

A Dirac delta-type orthogonality relation for the on-the-cut generalized associated Legendre functions of the first kind with imaginary second upper indices

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The integral $\int_{-1}^1 dx (1+x)^{-1} P_v^{\lambda, i\mu}(x) P_v^{\lambda, i\mu'}(x)$, involving the on-the-cut generalized associated Legendre functions of the first kind with $v \in \mathbb{C}$, $\text{Re} \lambda < 1$ or $\lambda \in \mathbb{N}_+$, $\mu, \mu' \in \mathbb{R}$, is evaluated in a closed form. It is found to be proportional to $2^{i\mu/2} \delta(\mu - \mu') + 2^{-i\mu/2} \delta(\mu + \mu')$, where $\delta(\mu \mp \mu')$ is the Dirac delta distribution.

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1. Introduction

Among various types of index transforms, [1] an important class is formed by those with kernels involving the Legendre functions or the associated Legendre functions. Examples encompass the well-known Mehler–Fock transform [2,3] (see also [1, chap. 3]) and its various generalizations (see, e.g. [4–6]).

In Ref. [7], Götze considered a particular generalization of the Mehler–Fock transform with an integral kernel involving the generalized associated Legendre function of the first kind $P_v^{\lambda, \mu}(x)$, introduced by Kuipers and Meulenbeld [8] (see also [9]). From the results presented in Ref. [7], one may deduce the following Dirac delta-type orthogonality relation:

$$\begin{aligned} \int_1^\infty dx P_{-1/2+i\kappa}^{\lambda, \mu}(x) P_{-1/2+i\kappa'}^{\lambda, \mu}(x) &= \frac{2^{\mu-\lambda+1} \pi^2 [\delta(\kappa - \kappa') + \delta(\kappa + \kappa')]}{\kappa \sinh(2\pi\kappa)} \\ &\times \Gamma^{-1}((1 - \lambda + \mu)/2 + i\kappa) \Gamma^{-1}((1 - \lambda + \mu)/2 - i\kappa) \\ &\times \Gamma^{-1}((1 - \lambda - \mu)/2 + i\kappa) \Gamma^{-1}((1 - \lambda - \mu)/2 - i\kappa) \\ &(\kappa, \kappa' \in \mathbb{R}; \text{Re} \lambda < 1 \text{ or } \lambda \in \mathbb{N}_+; \mu \in \mathbb{C}), \end{aligned} \quad (1.1)$$

where $\delta(\kappa \mp \kappa')$ is the Dirac delta distribution.

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It is the purpose of the present note to provide another Dirac delta-type orthogonality relation obeyed by the generalized associated Legendre functions of the first kind. Specifically, we shall prove that on the cut $-1 \leq x \leq 1$ these functions satisfy

$$\begin{aligned} \int_{-1}^1 dx \frac{P_v^{\lambda, i\mu}(x) P_v^{\lambda, i\mu'}(x)}{1+x} &= \frac{2^{i\mu/2-\lambda+2} \pi^2 [2^{i\mu/2} \delta(\mu - \mu') + 2^{-i\mu/2} \delta(\mu + \mu')]}{\mu \sinh(\pi \mu)} \\ &\times \Gamma^{-1}(v+1 - (\lambda + i\mu)/2) \Gamma^{-1}(v+1 - (\lambda - i\mu)/2) \\ &\times \Gamma^{-1}(-v - (\lambda + i\mu)/2) \Gamma^{-1}(-v - (\lambda - i\mu)/2) \\ &(\nu \in \mathbb{C}; \operatorname{Re} \lambda < 1 \text{ or } \lambda \in \mathbb{N}_+; \mu, \mu' \in \mathbb{R}). \end{aligned} \quad (1.2)$$

The method we shall use below to arrive at the relation in Equation (1.2) is analogous to the one we have employed in our previous works [10–12] on the Dirac delta-type orthogonality relations for some other special functions of mathematical physics.

