

Electrostatic Excitation of a Conducting Toroid: Exact Solution and Thin-Wire Approximation

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Laplace's equation is solved via separation of variables in toroidal coordinates for the electrostatic potential external to a conducting torus placed in a uniform electric field and excited by an arbitrarily located point charge. The accuracy of the static thin-wire kernel approximation in an integral equation applied to the circular loop is verified using the exact results in the limit as the toroid shrinks to a ring. An equivalent lineal charge density from the exact solution agrees remarkably well with the integral equation solution for the conducting ring. Since the singularity in the Helmholtz Green's function for the electrodynamic problem is the static singularity considered herein, the results confirm the applicability of the thin-wire kernel to the scattering and radiation problems of the circular loop.

Keywords toroidal coordinates, Legendre functions, thin-wire kernel, Laplace's equation

Introduction

An investigation into the accuracy and applicability of the thin-wire kernel approximation (Wu, 1962; King, 1969) in treating the *dynamic* electromagnetic wave interactions with a conducting circular loop leads directly to the *static* analysis of the present paper. Although the exact kernel can be integrated with sinusoidal ring currents to yield separated series expansions (Werner, 1996) at arbitrary field points, the simple and therefore *physically insightful* approximation provided by the thin-wire kernel is still an attractive tool in the numerical and asymptotic analysis of the loop antenna or scatterer. Both the low-frequency and even the high-frequency behavior of the Helmholtz Green's function $\exp(ikR)/R$ in a neighborhood of the singular point $R = |\vec{r} - \vec{r}'| \rightarrow 0$ is predominantly that of the static kernel $1/R$.

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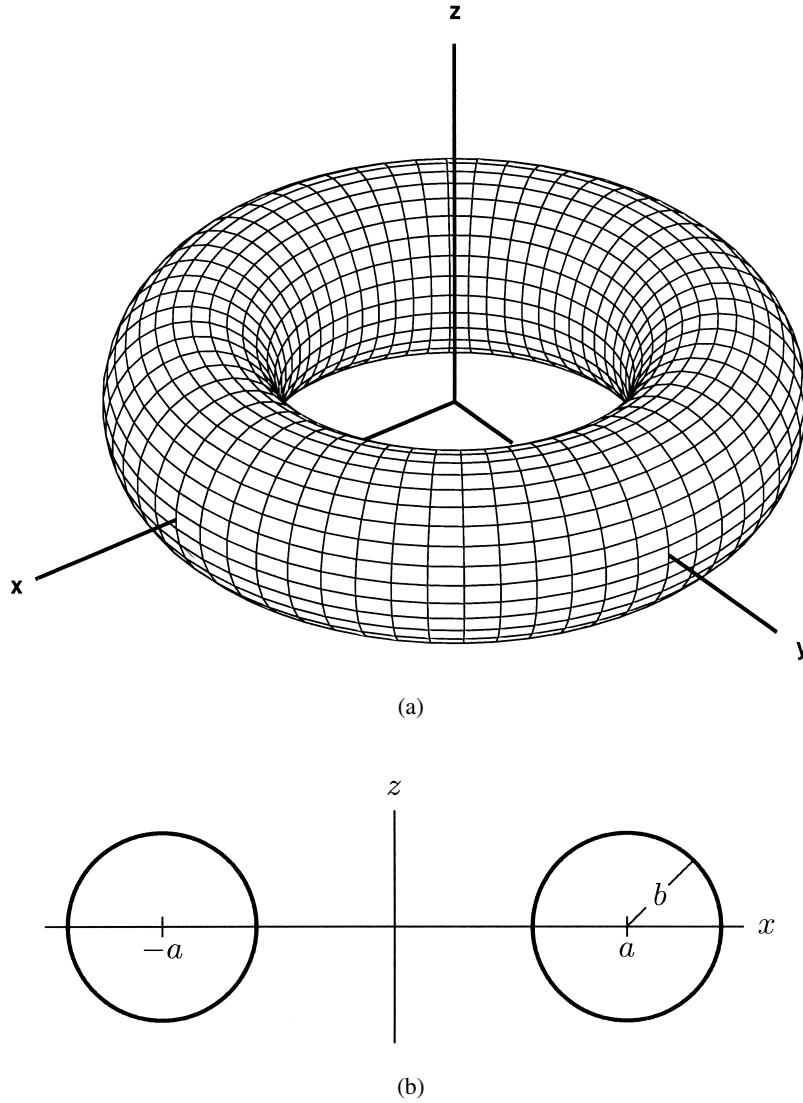


Figure 1. Toroid geometry. (a) Axonometric view; (b) cross section of toroidal surface in $\phi = 0, \pi$ plane.

The next section presents the exact solution in toroidal coordinates for the conducting toroid of Figure 1 excited by a uniform axial electric field, a uniform transverse electric field, and an arbitrarily located point source. The results of the uniform excitations are recovered from the general point-source result in the obvious limiting cases of properly positioned sources at infinity. Induced surface charge distributions (C/m^2) are collapsed to equivalent lineal or ring charge distributions (C/m) in the primary case of interest, the thin toroid. This is used to verify the results of the thin-wire approximation in the integral equation analysis of the third section (“Integral Equation for Ring Charge Density”). Detailed quantitative data is presented in the final section for the case of transverse electric field excitation, which is of primary relevance to the motivating wave problem.

The boundary value problems for several axisymmetric excitations of the conducting torus are stated, with partial answers, in a collection of problems (Lebedev, Skalskaya, & Uflyand, 1979). It is not surprising that both the axial and transverse electric field excitations are included in Smythe's formidable problem set (Smythe, 1968, 1974). Cade (1978a) performs the Kelvin transformation (image theory) for a point source on the axis of a thin toroid and then extends this to accommodate other axisymmetric source distributions. Cade comments that the use of toroidal harmonics is a "theory of much elegance but of questionable value, leading to solutions in terms of slowly converging series of functions which themselves, individually, are difficult to compute." The present paper resolves these difficulties. Cade (1978b) also studied general $\mathbb{R}^3$ toroids of revolution via a perturbative analysis that starts with the cross-sectional geometry in $\mathbb{R}^2$.

Exact Solution of Laplace's Equation in Toroidal Coordinates

Toroidal coordinates (ξ, η, ϕ) are related to cylindrical polar coordinates (ρ, ϕ, z) via

$$\rho + iz = c \coth \left[\frac{1}{2}(\xi - i\eta) \right] \quad (1)$$

or

$$\rho = \frac{c \sinh \xi}{\cosh \xi - \cos \eta} \quad \text{and} \quad z = \frac{c \sin \eta}{\cosh \xi - \cos \eta}. \quad (2)$$

The observations

$$(\rho - c \coth \xi)^2 + z^2 = (c \operatorname{csch} \xi)^2, \quad \rho^2 + (z - c \cot \eta)^2 = (c \csc \eta)^2 \quad (3)$$

enable the orthogonal circles of constant ξ and constant η to be drawn in the (ρ, z) plane as in Figure 2. Note that the origin of the (ρ, z) plane is mapped to the point $(\xi, \eta) = (0, \pi)$ and the point at infinity in the (ρ, z) plane is mapped to the point $(\xi, \eta) = (0, 0)$. As $\xi \rightarrow \infty$, η is ignorable and $(\rho, z) \rightarrow (c, 0)$.

