

Legendre Functions, Spherical Rotations, and Multiple Elliptic Integrals

Yajun Zhou

PROGRAM IN APPLIED AND COMPUTATIONAL MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544

A closed-form formula is derived for the generalized Clebsch-Gordan integral $\int_{-1}^1 [P_\nu(x)]^2 P_\nu(-x) dx$, with P_ν being the Legendre function of arbitrary complex degree $\nu \in \mathbb{C}$. The finite Hilbert transform of $P_\nu(x)P_\nu(-x)$, $-1 < x < 1$ is evaluated. An analytic proof is provided for a recently conjectured identity $\int_0^1 [\mathbf{K}(\sqrt{1-k^2})]^3 dk = 6 \int_0^1 [\mathbf{K}(k)]^2 \mathbf{K}(\sqrt{1-k^2}) k dk = [\Gamma(\frac{1}{4})]^8 / (128\pi^2)$ involving complete elliptic integrals of the first kind $\mathbf{K}(k)$ and Euler's gamma function $\Gamma(z)$.

0 Introduction

In a wide range of mathematical and physical contexts, one needs to evaluate *multiple elliptic integrals* where the integrands involve one or more factors of complete elliptic integrals. As a glimpse of some recent applications in number theory, high-energy physics, statistical mechanics and probability theory, we mention automorphic Green's functions [1], multi-loop Feynman diagrams [2], lattice Green's functions [3] and short uniform random walks [4, 5, 6]. Various techniques, of algebraic, geometric, combinatoric or analytic flavor, have been developed to express these multiple elliptic integrals in terms of familiar mathematical constants and special values of Euler's gamma function [1, 2, 3, 4, 5, 6].

When the integrands contain the products of three or more complete elliptic integrals, the evaluation can become analytically challenging. For example, the following relation

$$\frac{[\Gamma(\frac{1}{4})]^8}{128\pi^2} \stackrel{?}{=} \int_0^1 [\mathbf{K}(\sqrt{1-k^2})]^3 dk \stackrel{?}{=} 6 \int_0^1 [\mathbf{K}(k)]^2 \mathbf{K}(\sqrt{1-k^2}) k dk$$

has been recently conjectured by J. G. Wan in [6] based on numerical evidence.

In this work, we solve Wan's conjecture and compute a related family of multiple elliptic integrals with sophisticated appearance. We employ an approach that is probably overlooked in previous literature: connecting elliptic integrals to spherical functions and spherical rotations. Concretely speaking, the complete elliptic integral of the first kind $\mathbf{K}(k) := \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2} d\theta$ is related to fractional degree Legendre functions $P_{-1/2}, P_{-1/3}, P_{-1/4}, P_{-1/6}$, where

$$P_\nu(1) = 1; \quad P_\nu(\cos \theta) := \frac{2}{\pi} \int_0^\theta \frac{\cos \frac{(2\nu+1)\beta}{2}}{\sqrt{2(\cos \beta - \cos \theta)}} d\beta, \quad \theta \in (0, \pi), \nu \in \mathbb{C}$$

extend the familiar Legendre polynomials

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} [(x^2 - 1)^\ell], \quad \ell \in \mathbb{Z}_{\geq 0}$$

to arbitrary complex degree $\nu \in \mathbb{C}$. By a modest generalization of the "spherical harmonic coupling" for Legendre polynomials (Clebsch-Gordan theory) to Legendre functions of arbitrary degree, we are able to prove the integral identity

$$\int_{-1}^1 [P_\nu(x)]^2 P_\nu(-x) dx = \lim_{z \rightarrow \nu} \frac{1 + 2 \cos(\pi z)}{3} \frac{\pi \Gamma(\frac{z+1}{2}) \Gamma(\frac{3z+2}{2})}{[\Gamma(\frac{1-z}{2})]^2 [\Gamma(\frac{z+2}{2})]^3 \Gamma(\frac{3z+3}{2})}, \quad \nu \in \mathbb{C}$$

and compute the finite Hilbert transform

$$\mathcal{P} \int_{-1}^1 \frac{2P_\nu(\xi)P_\nu(-\xi)}{\pi(x-\xi)} d\xi = \frac{[P_\nu(x)]^2 - [P_\nu(-x)]^2}{\sin(\nu\pi)}, \quad \nu \in \mathbb{C} \setminus \mathbb{Z}.$$

