

by

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1. Introduction.

The functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, solutions of the differential equation:

$$(1-z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + \left\{ k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)} \right\} w = 0,$$

introduced in [1] by KUIPERS-MEULENBELD, have been defined for all points of the z -plane in which a cross-cut exists along the real axis from 1 to $-\infty$, and in [2] for the real values of z on the cross-cut for $-1 < z < 1$. These functions will be called *generalized Legendre functions*, whereas $P_k^m(z)$ and $Q_k^m(z)$ which are the special cases for $m=n$, will be denoted by *associated Legendre functions*.

For the sake of brevity we put:

$$\alpha = k + \frac{1}{2}(m+n), \quad \beta = k - \frac{1}{2}(m-n), \quad \gamma = k + \frac{1}{2}(m-n), \quad \delta = k - \frac{1}{2}(m+n).$$

In [3] is given a list of integral transforms whose kernels are associated Legendre functions. In this paper we want to extend this list to integral transforms with $P_k^{m,n}(x)$, $Q_k^{m,n}(x)$, or combinations of these as kernels. In section 2 the result is given without proofs. The evaluation of most of the integrals is carried out by expressing $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ in terms of hypergeometric functions in conjunction with the tables in [3, 20.2]. A more detailed presentation may be found in [4]. In section 3 some inverse transforms are given, making use of the inversion formulas of BRAAKSMA-MEULENBELD (see [5]). In the integrals, occurring in this section is integrated with respect to the lower parameter k . Finally, in section 4 we mention some useful relations, until now not yet published, between generalized and associated Legendre functions.

2. A. Laplace transforms with generalized Legendre functions.

$$\int_0^\infty e^{-pt} t^{c-1} (t+1)^{-\frac{1}{2}n} P_k^{m,n}(1+2t) dt = \frac{2^{\frac{1}{2}(n-m)} p^{\frac{1}{2}m-c}}{\Gamma(\beta+1)\Gamma(-\gamma)} E(\beta+1, -\gamma, c-\frac{1}{2}m; 1-m:p) \quad (1)$$

$\operatorname{Re} c > \frac{1}{2}\operatorname{Re} m, \operatorname{Re} p > 0.$

For $m=n$:

$$\int_0^\infty e^{-pt} t^{c-1} (t+1)^{-\frac{1}{2}m} P_k^m(1+2t) dt = -\frac{1}{\pi} p^{\frac{1}{2}m-c} \sin \pi k E(k+1, -k, c-\frac{1}{2}m; 1-m:p) \quad (2)$$

$\operatorname{Re} c > \frac{1}{2}\operatorname{Re} m, \operatorname{Re} p > 0.$

For $c=1-\frac{1}{2}m$ in (2):

$$\int_0^\infty e^{-pt} [t(1+t)]^{-\frac{1}{2}m} P_k^m(1+2t) dt = \pi^{-\frac{1}{2}} p^{m-\frac{1}{2}} e^{\frac{1}{2}p} K_{k+\frac{1}{2}}(\frac{1}{2}p)$$

$\operatorname{Re} m < 1, \operatorname{Re} p > 0.$

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This transform has already been found in [3, 4.13(1)].
For $m=0$ in (2):

$$\int_0^\infty e^{-pt} t^{c-1} P_k(1+2t) dt = -\frac{1}{\pi} p^{-c} \sin \pi k E(k+1, -k, c; 1:p)$$

$\operatorname{Re} c > 0, \operatorname{Re} p > 0.$

$$\left. \begin{aligned} & \int_0^\infty e^{-pt} t^{\lambda+\frac{1}{2}m-1} (t+2)^{\frac{1}{2}n} e^{-\pi i m} Q_k^{m,n}(1+t) dt = \\ & = \frac{\Gamma(\alpha+1)}{\Gamma(\delta+1)} \left\{ \frac{\sin \beta \pi}{2^{m-n+1} p^{\lambda+m} \sin m \pi} E(-\beta, \gamma+1, \lambda+m; m+1; 2p) + \right. \\ & \left. - \frac{\sin \alpha \pi}{2^{-n+1} p^\lambda \sin m \pi} E(-\alpha, \delta+1, \lambda; 1-m; 2p) \right\} \end{aligned} \right\} \quad (3)$$