The toroidal surface of Figure 1 is the surface of constant $\xi = \xi_0$, $-\pi \leq \eta \leq \pi$, and $-\pi \leq \phi \leq \pi$. With the original dimensions of the toroid

$$a = c \coth \xi_0 \quad \text{and} \quad b = c \operatorname{csch} \xi_0, \quad (4)$$

the introduced toroidal parameters are

$$c = \sqrt{a^2 - b^2} \quad \text{and} \quad \cosh \xi_0 = a/b. \quad (5)$$

Metric coefficients in the toroidal coordinates are

$$h_\xi = h_\eta = \frac{c}{\cosh \xi - \cos \eta}, \quad h_\phi = \frac{c \sinh \xi}{\cosh \xi - \cos \eta}. \quad (6)$$

Details of the separated differential equations are outlined in several texts (Lebedev, 1972; Moon & Spencer, 1961; Morse & Feshbach, 1953). The reciprocal distance between

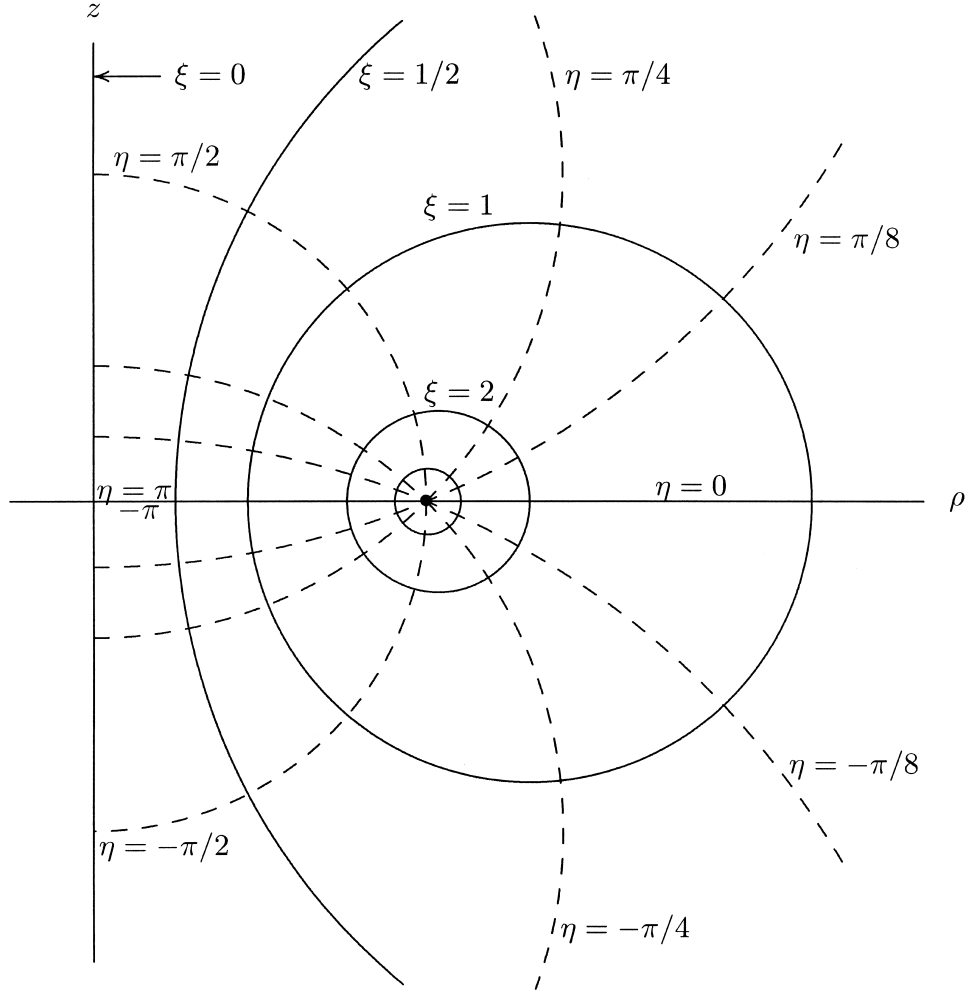


Figure 2. Circles of constant ξ and η in the (ρ, z) plane.

the points $\vec{r}$ and $\vec{r}'$ in toroidal coordinates is easily written using an addition theorem (Hobson, 1955, §223) and with a recursion relation (Abramowitz & Stegun, 1972, 8.2.5)

$$\begin{aligned}
 \frac{1}{|\vec{r} - \vec{r}'|} &= \frac{(\cosh \xi - \cos \eta)^{1/2} (\cosh \xi' - \cos \eta')^{1/2}}{c[2 \cosh \mu - 2 \cos(\eta - \eta')]^{1/2}} \\
 &= \frac{1}{c\pi} (\cosh \xi - \cos \eta)^{1/2} (\cosh \xi' - \cos \eta')^{1/2} \\
 &\quad \cdot \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon_n \epsilon_m (-1)^m \frac{\Gamma(n-m+1/2)}{\Gamma(n+m+1/2)} \\
 &\quad \cdot Q_{n-1/2}^m(\cosh \xi_{>}) P_{n-1/2}^m(\cosh \xi_{<}) \cos m(\phi - \phi') \cos n(\eta - \eta'),
 \end{aligned} \tag{7}$$

where

$$\cosh \mu \triangleq \cosh \xi \cosh \xi' - \sinh \xi \sinh \xi' \cos(\phi - \phi') \geq 1 \quad (8)$$

and with the Neumann number $\epsilon_n = 2 - \delta_{0n}$.

Axial Exciting Electric Field

In this case the boundary value problem is axisymmetric and the total harmonic potential is written as the incident plus scattered (or induced) potentials

$$\psi(\rho, z) = -E_0 z + \psi^s(\rho, z). \quad (9)$$

The disturbance caused by the introduction of the toroidal obstacle into the uniform field is of no consequence at infinity; hence $\psi^s(\infty) = 0$. The total potential must vanish on the conducting boundary $\xi = \xi_0$. The obvious odd symmetry in z translates to odd symmetry in η . An appropriate expansion for the scattered potential in the external region $0 \leq \xi \leq \xi_0$, $-\pi \leq \eta \leq \pi$ is

$$\psi^s(\xi, \eta) = E_0 c \sqrt{2 \cosh \xi - 2 \cos \eta} \sum_{n=1}^{\infty} a_n \frac{P_{n-1/2}(\cosh \xi)}{P_{n-1/2}(\cosh \xi_0)} \sin n\eta \quad (10)$$

in terms of half-degree toroidal or ring functions of the first kind. The Dirichlet boundary condition is now

$$\psi^s(\xi_0, \eta) = E_0 z = E_0 c \frac{\sin \eta}{\cosh \xi_0 - \cos \eta}, \quad (11)$$

so that a Fourier sine series is required in the form

$$\frac{\sin \eta}{[2 \cosh \xi_0 - 2 \cos \eta]^{3/2}} = \sum_{n=1}^{\infty} b_n \sin n\eta. \quad (12)$$

The Fourier coefficients b_n are evaluated by starting with an integral representation (Lebedev, 1972, 7.10.10) and performing an integration by parts:

$$\begin{aligned} Q_{n-1/2}(\cosh \xi) &= \int_0^\pi \frac{\cos nx}{[2 \cosh \xi - 2 \cos x]^{1/2}} dx \\ &= \frac{1}{n} \int_0^\pi \frac{\sin nx \sin x}{[2 \cosh \xi - 2 \cos x]^{3/2}} dx, \end{aligned} \quad (13)$$

such that the total potential is written entirely in toroidal coordinates as

$$\begin{aligned} \psi(\xi, \eta) &= \frac{4E_0 c}{\pi} \sqrt{2 \cosh \xi - 2 \cos \eta} \\ &\cdot \sum_{n=1}^{\infty} n \frac{Q_{n-1/2}(\cosh \xi_0) P_{n-1/2}(\cosh \xi) - Q_{n-1/2}(\cosh \xi) P_{n-1/2}(\cosh \xi_0)}{P_{n-1/2}(\cosh \xi_0)} \\ &\cdot \sin n\eta. \end{aligned} \quad (14)$$

This result can be transformed to Smythe's (1968, 1974) answers using Whipple's identity (Cohl et al., 2000). Charge density on the surface $\xi = \xi_0$ is

$$\sigma(\eta) = -\varepsilon_0 \frac{\partial}{\partial n} \psi(\xi_0, \eta) = \frac{\varepsilon_0}{h_\xi} \frac{\partial \psi(\xi_0, \eta)}{\partial \xi} = \frac{\varepsilon_0}{c} [\cosh \xi_0 - \cos \eta] \frac{\partial \psi(\xi_0, \eta)}{\partial \xi}. \quad (15)$$

Differentiation of the total potential gives, with the Wronskian (Lebedev, 1972, 7.7.2),

$$\sigma(\eta) = \frac{2\varepsilon_0 E_0}{\pi} \frac{[2 \cosh \xi_0 - 2 \cos \eta]^{3/2}}{\sinh \xi_0} \sum_{n=1}^{\infty} \frac{n}{P_{n-1/2}(\cosh \xi_0)} \sin n\eta. \quad (16)$$

As $b/a \rightarrow 0$, $\xi_0 \rightarrow \infty$ and the $n = 1$ term dominates, so that the surface charge density on a thin toroid is

$$\sigma(\eta) \xrightarrow{b/a \rightarrow 0} 2\varepsilon_0 E_0 \sin \eta. \quad (17)$$

This is the classical solution for the charge density on an infinitely long circular cylinder in the corresponding $\mathbb{R}^2$ potential problem.