The pair of formulae above, along with a couple of geometric transformations related to rotations on spheres, allow us to evaluate a slew of multiple elliptic integrals, which embody Wan’s conjectural identity as a special case.

The article is organized as follows. In §1, we review some classical results that express fractional degree Legendre functions as complete elliptic integrals, and give a new proof of Hobson’s coupling formula for two Legendre functions, based on scaling limits. We carry on the scaling analysis in §2, so as to interpolate the standard Clebsch-Gordan integrals for Legendre polynomials into a closed-form expression for $\int_{-1}^1 [P_\nu(x)]^2 P_\nu(-x) dx$ with arbitrary complex degree $\nu \in \mathbb{C}$. This is followed by a geometric intermezzo on spherical rotations in §3, providing tools to transform one multiple elliptic integral into another, as well as preparing some integral identities for §4. The finite Hilbert transform (also known as the Tricomi transform) of the function $P_\nu(x)P_\nu(-x)$ is studied in §4, in the spirit of “spherical harmonic coupling”. Lastly, §5 serves as a perspective on subsequent works on multiple elliptic integrals related to, or motivated by Legendre functions.

1 Fractional Degree Legendre Functions and Hansen-Heine Scaling Limits

The Legendre functions of the first kind can be defined via the Mehler-Dirichlet integral

$$P_\nu(1) = 1; \quad P_\nu(\cos \theta) := \frac{2}{\pi} \int_0^\theta \frac{\cos \frac{(2\nu+1)\beta}{2}}{\sqrt{2(\cos \beta - \cos \theta)}} d\beta, \quad \theta \in (0, \pi), \nu \in \mathbb{C}. \quad (1)$$

Alternatively, we may characterize $P_\nu(x)$, $-1 < x \leq 1$ as the unique $C^2(-1, 1)$ solution to

$$\frac{d}{dx} \left[(1-x^2) \frac{df(x)}{dx} \right] + \nu(\nu+1)f(x) = 0, \quad f(1) = 1. \quad (2)$$

By either definition, we have $P_\nu(x) \equiv P_{-\nu-1}(x)$. Later in this work, we shall also use the Legendre functions of the second kind, given by

$$Q_\nu(x) := \frac{\pi}{2\sin(\nu\pi)} [\cos(\nu\pi)P_\nu(x) - P_\nu(-x)], \quad \nu \in \mathbb{C} \setminus \mathbb{Z}; \quad Q_n(x) := \lim_{\nu \rightarrow n} Q_\nu(x), \quad n \in \mathbb{Z}_{\geq 0} \quad (3)$$

for $-1 < x < 1$. By convention, the Legendre function of the second kind $Q_n(x)$ is undefined for any negative integer degree $n \in \mathbb{Z}_{<0}$. While both $P_\nu(x)$ and $Q_\nu(x)$ solve the same Legendre differential equation of degree $\nu \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, the function $Q_\nu(x)$ blows up logarithmically as $x \rightarrow 1 - 0^+$, unlike the natural boundary condition for $P_\nu(1) \equiv 1$.