$\operatorname{Re} \lambda > 0, \operatorname{Re}(\lambda+m) > 0, \operatorname{Re} p > 0.$

For $m=n$ (3) is equal to [3, 4.13(16)].

$$\left. \begin{aligned} & \int_0^\infty e^{-pt} t^{-\frac{1}{2}m} (t+y)^{-k-1} (t+z)^k (t+y+z)^{\frac{1}{2}m} P_k^{m,n} \left(1 - \frac{2t(t+y+z)}{(t+y)(t+z)}\right) dt = \\ & = 2^{\frac{1}{2}(n-m)} e^{\frac{1}{2}(y+z)p} (yz)^{-\frac{1}{2}} p^{-1} W_{-k+\frac{1}{2}m-\frac{1}{2}, \frac{1}{2}n}(py) W_{k+\frac{1}{2}m+\frac{1}{2}, \frac{1}{2}n}(pz) \end{aligned} \right\} \quad (4)$$

$\operatorname{Re} m < 1, y, z \text{ real} > 0, \operatorname{Re} p > 0.$

For $y=z=1$ and $m=n=0$ we get [3, 4.13(6)].

$$\left. \begin{aligned} & \int_0^\infty e^{-pt} (1-e^{-t})^{\frac{1}{2}m} (1+e^{-t})^\sigma P_k^{-m, -n}(e^t) dt = \\ & = 2^{-p+\frac{1}{2}(m-n)} \frac{\Gamma(p-k)\Gamma(p+k+1)}{\Gamma(p+\frac{1}{2}m-\frac{1}{2}n+1)\Gamma(p+\frac{1}{2}m+\frac{1}{2}n+1)} \cdot \\ & {}_3F_2(p-k, p+\frac{1}{2}m+\sigma+1, p+k+1; p+\frac{1}{2}m-\frac{1}{2}n+1, p+\frac{1}{2}m+\frac{1}{2}n+1; \frac{1}{2}) \end{aligned} \right\} \quad (5)$$

$\operatorname{Re} m > -1, \operatorname{Re} p > -\operatorname{Re} k-1, \operatorname{Re} p > \operatorname{Re} k.$

For $m=n$ and $\sigma=\frac{1}{2}n$ we get [3, 4.13(11)].

$$\left. \begin{aligned} & \int_0^\infty e^{-pt} (e^t-1)^{\frac{1}{2}m} \left(\frac{\rho e^t}{\rho-2} - 1\right)^\sigma P_k^{-m, -n}(\rho e^t - \rho + 1) dt = \\ & = \rho^{p-\frac{1}{2}m} (\rho-2)^{-\sigma} \frac{\Gamma(p-\sigma-k-\frac{1}{2}m)\Gamma(p-\sigma+k-\frac{1}{2}m+1)}{\Gamma(p-\sigma-\frac{1}{2}n+1)\Gamma(p-\sigma+\frac{1}{2}n+1)} \cdot \\ & {}_3F_2(p-\sigma-k-\frac{1}{2}m, p+1, p-\sigma+k+1-\frac{1}{2}m; p-\sigma-\frac{1}{2}n+1, p-\sigma+\frac{1}{2}n+1; \frac{2-\rho}{2}) \end{aligned} \right\} \quad (6)$$

$\operatorname{Re} \rho > 0, \operatorname{Re} m > -1, \operatorname{Re} p > \operatorname{Re}(\frac{1}{2}m+\sigma-k)-1, \operatorname{Re} p > \operatorname{Re}(\frac{1}{2}m+\sigma+k).$

For $m=n$ and $\sigma=\frac{1}{2}n$ we get [3, 4.13(12)].