Equipotentials from (14) are drawn in Figure 3 for a reasonably fat ($b/a = 0.8$) toroid. Normalized surface charge density of Figure 4 shows that plus and minus charges tend to accumulate on opposing inner surfaces of the fat toroid ($b/a = 0.8$), while the thin toroid ($b/a = 0.1$) displays the limiting behavior of (17).

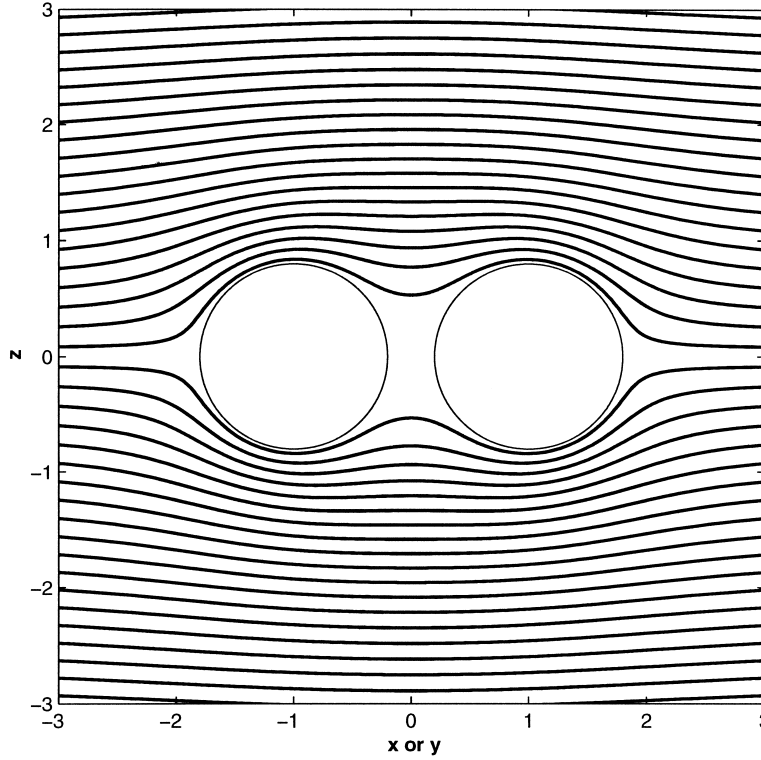


Figure 3. Equipotentials in the axial plane for axial excitation $\hat{z}E_0$. Case: $a = 1$, $b = 0.8$.

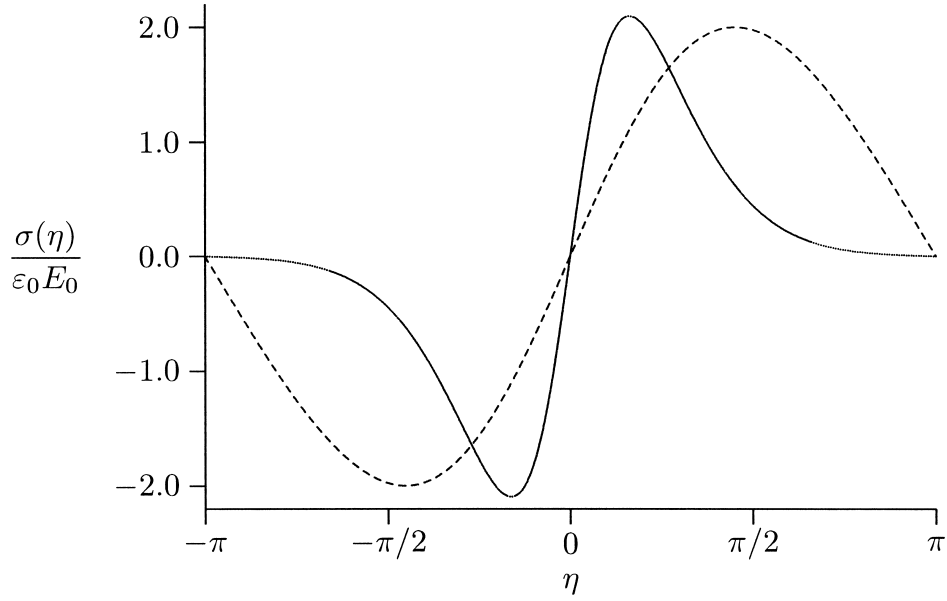


Figure 4. Normalized surface charge density for axial excitation. Solid curve: $b/a = 0.8$, dashed curve: $b/a = 0.1$.

Transverse Exciting Electric Field

With the source electric field polarized in the x direction, i.e., $\vec{E} = \hat{x}E_0$, the incident potential is $-E_0x = -E_0\rho \cos \phi$ and an appropriate expansion for the scattered potential is

$$\psi^s(\xi, \eta, \phi) = E_0c \cos \phi \sqrt{2 \cosh \xi - 2 \cos \eta} \sum_{n=0}^{\infty} c_n \frac{P_{n-1/2}^1(\cosh \xi)}{P_{n-1/2}^1(\cosh \xi_0)} \cos n\eta. \quad (18)$$

Vanishing of the total potential on the toroid surface $\xi = \xi_0$ now requires a Fourier series in the form

$$[2 \cosh \xi_0 - 2 \cos \eta]^{-3/2} = \sum_{n=0}^{\infty} \epsilon_n d_n \cos n\eta, \quad (19)$$

that is,

$$d_n = \frac{1}{\pi} \int_0^\pi \frac{\cos n\eta}{(2 \cosh \xi_0 - 2 \cos \eta)^{3/2}} d\eta = -\frac{1}{\pi} Q'_{n-1/2}(\cosh \xi_0) \quad (20)$$

by (13) above. The total potential is

$$\begin{aligned} \psi(\xi, \eta, \phi) &= -\frac{2E_0c}{\pi} \cos \phi \sqrt{2 \cosh \xi - 2 \cos \eta} \sum_{n=0}^{\infty} \epsilon_n \cos n\eta \\ &\quad \cdot \frac{\sinh \xi_0 Q'_{n-1/2}(\cosh \xi_0) P_{n-1/2}^1(\cosh \xi) - \sinh \xi Q'_{n-1/2}(\cosh \xi) P_{n-1/2}^1(\cosh \xi_0)}{P_{n-1/2}^1(\cosh \xi_0)}, \end{aligned} \quad (21)$$

in agreement with published results (Smythe, 1968, 1974) upon enlisting the Whipple and several recursion formulae. The Wronskian relationship

$$\begin{aligned} Q'_v(\zeta) P_v^1(\zeta) - Q''_v(\zeta) P_v^1(\zeta) &= W \left[Q_v^1(\zeta), P_v^1(\zeta) \right] + \frac{\zeta}{(\zeta^2 - 1)^{3/2}} Q_v^1(\zeta) P_v^1(\zeta) \\ &= -\frac{\nu(\nu + 1)}{(\zeta^2 - 1)^{3/2}} + \frac{\zeta}{(\zeta^2 - 1)^{3/2}} Q_v^1(\zeta) P_v^1(\zeta) \end{aligned} \quad (22)$$

permits the reduction

$$\begin{aligned} \frac{\partial}{\partial \xi} \left[\sinh \xi_0 Q'_{n-1/2}(\cosh \xi_0) P_{n-1/2}^1(\cosh \xi) - \sinh \xi Q'_{n-1/2}(\cosh \xi) P_{n-1/2}^1(\cosh \xi_0) \right]_{\xi=\xi_0} \\ = -\frac{(n - 1/2)(n + 1/2)}{\sinh \xi_0}. \end{aligned} \quad (23)$$