Additionally, the Legendre function of the first kind can be represented as the Laplace integral under some restrictions:

$$P_\nu(z) := \frac{1}{2\pi} \int_0^{2\pi} e^{\nu \log(z + i\sqrt{1-z^2} \cos \phi)} d\phi \equiv \frac{1}{2\pi} \int_0^{2\pi} e^{\nu \log(z - i\sqrt{1-z^2} \cos \phi)} d\phi, \quad \operatorname{Re} z > 0, \nu \in \mathbb{C}. \quad (4)$$

1.1 Kleiber-Ramanujan Correspondence

Highlights of the relation $P_{-1/2}(\cos \theta) = (2/\pi) \mathbf{K}(\sin(\theta/2))$ date back to J. Kleiber’s posthumous publication [7] in 1893, where systematic studies were carried out on connections between complete elliptic integrals and Legendre functions $P_{(2n+1)/2}$, $n \in \mathbb{Z}$, the latter of which are useful for solving differential equations in toroidal coordinates [Ref. 8, §§253-258].

In his seminal paper [9] and legendary notebooks [Ref. 10, Chap. 33], S. Ramanujan developed rich and elegant theories relating $P_{-1/3}, P_{-1/4}, P_{-1/6}$ to complete elliptic integrals of the first kind.

We recapitulate the aforementioned classical results in the following short lemma.

Lemma 1.1 (Some Fractional Degree Legendre Functions) (a) *We have the following identities for $0 \leq \theta < \pi$:*

$$P_{-1/2}(\cos \theta) = \frac{2}{\pi} \mathbf{K} \left(\sin \frac{\theta}{2} \right) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \sin^2(\theta/2) \sin^2 \psi}}, \quad (5)$$

$$P_{-1/4}(\cos \theta) = \frac{2}{\pi} \frac{1}{\sqrt{1 + \sin(\theta/2)}} \mathbf{K} \left(\sqrt{\frac{2 \sin(\theta/2)}{1 + \sin(\theta/2)}} \right) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \sin^2(\theta/2) \sin^4 \psi}}, \quad (6)$$

(b) For a parameter $p \in [0, 1)$, the following relations

$$P_{-1/6} \left(1 - 2 \frac{27p^2(1+p)^2}{4(1+p+p^2)^3} \right) = \sqrt{\frac{1+p+p^2}{1+2p}} P_{-1/2} \left(1 - 2 \frac{p(2+p)}{1+2p} \right) = \frac{2}{\pi} \sqrt{\frac{1+p+p^2}{1+2p}} \mathbf{K} \left(\sqrt{\frac{p(2+p)}{1+2p}} \right), \quad (7a)$$

$$P_{-1/6} \left(2 \frac{27p^2(1+p)^2}{4(1+p+p^2)^3} - 1 \right) = \sqrt{\frac{1+p+p^2}{1+2p}} P_{-1/2} \left(2 \frac{p(2+p)}{1+2p} - 1 \right) = \frac{2}{\pi} \sqrt{\frac{1+p+p^2}{1+2p}} \mathbf{K} \left(\sqrt{\frac{1-p^2}{1+2p}} \right); \quad (7b)$$

hold for degree $\nu = -1/6$, and the transformations

$$P_{-1/3} \left(1 - 2 \frac{27p^2(1+p)^2}{4(1+p+p^2)^3} \right) = \frac{1+p+p^2}{\sqrt{1+2p}} P_{-1/2} \left(1 - 2 \frac{p^3(2+p)}{1+2p} \right) = \frac{2}{\pi} \frac{1+p+p^2}{\sqrt{1+2p}} \mathbf{K} \left(\sqrt{\frac{p^3(2+p)}{1+2p}} \right), \quad (8a)$$

$$P_{-1/3} \left(2 \frac{27p^2(1+p)^2}{4(1+p+p^2)^3} - 1 \right) = \frac{1+p+p^2}{\sqrt{3+6p}} P_{-1/2} \left(2 \frac{p^3(2+p)}{1+2p} - 1 \right) = \frac{2}{\pi} \frac{1+p+p^2}{\sqrt{3+6p}} \mathbf{K} \left(\sqrt{\frac{(1-p)(1+p)^3}{1+2p}} \right). \quad (8b)$$

apply to degree $\nu = -1/3$.