$$\int_0^{\infty} e^{-pt} (1-e^{-t})^{\frac{1}{2}n} P_k^{m,n} (1-2e^{-t}) dt = 2^{\frac{1}{2}(n-m)} \frac{\Gamma(p-\frac{1}{2}m)\Gamma(-p-k-\frac{1}{2}n)\Gamma(-\delta)}{\Gamma(-p-\frac{1}{2}m+1)\Gamma(p-k+\frac{1}{2}n)\Gamma(-\alpha)} \quad (7)$$

$$\operatorname{Re} n > -1, \operatorname{Re} p > \frac{1}{2}\operatorname{Re} m.$$

$$\int_0^{\infty} e^{-pt} (1-e^{-t})^{-\frac{1}{2}m} P_k^{m,n} (2e^{-t}-1) dt = 2^{\frac{1}{2}(n-m)} \frac{\Gamma(p+\frac{1}{2}n)\Gamma(p-m-\frac{1}{2}n)}{\Gamma(p-k-\frac{1}{2}m)\Gamma(p+k-\frac{1}{2}m+1)} \quad (8)$$

$$\operatorname{Re} m < 1, \operatorname{Re} p > \frac{1}{2}|\operatorname{Re} n|.$$

B. Integral transforms with generalized Legendre functions.

$$\begin{aligned} \int_0^1 x^p (1-x)^q (1-zx)^{-\frac{1}{2}n} P_k^{m,n} (1-2xz) dx = \\ = 2^{\frac{1}{2}(n-m)} z^{-\frac{1}{2}m} \frac{\Gamma(p-\frac{1}{2}m+1)\Gamma(q+1)}{\Gamma(1-m)\Gamma(p+q-\frac{1}{2}m+2)}. \end{aligned} \quad (9)$$

$${}_3F_2(\beta+1, -\gamma, p-\frac{1}{2}m+1; 1-m, p+q-\frac{1}{2}m+2; z)$$

$$\operatorname{Re} p > \frac{1}{2}\operatorname{Re} m-1, \operatorname{Re} q > -1, z \text{ real}, |z| < 1.$$

$$\int_{-1}^{+1} (1-x^2)^{-\frac{1}{2}} P_k^{m,n}(x) dx = \frac{\pi^2 2^{\frac{1}{2}(m+n)}}{\cos \frac{1}{2}n\pi} \frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}-\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\beta+1)\Gamma(\frac{1}{2}\delta+1)} \quad (10)$$

$$\operatorname{Re} m < 1, |\operatorname{Re} n| < 1.$$

$$\int_{-1}^{+1} (1-x)^p (1+x)^{k-p-1} P_k^{m,n}(x) dx = 2^\beta \frac{\Gamma(k-p+\frac{1}{2}n)\Gamma(k-p-\frac{1}{2}n)\Gamma(p-\frac{1}{2}m+1)}{\Gamma(\beta+1)\Gamma(\delta+1)\Gamma(-p-\frac{1}{2}m)} \quad (11)$$

$$\operatorname{Re}(p-\frac{1}{2}m) > -1, \operatorname{Re}(k-p) > \frac{1}{2}|\operatorname{Re} n|.$$

$$\begin{aligned} \int_{-1}^{+1} (1-x)^p (1+x)^q P_k^{m,n}(x) dx = \\ = \frac{\Gamma(q+\frac{1}{2}n+1)\Gamma(p-\frac{1}{2}m+1)}{\Gamma(1-m)\Gamma(p+q-\frac{1}{2}(m-n)+2)} 2^{p+q-\frac{1}{2}(m-n)+1}. \end{aligned} \quad (12)$$

$${}_3F_2(\beta+1, -\gamma, p-\frac{1}{2}m+1; 1-m, p+q-\frac{1}{2}(m-n)+2; 1).$$

$$\operatorname{Re}(p-\frac{1}{2}m) > -1, \operatorname{Re} q+1 > \frac{1}{2}|\operatorname{Re} n|.$$

$$\begin{aligned} \int_{-1}^{+1} (1-x)^{-\frac{1}{2}m} (1+x)^q (u+x)^{-\sigma} P_k^{m,n}(x) dx = \\ = 2^{q+\frac{1}{2}n-m+1} \frac{\Gamma(q+\frac{1}{2}n+1)\Gamma(q-\frac{1}{2}n+1)}{\Gamma(q-k-\frac{1}{2}m+1)\Gamma(q+k-\frac{1}{2}m+2)} (u-1)^{-\sigma}. \end{aligned} \quad (13)$$