The induced surface charge density, as in (15), is hence

$$\sigma(\eta, \phi) = \frac{\varepsilon_0 E_0}{\pi} \cos \phi \frac{[2 \cosh \xi_0 - 2 \cos \eta]^{3/2}}{\sinh \xi_0} \sum_{n=0}^{\infty} \epsilon_n \frac{(n - 1/2)(n + 1/2)}{P_{n-1/2}^1(\cosh \xi_0)} \cos n\eta. \quad (24)$$

Only the $n = 0, 1$ terms are significant for a very thin toroid ($\cosh \xi_0 = a/b \rightarrow \infty$), resulting in

$$\sigma(\eta, \phi) \xrightarrow{b/a \rightarrow 0} \varepsilon_0 E_0 \cos \phi \left\{ \frac{\cosh \xi_0}{\xi_0 - 2(1 - \ln 2)} + 3 \cos \eta \left[1 - \frac{1}{2\xi_0 - 4(1 - \ln 2)} \right] \right\}. \quad (25)$$

An equivalent ring (lineal) charge density (C/m) is

$$\sigma_\ell(\phi) = 2\pi b \lim_{b/a \rightarrow 0} \sigma(\eta, \phi) = \frac{2\pi \varepsilon_0 E_0 a}{\xi_0 - 2(1 - \ln 2)} \cos \phi. \quad (26)$$

Equipotentials (21) in the meridian plane are drawn in Figure 5 for $b/a = 0.01$. The extremely thin toroid allows a noticeable potential variation in the “interior” region $\rho < a$. The curves in Figure 6 of normalized surface charge density (24) at $\phi = 0$ illustrate the importance of the constant (independent of η) term as b/a changes from the selected values 0.8 to 0.1.

Point Charge Excitation

No loss in generality is suffered by the Green's function in the presence of the axisymmetric toroid by placing the point charge q at the azimuthal coordinate $\phi = 0$. The spherical coordinates of the source point $(r_s, \theta_s, 0)$ transform to the toroidal coordinates

$$\coth \xi_s = \frac{r_s^2 + c^2}{2cr_s \sin \theta_s}, \quad \cot \eta_s = \frac{r_s^2 - c^2}{2cr_s \cos \theta_s}. \quad (27)$$

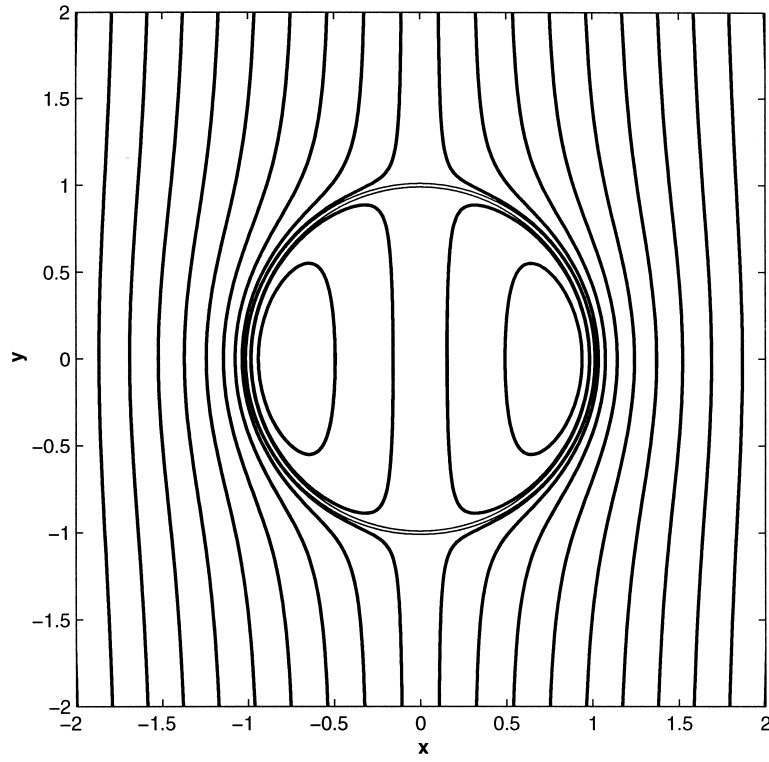


Figure 5. Equipotentials in the meridian ($z = 0$) plane for transverse excitation $\hat{x}E_0$. Case: $a = 1$, $b = 0.01$.

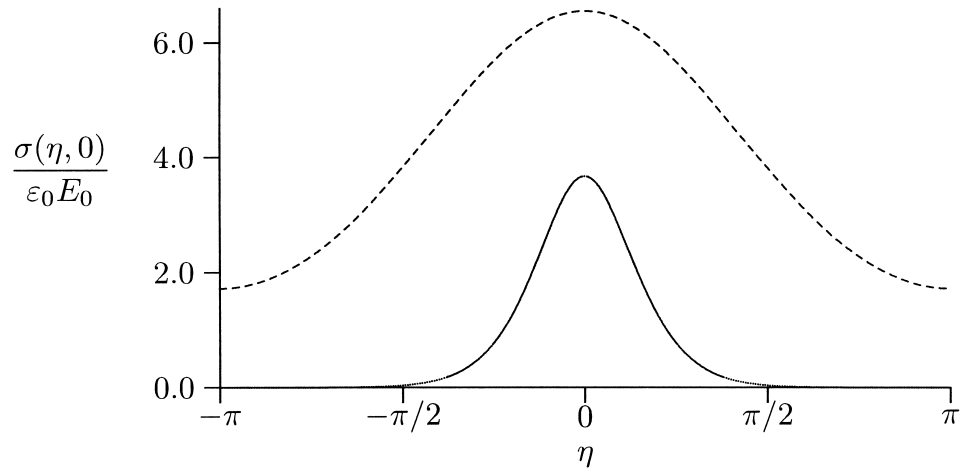


Figure 6. Normalized surface charge density for transverse excitation. Solid curve: $b/a = 0.8$, dashed curve: $b/a = 0.1$.