2. Summary of relevant properties of the generalized associated Legendre function of the first kind

In the complex plane cut along the real axis from $z = -\infty$ to $z = 1$, the generalized associated Legendre function of the first kind may be defined in terms of the Gauss' hypergeometric function as

$$P_v^{\lambda, \mu}(z) = \frac{1}{\Gamma(1-\lambda)} \frac{(z+1)^{\mu/2}}{(z-1)^{\lambda/2}} {}_2F_1 \left(v+1 - \frac{\lambda-\mu}{2}, -v - \frac{\lambda-\mu}{2}; 1-\lambda; \frac{1-z}{2} \right). \quad (2.1)$$

The function (2.1) obeys the differential identity

$$\frac{d}{dz} (1-z^2) \frac{dP_v^{\lambda, \mu}(z)}{dz} + \left[v(v+1) - \frac{\lambda^2}{2(1-z)} - \frac{\mu^2}{2(1+z)} \right] P_v^{\lambda, \mu}(z) = 0. \quad (2.2)$$

On that part of the cut for which $z = x \in [-1, 1]$, we define the on-the-cut function $P_v^{\lambda, \mu}(x)$ in terms of either of the limits $P_v^{\lambda, \mu}(x \pm i0)$ as

$$P_v^{\lambda, \mu}(x) = e^{\pm i\pi\lambda/2} P_v^{\lambda, \mu}(x \pm i0) \quad (-1 \leq x \leq 1). \quad (2.3)$$

Hence, with the aid of the relation

$$z-1 = e^{\pm i\pi} (1-z) \quad (\operatorname{Im} z \gtrless 0), \quad (2.4)$$

it follows that

$$\begin{aligned} P_v^{\lambda, \mu}(x) &= \frac{1}{\Gamma(1-\lambda)} \frac{(1+x)^{\mu/2}}{(1-x)^{\lambda/2}} {}_2F_1 \left(v+1 - \frac{\lambda-\mu}{2}, -v - \frac{\lambda-\mu}{2}; 1-\lambda; \frac{1-x}{2} \right) \\ &(-1 \leq x \leq 1). \end{aligned} \quad (2.5)$$

The on-the-cut function $P_v^{\lambda, \mu}(x)$ satisfies the differential identity obtained from Equation (2.2) through the formal replacement $z \rightarrow x$.

In Section 3, we shall rely on asymptotic representations of the on-the-cut function $P_v^{\lambda,\mu}(x)$ as $x \rightarrow \pm 1 \mp 0$. The representations valid for $x \rightarrow 1 - 0$ may be immediately deduced from Equation (2.5); these are

$$P_v^{\lambda,\mu}(x) \xrightarrow{x \rightarrow 1-0} \frac{2^{(\mu-\lambda)/2}}{\Gamma(1-\lambda)} \left(\frac{1-x}{2}\right)^{-\lambda/2} [1 + O(1-x)] \quad (\lambda \notin \mathbb{N}_+) \quad (2.6a)$$

and

$$P_v^{\lambda,\mu}(x) \xrightarrow{x \rightarrow 1-0} \frac{2^{(\mu-\lambda)/2}}{\Gamma(\lambda+1)} \frac{\Gamma(v+1+(\lambda+\mu)/2)\Gamma(-v+(\lambda+\mu)/2)}{\Gamma(v+1-(\lambda-\mu)/2)\Gamma(-v-(\lambda-\mu)/2)} \left(\frac{1-x}{2}\right)^{\lambda/2} \times [1 + O(1-x)] \quad (\lambda \in \mathbb{N}_+). \quad (2.6b)$$

The counterpart representation valid for $x \rightarrow -1 + 0$ is

$$P_v^{\lambda,\mu}(x) \xrightarrow{x \rightarrow -1+0} \frac{2^{(\mu-\lambda)/2}\Gamma(-\mu)}{\Gamma(v+1-(\lambda+\mu)/2)\Gamma(-v-(\lambda+\mu)/2)} \left(\frac{1+x}{2}\right)^{\mu/2} [1 + O(1+x)] + \frac{2^{(\mu-\lambda)/2}\Gamma(\mu)}{\Gamma(v+1-(\lambda-\mu)/2)\Gamma(-v-(\lambda-\mu)/2)} \left(\frac{1+x}{2}\right)^{-\mu/2} [1 + O(1+x)], \quad (2.7)$$