$${}_3F_2(q+\frac{1}{2}n+1, \sigma, q-\frac{1}{2}n+1; q-k-\frac{1}{2}m+1, q+k-\frac{1}{2}m+2; \frac{2}{1-u})$$

$$\operatorname{Re} m < 1, \operatorname{Re} q+1 > \frac{1}{2}|\operatorname{Re} n|, u \text{ not lying on the cut } (-1, +1).$$

$$\begin{aligned}
& \int_{-1}^{+1} (1-x)^{-\frac{1}{2}m} (1+x)^q e^{-ux} P_k^{m,n}(x) dx = \\
& = 2^{q+\frac{1}{2}n-m+1} \frac{\Gamma(q+\frac{1}{2}n+1)\Gamma(q-\frac{1}{2}n+1)}{\Gamma(q-k-\frac{1}{2}m+1)\Gamma(q+k-\frac{1}{2}m+2)} \cdot e^u. \quad (14) \\
& {}_2F_2(q+\frac{1}{2}n+1, q-\frac{1}{2}n+1; q-k-\frac{1}{2}m+1, q+k-\frac{1}{2}m+2; -2u) \\
& \operatorname{Re} m < 1, \operatorname{Re} q+1 > \frac{1}{2}|\operatorname{Re} n|.
\end{aligned}$$

$$\begin{aligned}
& \int_1^\infty (x-1)^{-\frac{1}{2}m} (x+1)^{-p} P_k^{m,n}(x) dx = 2^{1-m+\frac{1}{2}n-p} \frac{\Gamma(p+k+\frac{1}{2}m)\Gamma(p-k+\frac{1}{2}m-1)}{\Gamma(p-\frac{1}{2}n)\Gamma(p+\frac{1}{2}n)} \quad (15) \\
& \operatorname{Re} m < 1, \operatorname{Re}(p+k+\frac{1}{2}m) > 0, \operatorname{Re}(p-k+\frac{1}{2}m-1) > 0.
\end{aligned}$$

$$\begin{aligned}
& \int_1^\infty (x-1)^{-\frac{1}{2}m} (x+1)^{-\frac{1}{2}n} (x+t)^{-2\ell-1} P_k^{m,n}(x) dx = \\
& = (1+t)^{-\frac{1}{2}m} (1-t)^{-\frac{1}{2}n} \frac{\Gamma(\alpha+2\ell+1)\Gamma(-\delta+2\ell)}{\Gamma(2\ell+1)} P_k^{-n-2\ell, -m-2\ell}(t) \quad (16) \\
& \operatorname{Re} m < 1, |\operatorname{Re}(2k+1)| < \operatorname{Re}(m+n+4\ell+1), |t| < 1.
\end{aligned}$$

$$\begin{aligned}
& \int_1^\infty e^{-ax} (x-1)^{-\frac{1}{2}m} (x+1)^{-\frac{1}{2}n} P_k^{m,n}(x) dx = 2^{\frac{1}{2}(n-m)} a^{\frac{1}{2}(m+n)-1} W_{\frac{1}{2}(m-n), k+\frac{1}{2}}(2a) \quad (17) \\
& \operatorname{Re} m < 1, \operatorname{Re} a > 0.
\end{aligned}$$

$$\begin{aligned}
& \int_1^\infty e^{-ax} (x-1)^{c-1} (x+1)^{-\frac{1}{2}n} P_k^{m,n}(x) dx = \\
& = \frac{a^{\frac{1}{2}m-c} e^{-a}}{\Gamma(\beta+1)\Gamma(-\gamma)} E(\beta+1, -\gamma, c-\frac{1}{2}m; 1-m; 2a) \quad (18) \\
& \operatorname{Re} c > \frac{1}{2}\operatorname{Re} m, \operatorname{Re} a > 0.
\end{aligned}$$