The free-field or source potential due to a point charge q at $(\xi_s, \eta_s, 0)$, in the absence of the toroid, is from (7),

$$\begin{aligned} \psi^i(\xi, \eta, \phi) &= \frac{q}{4\pi^2 \epsilon_0 c} (\cosh \xi - \cos \eta)^{1/2} (\cosh \xi_s - \cos \eta_s)^{1/2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon_n \epsilon_m (-1)^m \\ &\cdot \frac{\Gamma(n-m+1/2)}{\Gamma(n+m+1/2)} Q_{n-1/2}^m(\cosh \xi_>) P_{n-1/2}^m(\cosh \xi_<) \cos m\phi \cos n(\eta - \eta_s). \end{aligned} \quad (28)$$

The total potential, that is, the sum of this exciting field plus an induced or scattered potential, can now be written by inspection in these natural toroidal coordinates. As in all the cases of this paper, the scattered potential must be bounded external to the torus where $0 \leq \xi \leq \xi_0$, and the total potential vanishes on the conducting surface $\xi = \xi_0$. Thus, the total potential is

$$\begin{aligned} \psi(\xi, \eta, \phi) &= \frac{q}{4\pi^2 \epsilon_0 c} (\cosh \xi - \cos \eta)^{1/2} (\cosh \xi_s - \cos \eta_s)^{1/2} \\ &\cdot \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon_n \epsilon_m (-1)^m \frac{\Gamma(n-m+1/2)}{\Gamma(n+m+1/2)} \frac{\cos m\phi \cos n(\eta - \eta_s)}{P_{n-1/2}^m(\cosh \xi_0)} \\ &\cdot \left[P_{n-1/2}^m(\cosh \xi_0) Q_{n-1/2}^m(\cosh \xi_>) P_{n-1/2}^m(\cosh \xi_<) \right. \\ &\quad \left. - Q_{n-1/2}^m(\cosh \xi_0) P_{n-1/2}^m(\cosh \xi_s) P_{n-1/2}^m(\cosh \xi) \right]. \end{aligned} \quad (29)$$

It is advantageous to compute the point-source potential directly, as in (7), and separately evaluate the double series contribution for the scattered potential. The surface charge density, as in (15), simplifies nicely because the gamma functions in the applicable Wronskian cancel those in the series such that

$$\begin{aligned} \sigma(\eta, \phi) &= -\frac{q}{4\pi^2 c^2 \sinh \xi_0} (\cosh \xi_s - \cos \eta_s)^{1/2} (\cosh \xi_0 - \cos \eta)^{3/2} \\ &\cdot \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon_n \epsilon_m \frac{P_{n-1/2}^m(\cosh \xi_s)}{P_{n-1/2}^m(\cosh \xi_0)} \cos m\phi \cos n(\eta - \eta_s). \end{aligned} \quad (30)$$

As $b/a \rightarrow 0$ the appropriate asymptotic forms of the denominator functions are

$$\begin{aligned} P_{-1/2}(\cosh \xi_0) &\xrightarrow{\xi_0 \rightarrow \infty} \frac{2}{\pi} (\xi_0 + 2 \ln 2) e^{-\xi_0/2}, \\ P_{-1/2}^m(\cosh \xi_0) &\xrightarrow{\xi_0 \rightarrow \infty} \frac{2}{\sqrt{\pi}} \frac{\xi_0 - \ln 2}{\Gamma(1/2 - m)} e^{-\xi_0/2} \quad (m = 1, 2, \dots), \\ P_{n-1/2}^m(\cosh \xi_0) &\xrightarrow{\xi_0 \rightarrow \infty} \frac{\Gamma(n)}{\sqrt{\pi} \Gamma(n-m+1/2)} e^{\xi_0(n-1/2)} \quad \begin{pmatrix} m = 0, 1, \dots \\ n = 1, 2, \dots \end{pmatrix}. \end{aligned} \quad (31)$$

Obviously, then, only the terms with $n = 0$ matter in the case of an extremely thin toroid, and the approximate lineal charge density, independent of η as expected, is

$$\begin{aligned}\sigma_\ell(\phi) &= 2\pi b \lim_{b/a \rightarrow 0} \sigma(\eta, \phi) \\ &= -\frac{q}{2\pi a} \sqrt{\frac{\sinh \xi_0}{\sinh \xi_s}} (\cosh \xi_s - \cos \eta_s)^{1/2} \sum_{m=0}^{\infty} \epsilon_m \frac{Q_{m-1/2}(\coth \xi_s)}{Q_{m-1/2}(\coth \xi_0)} \cos m\phi,\end{aligned}\quad (32)$$

where

$$\frac{b(\cosh \xi_0 - \cos \eta)^{3/2}}{c^2 \sinh \xi_0} \xrightarrow{b/a \rightarrow 0} \frac{1}{a}, \quad (33)$$

and Whipple's relation (Cohl et al., 2000) has been used to transform the ratio of first-kind $P_{-1/2}^m$ functions of varying order to a ratio of second-kind $Q_{m-1/2}$ functions of varying degree and order zero. In this limiting case, the argument of the denominator function is

$$\coth \xi_0 = \left[1 - (b/a)^2\right]^{-1/2} \approx 1 + \frac{1}{2}(b/a)^2 = 1 + 2\alpha^2, \quad (34)$$

where the small parameter $\alpha = b/2a$ arises naturally in the thin-wire approximation of the next section. Using the above toroidal-to-spherical coordinate transformations, this equivalent lineal charge density becomes, since $c \rightarrow a$ and $\xi_0 \rightarrow \infty$ as $b/a \rightarrow 0$,

$$\sigma_\ell(\phi) = -\frac{q}{2\pi \sqrt{a r_s} \sin \theta_s} \sum_{m=0}^{\infty} \epsilon_m \frac{Q_{m-1/2}(\coth \xi_s)}{Q_{m-1/2}(1 + 2\alpha^2)} \cos m\phi \quad (-\pi \leq \phi \leq \pi). \quad (35)$$

This expression is in exact agreement with the independent solution (62) of the thin-wire integral equation of the subsection on "Point Charge Excitation."

Equipotentials are drawn in the source plane ($y = 0$ or $\phi = 0, \pi$) in Figure 7 for a representative toroid excited by a point source at $(\rho_s, \phi_s, z_s) = (4, 0, 4)$. The scaled lineal charge density (32) of a thin toroid with $b/a = 0.001$ is graphed as a function of ϕ in Figure 8 when the exciting point charge lies in the plane of the ring ($\theta_s = \pi/2$). Three cases are presented: the point charge half a radius inside ($r_s = 0.5$), half a radius outside ($r_s = 1.5$), and close to the outside ($r_s = 1.1$) of the ring of unit radius $a = 1$.

The uniform exciting fields of this section are recovered as special cases of this more general point-source excitation by considering a pair of equal and opposite point charges (Jackson, 1999). The uniform axial field derives from placing $+q/2$ a distance r_s on the positive z axis, i.e., at the spherical coordinates $(r_s, 0, 0)$ accompanied by $-q/2$ the same distance r_s along the negative z axis, i.e., at the spherical coordinates $(r_s, \pi, 0)$. Then, as $r_s \rightarrow \infty$:

$$\begin{aligned}\xi_s &\rightarrow 0, & \cot \eta_s^\pm &\rightarrow \pm \frac{r_s}{2c}, & \sin \eta_s^\pm &\rightarrow \pm \frac{2c}{r_s}, & \cos \eta_s &\rightarrow 1 - 2(c/r_s)^2, \\ (\cosh \xi_s - \cos \eta_s)^{1/2} &\rightarrow \sqrt{2} \frac{c}{r_s}, & P_{n-1/2}^m(\cosh \xi_s) &\rightarrow \delta_{m,0}, \\ \cos n(\eta - \eta_s) - \cos n(\eta + \eta_s) &= 2 \sin n\eta \sin n\eta_s \rightarrow \frac{4cn}{r_s} \sin n\eta.\end{aligned}\quad (36)$$

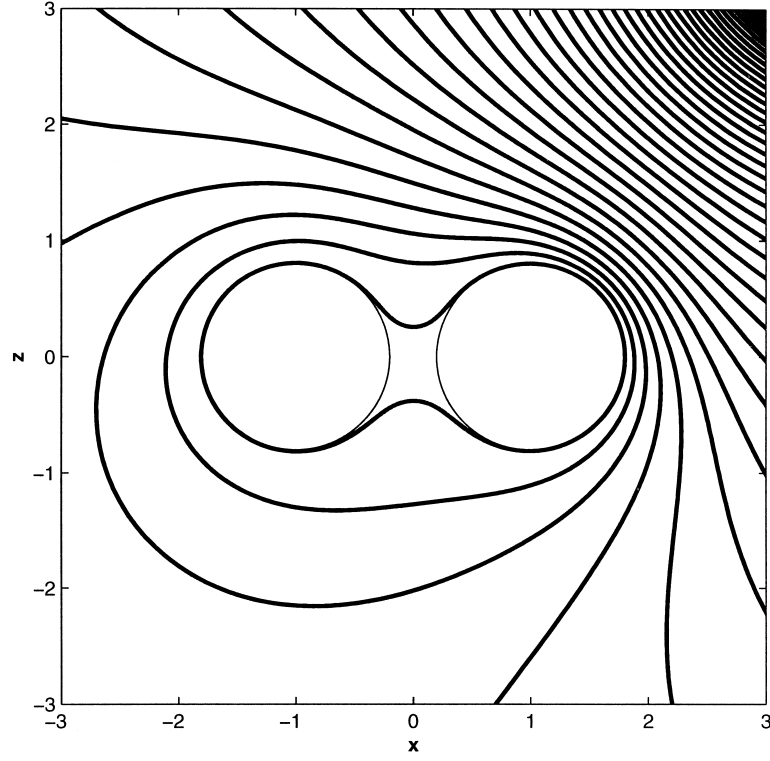


Figure 7. Equipotentials in the $y = 0$ plane for point-source excitation. Case: $a = 1$, $b = 0.8$; source coordinates $(x_s, y_s, z_s) = (4, 0, 4)$.