as may be inferred from Equation (2.5) and from the Gauss' relation [13, Equation (9.131.2)]

$$\begin{aligned} {}_2F_1(a_1, a_2; b; z) &= \frac{\Gamma(b)\Gamma(b-a_1-a_2)}{\Gamma(b-a_1)\Gamma(b-a_2)} {}_2F_1(a_1, a_2; a_1+a_2-b+1; 1-z) \\ &\quad + (1-z)^{b-a_1-a_2} \frac{\Gamma(b)\Gamma(a_1+a_2-b)}{\Gamma(a_1)\Gamma(a_2)} \\ &\quad \times {}_2F_1(b-a_1, b-a_2; b-a_1-a_2+1; 1-z). \end{aligned} \quad (2.8)$$

3. A Dirac delta-type orthogonality relation involving the on-the-cut generalized associated Legendre functions of the first kind with imaginary second upper indices

Throughout this section, it will be assumed that

$$-1 \leq x \leq 1, \quad v \in \mathbb{C}, \quad \mu, \mu' \in \mathbb{R}. \quad (3.1)$$

For the time being, we also assume that $\lambda \in \mathbb{C}$. Later on, some necessary restrictions on λ will be imposed.

Consider the two on-the-cut functions $P_v^{\lambda,i\mu}(x)$ and $P_v^{\lambda,i\mu'}(x)$. According to what has been said in Section 2, they satisfy the differential identities

$$\frac{d}{dx}(1-x^2) \frac{dP_v^{\lambda,i\mu}(x)}{dx} + \left[v(v+1) - \frac{\lambda^2}{2(1-x)} + \frac{\mu^2}{2(1+x)} \right] P_v^{\lambda,i\mu}(x) = 0 \quad (3.2)$$

and

$$\frac{d}{dx}(1-x^2) \frac{dP_v^{\lambda,i\mu'}(x)}{dx} + \left[v(v+1) - \frac{\lambda^2}{2(1-x)} + \frac{\mu'^2}{2(1+x)} \right] P_v^{\lambda,i\mu'}(x) = 0. \quad (3.3)$$

Premultiplying Equation (3.2) by $P_v^{\lambda, i\mu'}(x)$, Equation (3.3) by $P_v^{\lambda, i\mu}(x)$, subtracting and integrating the resulting equation over x from x_1 to x_2 , where $-1 < x_1 < x_2 < 1$, yields

$$\begin{aligned} & \int_{x_1}^{x_2} dx \frac{P_v^{\lambda, i\mu}(x) P_v^{\lambda, i\mu'}(x)}{1+x} \\ &= \frac{2}{\mu'^2 - \mu^2} (1 - x^2) \left[P_v^{\lambda, i\mu'}(x) \frac{dP_v^{\lambda, i\mu}(x)}{dx} - P_v^{\lambda, i\mu}(x) \frac{dP_v^{\lambda, i\mu'}(x)}{dx} \right] \Bigg|_{x=x_1}^{x=x_2}. \end{aligned} \quad (3.4)$$

It follows from Equations (2.6a) and (2.6b) that the integral on the left-hand side of Equation (3.4) converges as $x_2 \rightarrow 1 - 0$, provided that λ is constrained to obey

$$\operatorname{Re} \lambda < 1 \quad \text{or} \quad \lambda \in \mathbb{N}_+. \quad (3.5)$$

Henceforth, the restrictions (3.5) will be assumed to hold. Then, again with the aid of Equations (2.6a) and (2.6b), one finds that the expression on the right-hand side of Equation (3.4) vanishes for $x_2 = 1$. Hence, it follows that

$$\begin{aligned} & \int_{x_1}^1 dx \frac{P_v^{\lambda, i\mu}(x) P_v^{\lambda, i\mu'}(x)}{1+x} \\ &= \frac{2}{\mu'^2 - \mu^2} (1 - x_1^2) \left[P_v^{\lambda, i\mu}(x_1) \frac{dP_v^{\lambda, i\mu'}(x_1)}{dx_1} - P_v^{\lambda, i\mu'}(x_1) \frac{dP_v^{\lambda, i\mu}(x_1)}{dx_1} \right]. \end{aligned} \quad (3.6)$$