$$\begin{aligned}
& \int_1^\infty (x-1)^{-\frac{1}{2}m} (x+1)^{-n} e^{xt} K_{\frac{1}{2}n}[(1+x)t] P_k^{m,n}(x) dx = \\
& = 2^{-\frac{1}{2}n-1} e^t \pi^{-\frac{1}{2}} \cos \frac{1}{2}n\pi \Gamma(-k+\frac{1}{2}m-\frac{1}{2})\Gamma(k+\frac{1}{2}m+\frac{1}{2}) t^{\frac{1}{2}m-1} W_{\frac{1}{2}-\frac{1}{2}m, -k-\frac{1}{2}}(4t) \quad (19) \\
& \operatorname{Re} m < 1, \operatorname{Re}(-k+\frac{1}{2}m-\frac{1}{2}) > 0, \operatorname{Re}(k+\frac{1}{2}m+\frac{1}{2}) > 0, |\arg t| < \frac{3\pi}{2}.
\end{aligned}$$

$$\begin{aligned}
& \int_1^\infty e^{-\pi i m} Q_k^{m,n}(x) dx = \\
& = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(2k+2)} \frac{\Gamma(1-\frac{1}{2}m)\Gamma(k)}{\Gamma(k-\frac{1}{2}m+1)} 2^{-\frac{1}{2}m+\frac{1}{2}n} \quad (20) \\
& {}_3F_2(\beta+1, \delta+1, k; 2k+2, k-\frac{1}{2}m+2; 1) \\
& \operatorname{Re} k > 0, |\operatorname{Re} m| < 2.
\end{aligned}$$

$$\int_1^{\infty} (x^2-1)^{-\frac{1}{2}} e^{-\pi i m} Q_k^{m,n}(x) dx =$$

$$= 2^{-2k-\frac{1}{2}m+\frac{1}{2}n} \frac{\pi^2}{\cos \frac{1}{2}m\pi} \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\frac{1}{2}\alpha+1)\Gamma(\frac{1}{2}\beta+1)\Gamma(\frac{1}{2}\gamma+1)\Gamma(\frac{1}{2}\delta+1)} \quad (21)$$

$$\operatorname{Re} k > -1, \quad |\operatorname{Re} m| < 1.$$

$$\int_1^{\infty} (x-1)^p (x+1)^{-p-k-2} e^{-\pi i m} Q_k^{m,n}(x) dx =$$

$$= 2^{-k-\frac{1}{2}m+\frac{1}{2}n-2} \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(p-\frac{1}{2}m+1)\Gamma(p+\frac{1}{2}m+1)}{\Gamma(k+p-\frac{1}{2}n+2)\Gamma(k+p+\frac{1}{2}n+2)} \quad (22)$$

$$\operatorname{Re} k > -1, \quad \operatorname{Re} p > \frac{1}{2} |\operatorname{Re} m| - 1.$$

$$\int_1^{\infty} (x-1)^p (x+1)^q e^{-\pi i m} Q_k^{m,n}(x) dx =$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(2k+2)} \frac{\Gamma(p-\frac{1}{2}m+1)\Gamma(-p-q+k)}{\Gamma(k-q-\frac{1}{2}m+1)} 2^{p+q-\frac{1}{2}m+\frac{1}{2}n} \quad (23)$$

$$\cdot {}_3F_2(\beta+1, \delta+1, k-p-q; 2k+2, k-q-\frac{1}{2}m+1; 1)$$

$$\frac{1}{2} |\operatorname{Re} m| - 1 < \operatorname{Re} p < \operatorname{Re} (-q+k).$$

$$\int_1^{\infty} P_k^{m,n}(x) Q_\ell^{m,n}(x) dx =$$

$$= \frac{e^{\pi i m} 2^{-m+n}}{(\ell-k)(\ell+k+1)} \frac{\Gamma(\ell+\frac{1}{2}(m+n)+1)\Gamma(\ell+\frac{1}{2}(m-n)+1)}{\Gamma(\ell-\frac{1}{2}(m+n)+1)\Gamma(\ell-\frac{1}{2}(m-n)+1)} \quad (24)$$

$$\operatorname{Re}(\ell-k) > 0, \quad \operatorname{Re}(\ell+k) > -1, \quad \operatorname{Re} m < 1.$$

One of the theorems of BRAAKSMA-MEULENBELD [5] is:

Let k_1 be a real number with $k_1 > \frac{1}{2}\operatorname{Re} m + \frac{1}{2}|\operatorname{Re} n| - 1$, and $\varphi(t)$ a function such that for all $a > 1$:

$$\varphi(t) (t-1)^{-\frac{1}{2}-\frac{1}{2}|\operatorname{Re} m|} \in L(1, a) \quad \text{if } \operatorname{Re} m \neq 0,$$

$$\varphi(t) (t-1)^{-\frac{1}{2}} \log(t-1) \in L(1, a) \quad \text{if } \operatorname{Re} m = 0,$$

$$\varphi(t) t^{-1-k_1} \in L(a, \infty).$$

Let further $\varphi(t)$ be a function of bounded variation in a neighbourhood of $t=x(x > 1)$. Then $\varphi(t)$ satisfies the relations:

$$\frac{1}{2\pi i} \int_{k_1-i\infty}^{k_1+i\infty} dk (2k+1) P_k^{m,n}(x) \int_1^{\infty} \varphi(t) e^{\pi i m} Q_k^{-m,-n}(t) dt = \frac{1}{2} \{ \varphi(x-0) + \varphi(x+0) \}, \quad (25)$$

and

$$\frac{1}{2\pi i} \int_{k_1-i\infty}^{k_1+i\infty} dk(2k+1)e^{\pi i m} Q_k^{-m, -n}(x) \int_1^\infty \varphi(t) P_k^{m, n}(t) dt = \frac{1}{2} \{ \varphi(x-0) + \varphi(x+0) \}. \quad (26)$$

If we apply (25) to (24), we find after some simplifications:

$$\begin{aligned} & \frac{1}{2\pi i} \int_{k_1-i\infty}^{k_1+i\infty} \left(\frac{1}{\ell-k} - \frac{1}{\ell+k+1} \right) \frac{\Gamma(k+\frac{1}{2}(m+n)+1)\Gamma(k+\frac{1}{2}(m-n)+1)}{\Gamma(k-\frac{1}{2}(m+n)+1)\Gamma(k-\frac{1}{2}(m-n)+1)} P_k^{-m, -n}(x) dk \\ & = 2^{m-n} P_\ell^{m, n}(x) \end{aligned} \quad (27)$$

$$k_1 > -\frac{1}{2}\operatorname{Re} m + \frac{1}{2}|\operatorname{Re} n| - 1, \quad 2k_1 + 1 > |2\operatorname{Re} \ell + 1|, \quad |\operatorname{Re} m| < \frac{3}{4}, \quad x > 1.$$

Application of (26) yields:

$$\frac{1}{2\pi i} \int_{k_1-i\infty}^{k_1+i\infty} \left(\frac{1}{\ell-k} - \frac{1}{\ell+k+1} \right) Q_k^{m, n}(x) dk = Q_\ell^{m, n}(x) \quad (28)$$

$$k_1 > \frac{1}{2}\operatorname{Re} m + \frac{1}{2}|\operatorname{Re} n| - 1, \quad 2\operatorname{Re} \ell + 1 > |2k_1 + 1|, \quad |\operatorname{Re} m| < \frac{3}{4}, \quad x > 1.$$

This result can be found in a direct way by applying Cauchy's integral formula, since the asymptotic behaviour of $Q_k^{m, n}(x)$ as $k \rightarrow \infty$ on $|\arg k| \leq \pi - \eta$ with $0 < \eta < \pi$ is given by:

$$Q_k^{m, n}(x) = k^{m-\frac{1}{2}} (x + \sqrt{x^2 - 1})^{-k} O(1) \quad (x > 1).$$

Remark. The formula (27) may be considered as a continuous analogue of the classical Dougall series expansions.