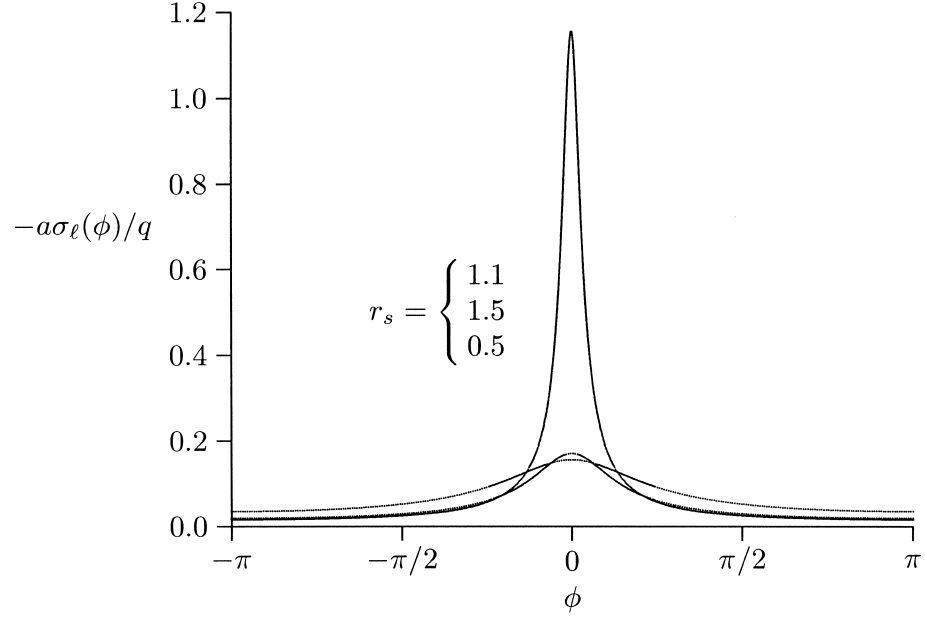


Figure 8. Lineal charge density for point-source excitation. $\theta_s = \pi/2$, $\phi_s = 0$, $a = 1$, $b = 0.001$, $r_s = 0.5, 1.1, 1.5$.

This superposition of the two appropriately adjusted potentials is

$$\psi^s \rightarrow -\frac{qc}{\pi^2 \epsilon_0 r_s^2} (2 \cosh \xi - 2 \cos \eta)^{1/2} \sum_{n=1}^{\infty} n \frac{Q_{n-1/2}(\cosh \xi_0)}{P_{n-1/2}(\cosh \xi_0)} P_{n-1/2}(\cosh \xi) \sin n\eta, \quad (37)$$

exactly equal to the scattered component of (14) in the previous subsection, where

$$E_0 = \frac{-q}{4\pi \epsilon_0 r_s^2} \quad (38)$$

is the magnitude of the z -directed uniform electric field.

The x -directed uniform exciting field is likewise obtained by placing $+q/2$ at $(r_s, \pi/2, 0)$ (along the $+x$ axis) and $-q/2$ at the diametrically opposite point $(r_s, \pi/2, \pi)$. The required limiting geometrical forms in this case are, as $r_s \rightarrow \infty$ and with $\eta_s = 0$:

$$\begin{aligned} \coth \xi_s &\rightarrow \frac{r_s}{2c}, & \cosh \xi_s &\rightarrow 1 + 2(c/r_s)^2, \\ \sinh \xi_s &\rightarrow \frac{2c}{r_s}, & \cosh \xi_s - \cos \eta_s &\rightarrow 2(c/r_s)^2. \end{aligned} \quad (39)$$

The combination $\cos m\phi - \cos m(\phi - \pi)$ furnishes the sum of contributions

$$\begin{aligned} \psi^s &\rightarrow \frac{q}{2\pi^2 \epsilon_0 r_s} (2 \cosh \xi - 2 \cos \eta)^{1/2} \sum_{n=0}^{\infty} \epsilon_n \sum_{m=1,3,\dots}^{\infty} \frac{\Gamma(n-m+1/2)}{\Gamma(n+m+1/2)} \cos m\phi \cos n\eta \\ &\cdot \frac{Q_{n-1/2}^m(\cosh \xi_0) P_{n-1/2}^m(\cosh \xi_s) P_{n-1/2}^m(\cosh \xi)}{P_{n-1/2}^m(\cosh \xi_0)}. \end{aligned} \quad (40)$$

The leading term in an adaptation of an asymptotic form (Olver, 1997)

$$P_{n-1/2}^m(\cosh \xi_s) \xrightarrow{r_s \rightarrow \infty} \frac{\Gamma(n+m+1/2)}{\Gamma(m+1) \Gamma(n-m+1/2)} \left(\frac{c}{r_s}\right)^m \quad (41)$$

shows that only the $m = 1$ term is of any consequence:

$$\begin{aligned} \psi^s &\rightarrow \frac{qc}{2\pi^2 \epsilon_0 r_s^2} \cos \phi (2 \cosh \xi - 2 \cos \eta)^{1/2} \\ &\cdot \sum_{n=0}^{\infty} \epsilon_n \frac{Q_{n-1/2}^1(\cosh \xi_0) P_{n-1/2}^1(\cosh \xi)}{P_{n-1/2}^1(\cosh \xi_0)} \cos n\eta. \end{aligned} \quad (42)$$

This result is exactly equal to the scattered term of (21) with

$$Q_{n-1/2}^1(\cosh \xi_0) = \sinh \xi_0 Q'_{n-1/2}(\cosh \xi_0) \quad (43)$$

and with the x -directed incident electric field E_0 also given by (38).

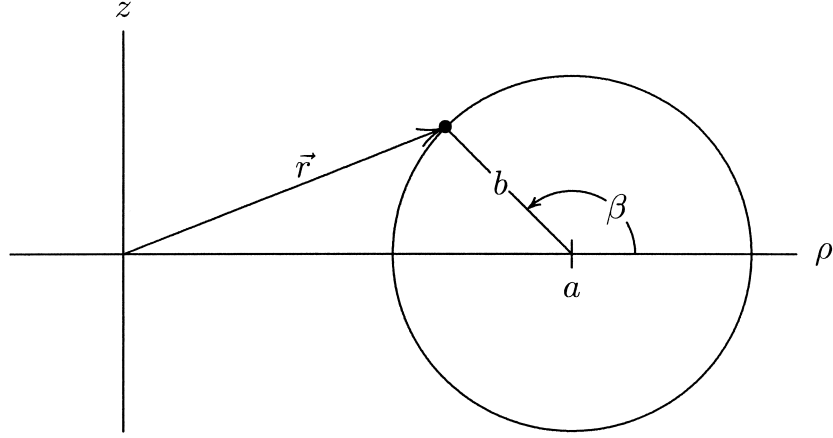


Figure 9. Toroid geometry in bastardized cylindrical polar coordinates.