To proceed further, we observe that Equation (2.7) implies the asymptotic formula

$$P_v^{\lambda, i\mu}(x) \xrightarrow{x \rightarrow -1+0} 2^{(i\mu - \lambda)/2} \left[A_v^{\lambda, \mu} \exp\left(\frac{i\mu}{2} \ln \frac{1+x}{2}\right) + A_v^{\lambda, -\mu} \exp\left(-\frac{i\mu}{2} \ln \frac{1+x}{2}\right) \right], \quad (3.7)$$

where

$$A_v^{\lambda, \pm\mu} = \frac{\Gamma(\mp i\mu)}{\Gamma(v+1 - (\lambda \pm i\mu)/2) \Gamma(-v - (\lambda \pm i\mu)/2)}. \quad (3.8)$$

Making use of Equation (3.7) on the right-hand side of Equation (3.6) results in

$$\begin{aligned} & \int_{-1}^1 dx \frac{P_v^{\lambda, i\mu}(x) P_v^{\lambda, i\mu'}(x)}{1+x} = 2^{i(\mu + \mu')/2 - \lambda + 1} \\ & \times \lim_{x \rightarrow -1+0} \left[(A_v^{\lambda, \mu} A_v^{\lambda, -\mu'} + A_v^{\lambda, -\mu} A_v^{\lambda, \mu'}) \frac{\sin(-((\mu - \mu')/2) \ln((1+x)/2))}{\mu - \mu'} \right. \\ & + (A_v^{\lambda, \mu} A_v^{\lambda, \mu'} + A_v^{\lambda, -\mu} A_v^{\lambda, -\mu'}) \frac{\sin(-((\mu + \mu')/2) \ln((1+x)/2))}{\mu + \mu'} \\ & + i \frac{A_v^{\lambda, \mu} A_v^{\lambda, -\mu'} - A_v^{\lambda, -\mu} A_v^{\lambda, \mu'}}{\mu - \mu'} \cos\left(\frac{\mu - \mu'}{2} \ln \frac{1+x}{2}\right) \\ & \left. + i \frac{A_v^{\lambda, \mu} A_v^{\lambda, \mu'} - A_v^{\lambda, -\mu} A_v^{\lambda, -\mu'}}{\mu + \mu'} \cos\left(\frac{\mu + \mu'}{2} \ln \frac{1+x}{2}\right) \right]. \end{aligned} \quad (3.9)$$

To take the limit on the right-hand side of Equation (3.9), we observe that in the distributional sense it holds that (cf., e.g. [14, p. 35] or [15, vol. I, p. 43])

$$\lim_{a \rightarrow \infty} \frac{\sin a\eta}{\pi\eta} = \frac{1}{2\pi} \lim_{a \rightarrow \infty} \int_{-a}^a d\xi e^{i\xi\eta} = \delta(\eta), \quad (3.10a)$$

where $\delta(\eta)$ is the Dirac delta distribution, and

$$\lim_{a \rightarrow \infty} \cos a\eta = 0 \quad (3.10b)$$

(the latter identity is a corollary from the Riemann–Lebesgue lemma). Since for $x \rightarrow -1 + 0$ one has $\ln[(1+x)/2] \rightarrow -\infty$, application of Equations (3.10a) and (3.10b) transforms Equation (3.9) into the distributional identity

$$\begin{aligned} \int_{-1}^1 dx \frac{P_v^{\lambda, i\mu}(x) P_v^{\lambda, i\mu'}(x)}{1+x} &= 2^{i(\mu+\mu')/2-\lambda+1} \pi [(A_v^{\lambda, \mu} A_v^{\lambda, -\mu'} + A_v^{\lambda, -\mu} A_v^{\lambda, \mu'}) \delta(\mu - \mu') \\ &\quad + (A_v^{\lambda, \mu} A_v^{\lambda, \mu'} + A_v^{\lambda, -\mu} A_v^{\lambda, -\mu'}) \delta(\mu + \mu')]. \end{aligned} \quad (3.11)$$

