generalized Functions via the Method of Adjoints

Benito J. González and Emilio R. Negrin
Proceedings of the American Mathematical Society
Vol. 125, No. 11 (Nov., 1997), pp. 3243-3253

Rosenthal has a later 1974 paper where he refers to Sneddon's integral transforms book and shows the class of functions which work in MF.

8. I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, 1972.

but this R paper has nothing useful to me.

Next I note an "Index Transforms" book on google of Semen and Yakubovich

We start to discuss here the Mehler-Fock transform of the following type

$$MF[f](\tau) = \frac{\pi}{2}\int_0^\infty P_{-1/2+i\tau/2}(2y^2+1)f(y)dy, \quad \tau \geq 0, \qquad (3.1)$$

which is a new form for me. I had

$g(y) = \int_0^\infty d\tau\, P^{\mu}_{-1/2+i\tau}(y)\, f(\tau)$

Notice the index has iτ/2 ! This source gets off on the K function stuff. a VERY long book section with stuff not helpful to me. Kontorovich-Lebedev pathway.

Footnote: MacDonald's function is just the Bessel K function, modified second kind.

I finally found something with delta functions!!! Saved the PDF. Here it is

$$\int_1^\infty dx P^{\lambda,\mu}_{-1/2+i\kappa}(x)P^{\lambda,\mu}_{-1/2+i\kappa'}(x) = \frac{2^{\mu-\lambda+1}\pi^2[\delta(\kappa-\kappa')+\delta(\kappa+\kappa')]}{\kappa\sinh(2\pi\kappa)}$$
$$\times \Gamma^{-1}((1-\lambda+\mu)/2+i\kappa)\Gamma^{-1}((1-\lambda+\mu)/2-i\kappa)$$
$$\times \Gamma^{-1}((1-\lambda-\mu)/2+i\kappa)\Gamma^{-1}((1-\lambda-\mu)/2-i\kappa)$$
$$(\kappa,\kappa' \in \mathbb{R}; \mathrm{Re}\lambda < 1 \text{ or } \lambda \in \mathbb{N}_+; \mu \in \mathbb{C}),$$

But the fancy P functions are not clearly defined! They are defined here

2. L. Kuipers, *Generalized Legendre's associated functions (Integral Theorems, Recurrence formulas)*, Monatsh. Math. 62 (1958), 24–31.

OK, here is the connection

1. *Introduction.*

The functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, solutions of the differential equation:

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left\{k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)}\right\} w = 0,$$

introduced in [1] by KUIPERS-MEULENBELD, have been defined for all points of the z-plane in which a cross-cut exists along the real axis from 1 to $-\infty$, and in [2] for the real values of z on the cross-cut for $-1 < z < 1$. These functions will be called *generalized Legendre functions*, whereas $P_k^m(z)$ and $Q_k^m(z)$ which are the special cases for m=n, will be denoted by *associated Legendre functions*.

For the sake of brevity we put:

$$\alpha = k+\tfrac{1}{2}(m+n), \quad \beta = k-\tfrac{1}{2}(m-n), \quad \gamma = k+\tfrac{1}{2}(m-n), \quad \delta = k-\tfrac{1}{2}(m+n).$$

So go back the previous and set $\lambda = \mu$

$$\int_1^\infty dx P^{\lambda,\mu}_{-1/2+i\kappa}(x) P^{\lambda,\mu}_{-1/2+i\kappa'}(x) = \frac{2^{\mu-\lambda+1}\pi^2[\delta(\kappa-\kappa') + \delta(\kappa+\kappa')]}{\kappa \sinh(2\pi\kappa)}$$
$$\times \Gamma^{-1}((1-\lambda+\mu)/2 + i\kappa)\Gamma^{-1}((1-\lambda+\mu)/2 - i\kappa)$$
$$\times \Gamma^{-1}((1-\lambda-\mu)/2 + i\kappa)\Gamma^{-1}((1-\lambda-\mu)/2 - i\kappa)$$
$$(\kappa, \kappa' \in \mathbb{R}; \mathrm{Re}\lambda < 1 \text{ or } \lambda \in \mathbb{N}_+; \mu \in \mathbb{C}),$$

$\int_1^\infty dy P^{\mu}_{-1/2+i\tau}(y)\, P^{\mu}_{-1/2+i\tau'}(y) = \delta(\tau-\tau')/\, f(\tau,\mu)$ // orthogonality (mine)

so they are then claiming

$\int_1^\infty dx\, P^{\mu}_{-1/2+i\tau}(x)\, P^{\mu}_{-1/2+i\tau'}(x) = 2\,\pi^2\delta(\tau-\tau')[1/\tau\, \mathrm{sh}(2\pi\tau)\,] * 1/\mathrm{fourgamma}$

$\mathrm{fourgamma} = \Gamma(1/2+i\tau)\Gamma(1/2-i\tau)\Gamma(1/2-\mu+i\kappa)\Gamma(1/2-\mu-i\kappa) =$

But recall $\Gamma(1/2+i\tau)\Gamma(1/2-i\tau) = \pi/\cosh(\pi\tau)$ so we then have

$\mathrm{fourgamma} = \pi/\cosh(\pi\tau) * \Gamma(1/2-\mu+i\kappa)\Gamma(1/2-\mu-i\kappa)$

So the right side of the above is then

$2\,\pi^2\delta(\tau-\tau')[1/\tau\, \mathrm{sh}(2\pi\tau)\,] * 1/[\,\pi/\cosh(\pi\tau) * \Gamma(1/2-\mu+i\kappa)\Gamma(1/2-\mu-i\kappa)]$

$= 2\,\pi^2\delta(\tau-\tau')[1/\tau\, 2\mathrm{sh}(\pi\tau)\mathrm{ch}(\pi\tau)\,] * 1/[\,\pi/\mathrm{ch}(\pi\tau) * \Gamma(1/2-\mu+i\kappa)\Gamma(1/2-\mu-i\kappa)]$

$= \pi^2\delta(\tau-\tau')[1/\tau\ \mathrm{sh}(\pi\tau)\ ] * 1/[\ \pi * \Gamma(1/2-\mu+i\kappa)\Gamma(1/2-\mu-i\kappa)]$

$= \pi^2\delta(\tau-\tau')[1/(\pi\tau\ \mathrm{sh}(\pi\tau))\ ] * 1/[\Gamma(1/2-\mu+i\kappa)\Gamma(1/2-\mu-i\kappa)]$

$= \pi^2\delta(\tau-\tau') / [\pi\tau\ \mathrm{sh}(\pi\tau)\Gamma(1/2-\mu+i\kappa)\Gamma(1/2-\mu-i\kappa)]$

$= \delta(\tau-\tau') / [\pi^{-1}\tau\ \mathrm{sh}(\pi\tau)\Gamma(1/2-\mu+i\kappa)\Gamma(1/2-\mu-i\kappa)]$

But recall from earlier that

$f(\tau,\mu) = (\tau/\pi)\ \mathrm{sh}(\pi\tau)\ \Gamma(1/2-\mu+i\tau)\ \Gamma(1/2-\mu-i\tau)$

so their result is

$= V$

Hurray! Their paper is this

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A Dirac delta-type orthogonality relation for the on-the-cut generalized associated Legendre functions of the first kind with imaginary second upper indices

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