Integral Equation for Ring Charge Density

Treatment of the conducting ring of finite thickness is facilitated by considering the exaggerated geometry of Figure 9, where the position of an arbitrary point on the ring surface is

$$\vec{r} = (a + b \cos \beta) \hat{\rho} + b \sin \beta \hat{z} \quad (44)$$

in terms of an angle β that describes the location around the circumference of the reasonably thin ring. Note that β is not, in general, equal to η of the toroidal coordinates, but it is in the limit of a vanishingly thin ring. With the field point $\vec{r}$ on the surface of the thin toroid (44) and with the source point right on the axis $\vec{r}' = a\hat{\rho}'$ according to this equivalent line charge model, the subject distance function is

$$|\vec{r} - \vec{r}'|^2 = 4a^2 \sin^2 \left[\frac{1}{2}(\phi - \phi') \right] + 4ab \sin^2 \left[\frac{1}{2}(\phi - \phi') \right] \cos \beta + b^2. \quad (45)$$

The average of this over β eliminates the $\cos \beta$ term, which is evidently one source of the static form of the thin-wire kernel approximation (Wu, 1962)

$$|\vec{r} - \vec{r}'| \approx 2a \left\{ \sin^2 \left[\frac{1}{2}(\phi - \phi') \right] + (b/2a)^2 \right\}^{1/2} \quad (46)$$

that was also adopted for subsequent loop antenna and scatterer solutions (King, 1969).

Vanishing of the total (incident plus scattered) potential on the surface of the thin ring gives the integral equation

$$\frac{1}{4\pi\epsilon_0} \int_C \frac{\sigma_\ell(\vec{r}')}{|\vec{r} - \vec{r}'|} d\ell' = -\psi^i(\vec{r}) \quad (\vec{r} \in C) \quad (47)$$

for the induced lineal charge density, where $d\ell' = a d\phi'$.

Transverse Uniform Electric Field Excitation

In this case all the fields vary like $\cos \phi$, so the single unknown is the magnitude $\sigma_{\ell 0}$ of the induced lineal charge density

$$\sigma_{\ell}(\vec{r}') = \sigma_{\ell 0} \cos \phi'. \quad (48)$$

The integral equation is therefore

$$\int_{-\pi}^{\pi} \frac{\cos \phi'}{\left\{ \sin^2 \left[\frac{1}{2}(\phi' - \phi) \right] + (b/2a)^2 \right\}^{1/2}} d\phi' = \frac{8\pi \varepsilon_0 E_0 a}{\sigma_{\ell 0}} \cos \phi \quad (-\pi \leq \phi \leq \pi). \quad (49)$$

A change of integration variable $\theta = \phi' - \phi$, with $\alpha = b/2a$, produces

$$I(\alpha) = \int_0^{\pi} \frac{\cos \theta}{\left[\sin^2 \frac{1}{2}\theta + \alpha^2 \right]^{1/2}} d\theta = \frac{4\pi \varepsilon_0 E_0 a}{\sigma_{\ell 0}}. \quad (50)$$

This integral is our familiar toroidal function of degree 1/2, which can alternately be expressed in terms of the complete elliptic integral of the first and second kinds to exploit tabulated asymptotic behavior

$$\begin{aligned} I(\alpha) &= 2 \int_0^{\pi} \frac{\cos \theta}{[2(1 + 2\alpha^2) - 2\cos \theta]^{1/2}} d\theta \\ &= 2Q_{1/2}(1 + 2\alpha^2) \\ &= \frac{4\alpha^2 + 2}{\sqrt{\alpha^2 + 1}} \mathbf{K}[(\alpha^2 + 1)^{-1/2}] - 4\sqrt{\alpha^2 + 1} \mathbf{E}[(\alpha^2 + 1)^{-1/2}] \\ &= 2[\ln(4/\alpha) - 2] + \frac{1}{2}[3\ln(4/\alpha) - 1]\alpha^2 + \frac{1}{32}[15\ln(\alpha/4) + 23/2]\alpha^4 + \mathcal{O}(\alpha^6). \end{aligned} \quad (51)$$

The leading term is most useful in the present case of a thin toroid, whereupon the desired coefficient of the lineal charge density is

$$\sigma_{\ell 0} \approx \frac{2\pi \varepsilon_0 E_0 a}{\ln(4/\alpha) - 2} \approx \frac{2\pi \varepsilon_0 E_0 a}{\xi_0 - 2(1 - \ln 2)} \quad (52)$$

in agreement with (35), since for $b/a \rightarrow 0$,

$$\xi_0 = \cosh^{-1} \frac{a}{b} \approx \ln \frac{2a}{b} = -\ln \alpha. \quad (53)$$

Point Charge Excitation

The forcing term in the integral equation (47) is now the incident potential

$$\psi^i(\vec{r}) = \frac{q}{4\pi\epsilon_0|\vec{r} - \vec{r}_s|}, \quad (54)$$

where the source charge q is located at

$$\vec{r}_s = (r_s, \theta_s, 0) = (\rho_s, 0, z_s) = (\xi_s, \eta_s, 0) \quad (55)$$

in spherical, cylindrical polar, and toroidal coordinates, respectively. A convenient representation (Morse & Feshbach, 1953) for the reciprocal distance of the exciting potential is

$$\frac{1}{|\vec{r} - \vec{r}_s|} = \frac{1}{\pi\sqrt{\rho\rho_s}} \sum_{m=0}^{\infty} \epsilon_m Q_{m-1/2}(\cosh \chi) \cos m\phi, \quad (56)$$

where

$$\cosh \chi \triangleq \frac{\rho^2 + \rho_s^2 + (z - z_s)^2}{2\rho\rho_s}. \quad (57)$$

The integral equation, with the thin-wire kernel approximation, is therefore

$$\begin{aligned} & \int_{-\pi}^{\pi} \frac{\sigma_\ell(\phi')}{\left\{ \sin^2 \left[\frac{1}{2}(\phi - \phi') \right] + (b/2a)^2 \right\}^{1/2}} d\phi' \\ &= -\frac{2q}{\pi\sqrt{ar_s \sin \theta_s}} \sum_{m=0}^{\infty} \epsilon_m Q_{m-1/2} \left(\frac{a^2 + r_s^2}{2ar_s \sin \theta_s} \right) \cos m\phi \quad (-\pi \leq \phi \leq \pi). \end{aligned} \quad (58)$$

Expanding the unknown charge density in a Fourier series

$$\sigma_\ell(\phi') = -\frac{2q}{\pi\sqrt{ar_s \sin \theta_s}} \sum_{n=0}^{\infty} \epsilon_n A_n Q_{n-1/2} \left(\frac{a^2 + r_s^2}{2ar_s \sin \theta_s} \right) \cos n\phi' \quad (-\pi \leq \phi' \leq \pi) \quad (59)$$

requires evaluation of

$$I(n, \alpha) = \int_{-\pi}^{\pi} \frac{\cos n\phi'}{\sqrt{\sin^2 \left[\frac{1}{2}(\phi' - \phi) \right] + \alpha^2}} d\phi', \quad (60)$$

where $\alpha = b/2a$ is always the important parameter. Letting $\theta = \phi' - \phi$ and noting the periodicity and symmetry in the integrand shows

$$\begin{aligned} I(n, \alpha) &= 2 \cos n\phi \int_0^{\pi} \frac{\cos n\theta}{\sqrt{\sin^2 \frac{1}{2}\theta + \alpha^2}} d\theta \\ &= 4 \cos n\phi \int_0^{\pi} \frac{\cos n\theta}{[2(1 + 2\alpha^2) - 2 \cos \theta]^{1/2}} d\theta \\ &= 4 \cos n\phi Q_{n-1/2}(1 + 2\alpha^2). \end{aligned} \quad (61)$$