However, it follows from elementary properties of the Dirac delta that

$$A_v^{\lambda, \mu'} \delta(\mu \mp \mu') = A_v^{\lambda, \pm\mu} \delta(\mu \mp \mu'). \quad (3.12)$$

With this, after employing Equation (3.8) and using the well-known relation

$$|\Gamma(i\mu)| = \sqrt{\frac{\pi}{\mu \sinh(\pi\mu)}} \quad (\mu \in \mathbb{R}), \quad (3.13)$$

Equation (3.11) is transformed into the Dirac delta-type orthogonality relation

$$\begin{aligned} \int_{-1}^1 dx \frac{P_v^{\lambda, i\mu}(x) P_v^{\lambda, i\mu'}(x)}{1+x} &= \frac{2^{i\mu/2-\lambda+2} \pi^2 [2^{i\mu/2} \delta(\mu - \mu') + 2^{-i\mu/2} \delta(\mu + \mu')]}{\mu \sinh(\pi\mu)} \\ &\quad \times \Gamma^{-1}(v+1-(\lambda+i\mu)/2) \Gamma^{-1}(v+1-(\lambda-i\mu)/2) \\ &\quad \times \Gamma^{-1}(-v-(\lambda+i\mu)/2) \Gamma^{-1}(-v-(\lambda-i\mu)/2) \\ &\quad (v \in \mathbb{C}; \operatorname{Re} \lambda < 1 \text{ or } \lambda \in \mathbb{N}_+; \mu, \mu' \in \mathbb{R}), \end{aligned} \quad (3.14)$$

that has been already announced in the introduction to be the central result of this work.

Concluding, we observe that the relation in Equation (3.14) may be split into

$$\begin{aligned} \int_{-1}^1 dx \frac{P_v^{\lambda, i\mu}(x) P_v^{\lambda, i\mu'}(x)}{1+x} &= \frac{2^{i\mu-\lambda+2} \pi^2 \delta(\mu - \mu')}{\mu \sinh(\pi\mu)} \\ &\quad \times \Gamma^{-1}(v+1-(\lambda+i\mu)/2) \Gamma^{-1}(v+1-(\lambda-i\mu)/2) \\ &\quad \times \Gamma^{-1}(-v-(\lambda+i\mu)/2) \Gamma^{-1}(-v-(\lambda-i\mu)/2) \\ &\quad (v \in \mathbb{C}; \operatorname{Re} \lambda < 1 \text{ or } \lambda \in \mathbb{N}_+; \mu, \mu' \in \mathbb{R}; \operatorname{sign} \mu = \operatorname{sign} \mu') \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} \int_{-1}^1 dx \frac{P_v^{\lambda, i\mu}(x) P_v^{\lambda, i\mu'}(x)}{1+x} &= \frac{2^{-\lambda+2} \pi^2 \delta(\mu + \mu')}{\mu \sinh(\pi\mu)} \\ &\quad \times \Gamma^{-1}(v+1-(\lambda+i\mu)/2) \Gamma^{-1}(v+1-(\lambda-i\mu)/2) \\ &\quad \times \Gamma^{-1}(-v-(\lambda+i\mu)/2) \Gamma^{-1}(-v-(\lambda-i\mu)/2) \\ &\quad (v \in \mathbb{C}; \operatorname{Re} \lambda < 1 \text{ or } \lambda \in \mathbb{N}_+; \mu, \mu' \in \mathbb{R}; \operatorname{sign} \mu \neq \operatorname{sign} \mu'). \end{aligned} \quad (3.16)$$

It is easy to see that the two relations are not independent as Equation (3.16) may be deduced from Equation (3.15) after one replaces in the latter μ by $-\mu$ and uses the identity [9, Equation (4.2)]

$$P_v^{\lambda, -i\mu}(x) = 2^{-i\mu} P_v^{\lambda, i\mu}(x). \quad (3.17)$$

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