By applying (26) to (15) we obtain:

$$\begin{aligned} & \frac{1}{2\pi i} \int_{k_1-i\infty}^{k_1+i\infty} (2k+1) (p+k+\frac{1}{2}m) \Gamma(p-k+\frac{1}{2}m-1) e^{\pi i m} Q_k^{-m, -n}(x) dk = \\ & = 2^{-1+m-\frac{1}{2}n+p} \Gamma(p-\frac{1}{2}n) \Gamma(p+\frac{1}{2}n) (x-1)^{-\frac{1}{2}m} (x+1)^{-p} \end{aligned} \quad (29)$$

$$k_1 > \frac{1}{2}\operatorname{Re} m + \frac{1}{2}|\operatorname{Re} n| - 1, \quad \operatorname{Re}(p+\frac{1}{2}m) - \frac{1}{2} > |k_1 + \frac{1}{2}|, \quad \operatorname{Re} m < \frac{3}{4}, \quad x > 1.$$

Application of (26) to (17) yields:

$$\begin{aligned} & \frac{1}{2\pi i} \int_{k_1-i\infty}^{k_1+i\infty} (2k+1) W_{\frac{1}{2}(n-m), k+\frac{1}{2}}(2a) e^{-\pi i m} Q_k^{m, n}(x) dk = \\ & = 2^{\frac{1}{2}(n-m)} a^{1+\frac{1}{2}(m+n)} e^{-ax} (x-1)^{\frac{1}{2}m} (x+1)^{\frac{1}{2}n} \end{aligned} \quad (30)$$

$$k_1 > -\frac{1}{2}\operatorname{Re} m + \frac{1}{2}|\operatorname{Re} n| - 1, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} m > -\frac{3}{4}, \quad x > 1.$$

For $m=n$ (30) is transformed into [5, (8.27)].

Some useful relations between the generalized and the associated Legendre functions are the following:

$$P_k^{m, \frac{1}{2}}(2z^2 - 1) = 2^{-\frac{3}{2}m + \frac{1}{4}} z^{-\frac{1}{2}} P_{2k+\frac{1}{2}}^m(z). \quad (31)$$

$$P_{-\frac{1}{4}}^{m, n}\left(\frac{2}{z^2} - 1\right) = 2^{-\frac{3m}{2} + \frac{n}{2}} z^{\frac{1}{2}} e^{\pm \frac{1}{2}\pi i m} P_{n-\frac{1}{2}}^m(z) \quad (32)$$

(the upper or lower sign according as $\text{Im} z \gtrless 0$).

$$Q_k^{m, \frac{1}{2}}(2z^2 - 1) = 2^{-\frac{3}{2}m + \frac{1}{4}} z^{-\frac{1}{2}} Q_{2k+\frac{1}{2}}^m(z). \quad (33)$$

$$P_k^{m, \frac{1}{2}}\left(\frac{z^2 + 1}{z^2 - 1}\right) = - \frac{2^{-\frac{3}{2}m + \frac{3}{4}} e^{2\pi i k} z^{-\frac{1}{2}} (z^2 - 1)^{\frac{1}{2}}}{\sqrt{\pi} \Gamma(-m - 2k - \frac{1}{2})} Q_{-\frac{1}{2}-m}^{-2k-1}(z). \quad (34)$$

Formulas (31) and (33) hold for $\text{Re } z \neq 0$, z not on the cross-cut, and for $-1 < z < 1$, $z \neq 0$. Formula (32) holds for $\text{Re } z \neq 0$, z not on the cross-cut, and for $-1 < z < 1$, $z \neq 0$ after omitting the factor $e^{\pm \frac{1}{2}\pi i m}$. Formula (34) holds for z not on the cross-cut.

References.

1. L. KUIPERS and B. MEULENBELD On a generalization of Legendre's associated differential equation I and II, Proc. Kon. Ned. Ak. v. Wet., Amsterdam **60**, 436-450 (1957).
2. B. MEULENBELD Generalized Legendre's associated functions for real values of the argument numerically less than unity, Proc. Kon. Ned. Ak. v. Wet., Amsterdam **61**, 557-563 (1958).
3. A. ERDÉLYI a.o. Tables of integral transforms, vol. 1, New York, 1954.
4. H. S. V. DE SNOO Integral transforms and series involving generalized Legendre functions, Report Technological University Delft, 1967.
5. B. L. J. BRAAKSMA and B. MEULENBELD Integral transforms with generalized Legendre functions as kernels, Compositio Mathematica (will appear in 1967).

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