Equality of the Fourier series in ϕ on the left- and right-hand sides of the integral equation (58) now reveals the complete solution for the ring charge density

$$\sigma_\ell(\phi) = -\frac{q}{2\pi\sqrt{ar_s\sin\theta_s}} \sum_{n=0}^{\infty} \epsilon_n \frac{Q_{n-1/2}\left(\frac{a^2 + r_s^2}{2ar_s\sin\theta_s}\right)}{Q_{n-1/2}(1 + 2\alpha^2)} \cos n\phi \quad (-\pi \leq \phi \leq \pi). \quad (62)$$

Since $\alpha \ll 1$, the interpretation of some limiting cases is assisted by the approximations

$$Q_{-1/2}(1 + 2\alpha^2) \xrightarrow{b/a \rightarrow 0} \ln \frac{2a}{b} \approx \xi_0 \quad (63)$$

and

$$Q_{1/2}(1 + 2\alpha^2) \xrightarrow{b/a \rightarrow 0} 2\ln 2 - 2 + \ln \frac{2a}{b} \approx \xi_0 - 2(1 - \ln 2). \quad (64)$$

If $\theta_s \rightarrow 0$, then $\coth \xi_s \rightarrow \infty$ and only the $n = 0$ term is significant and

$$\sigma_\ell(\phi) \rightarrow -\frac{q}{2\xi_0\sqrt{r_s^2 + a^2}}. \quad (65)$$

This term $\propto 1/\xi_0 \approx 0$ to the order of approximation here. In this case the incident electric field is axial to the ring and the dominant term in the surface charge varies as $\sin \eta$, known from the exact solution (17). Accordingly, the thin-wire approximation of (46) in (47) is trivial in such a case where the incident electric field has no component tangential to the infinitesimally thin ring.

If $\theta_s = \pi/2$ and $r_s \rightarrow \infty$, then $\coth \xi_s \rightarrow r_s/2a$ and

$$\begin{aligned} \sigma_\ell(\phi) &\xrightarrow{r_s \rightarrow \infty} -\frac{q}{2r_s} \left[\frac{1}{Q_{-1/2}(1 + 2\alpha^2)} + \frac{a/r_s}{Q_{1/2}(1 + 2\alpha^2)} \cos \phi \right] \\ &\xrightarrow{r_s \rightarrow \infty} -\frac{q}{2r_s\xi_0} - \frac{qa}{2r_s^2[\xi_0 - 2(1 - \ln 2)]} \cos \phi \\ &= -\frac{q}{2r_s\xi_0} + \frac{2\pi\epsilon_0 E_0 a}{\xi_0 - 2(1 - \ln 2)} \cos \phi \end{aligned} \quad (66)$$

where the familiar (38) is the magnitude of the incident electric field, which is essentially uniform and polarized in the x direction, transverse to the ring. The $\cos \phi$ term agrees with (26), as required. The first term that is constant in ϕ is not present in the analysis in the subsection on the ‘‘Transverse Exciting Electric Field,’’ where the ϕ -dependence of all the fields is $\cos \phi$ at the outset. Following again the symmetric limiting procedure (Jackson, 1999), the uniform transverse excitation is correctly recovered from the point-source solution by considering a *pair* of point sources, $+q/2$ at $(r_s, \pi/2, 0)$ and $-q/2$ at $(r_s, \pi/2, \pi)$ as $r_s \rightarrow \infty$. The constant terms from each of these contributions then cancel.

Summary of Main Implication for the Dynamic Problem

One important implication of the electrostatic analysis of this work is its support of the thin-wire kernel approximation (Wu, 1962). The case of a transverse electric field exhibits the most interaction with a thin ring and displays the dominant, static features of the motivating electrodynamic problem. Therefore, the differences between the exact and approximate solutions for the induced charge density in this case merit quantification. The known $\cos \phi$ variation is disregarded in the normalization of the thin-wire result (52)

$$\frac{\sigma_\ell(\phi)}{\varepsilon_0 E_0 a \cos \phi} = \frac{2\pi}{\ln(8a/b) - 2} = B_0, \quad (67)$$

where the constant B_0 is a convenient label. A comparison with the exact surface charge density of (24) is effected through the construction

$$\begin{aligned} \frac{2\pi b \sigma(\eta, \phi)}{\varepsilon_0 E_0 a \cos \phi} &= \frac{2b}{a} \frac{(2 \cosh \xi_0 - 2 \cos \eta)^{3/2}}{\sinh \xi_0} \sum_{n=0}^{\infty} \epsilon_n \frac{(n-1/2)(n+1/2)}{P_{n-1/2}^1(\cosh \xi_0)} \cos n\eta \\ &= C_0(\eta) + C_1(\eta) \cos \eta + C_2(\eta) \cos 2\eta + \cdots, \end{aligned} \quad (68)$$

which is *not* a Fourier series in η owing to the factor before the summation. However, a meaningful understanding of the relative importance of the terms is achieved by noting that the coefficient functions called $C_0(\eta)$ and $C_2(\eta)$ are most important when $\eta = \pi/2$, for example. Similarly, the coefficient $C_1(\eta)$ has its biggest effect when $\eta = 0$. The variation of these coefficients $C_0(\pi/2)$, $C_2(\pi/2)$, $C_1(0)$ with b/a is graphed in Figure 10, along with the thin-wire result, presently B_0 in this normalization. The striking feature of this result is the accuracy of the thin-wire result B_0 compared to the exact result embodied in C_0 , even for toroids that are definitely fat. Of course, the constant (independent of η or β) term of the thin-wire approximation is insufficient to completely

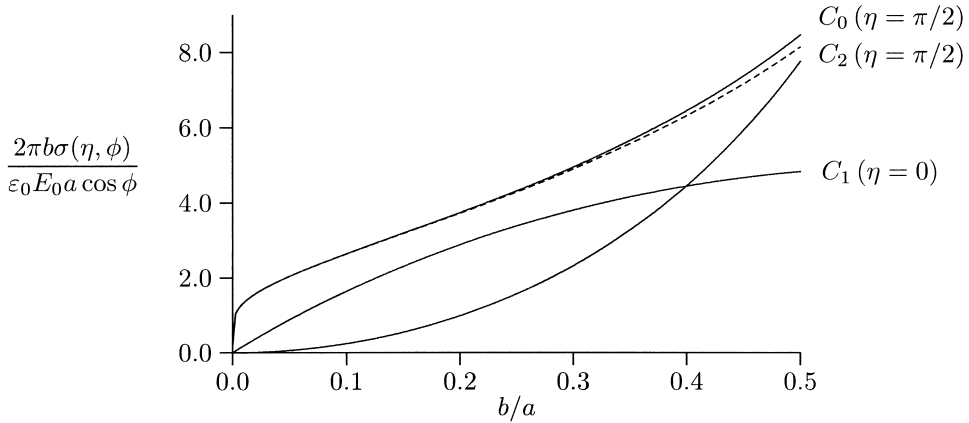


Figure 10. Coefficients of normalized charge density for transverse excitation. Solid curves: exact solution; Dashed curve: B_0 from thin-wire kernel approximation.

Table 1
Relative error between B_0 and $C_0(\pi/2)$

b/a	ξ_0	Error (—%)
10^{-6}	14.5	6.8×10^{-12}
10^{-5}	12.2	6.8×10^{-10}
10^{-4}	9.9	6.9×10^{-8}
10^{-3}	7.6	7.1×10^{-6}
10^{-2}	5.3	7.6×10^{-4}
10^{-1}	3.0	8.9×10^{-2}
0.2	2.3	4.0×10^{-1}
0.5	1.3	3.7×10^0

characterize a toroid of nonnegligible thickness, where the terms involving C_1 and C_2 become significant. Numerical values of the percentage difference or

$$\text{relative error} = \frac{B_0 - C_0(\pi/2)}{C_0(\pi/2)} \times 100\% \quad (69)$$

are given in Table 1.

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