Proof (a) Let $\mathbf{E}(k) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \phi} d\phi$ be the complete elliptic integral of the second kind, then we have the differential relations

$$\frac{d\mathbf{K}(k)}{dk} = \frac{\mathbf{E}(k)}{k(1-k^2)} - \frac{\mathbf{K}(k)}{k}, \quad \frac{d\mathbf{E}(k)}{dk} = \frac{\mathbf{E}(k) - \mathbf{K}(k)}{k}.$$

It is thus routine to check that the proposed forms of elliptic integrals in Eqs. 5 and 6 satisfy the said Legendre differential equations and natural boundary conditions (Eq. 2).

The last integral in Eq. 5 is the standard definition for the complete elliptic integral of the first kind. With a variable substitution $\theta = \arctan(\sqrt{1+\sin(\theta/2)} \tan \psi)$ [Ref. 11, p. 105], one can convert the last expression in Eq. 6 into the standard form.

(b) As in the proof of part (a), one can verify the relations between the respective fractional degree Legendre functions and complete elliptic integrals by a routine, though time-consuming procedure of Legendre differential equations.

Equations 7a and 7b have appeared in Ramanujan's work [Ref. 10, pp. 161-163] on the elliptic function theory for signature 6. As suggested by [10], Eqs. 7a, 7b can be constructed from first principles, using a hypergeometric transformation that identifies $P_{-1/6}(\cos \theta) = {}_2F_1 \left(\begin{smallmatrix} 1/6, 5/6 \\ 1 \end{smallmatrix} \middle| \sin^2 \frac{\theta}{2} \right)$, $0 \leq \theta \leq \pi/2$ with the Fricke-Klein function ${}_2F_1 \left(\begin{smallmatrix} 1/12, 5/12 \\ 1 \end{smallmatrix} \middle| \sin^2 \theta \right)$, $0 \leq \theta \leq \pi/2$ [Ref. 12, p. 334], the latter of which has a well-known connection to complete elliptic integrals of the first kind [1].

We may combine Eqs. 7a and 7b into a single formula

$$P_{-1/6} \left(\frac{x(9-x^2)}{(3+x^2)^{3/2}} \right) = \sqrt{\frac{\sqrt{3+x^2}}{2}} P_{-1/2}(x) = \frac{2}{\pi} \sqrt{\frac{\sqrt{3+x^2}}{2}} \mathbf{K} \left(\sqrt{\frac{1-x}{2}} \right), \quad x \in (-1, 1], \quad (9)$$

a form that is handier for conversion between integrals involving $P_{-1/6}$ and $\mathbf{K}$.

Ramanujan's studies of the elliptic function theory for signature 3 bring us Eqs. 8a and 8b. One may consult [Ref. 10, pp. 112-114] for the detailed theoretical arguments leading to the discovery of the algebraic relations between $P_{-1/3}$ and $\mathbf{K}$. ■

According to the recursion relation for Legendre functions $(x^2-1)dP_\nu(x)/dx = \nu x P_\nu(x) - \nu P_{\nu-1}(x)$ and the symmetry $P_\nu(x) = P_{-\nu-1}(x)$, one can relate some fractional degree Legendre functions to complete elliptic integrals of the first and the second kinds, such as

$$\begin{aligned} P_{1/2}(\cos \theta) &= \frac{2}{\pi} \left[2\mathbf{E} \left(\sin \frac{\theta}{2} \right) - \mathbf{K} \left(\sin \frac{\theta}{2} \right) \right], \quad P_{3/2}(\cos \theta) = \frac{2}{3\pi} \left[8\cos \theta \mathbf{E} \left(\sin \frac{\theta}{2} \right) - (4\cos \theta + 1)\mathbf{K} \left(\sin \frac{\theta}{2} \right) \right], \\ P_{1/4}(\cos \theta) &= \frac{2}{\pi} \left[2\sqrt{1+\sin(\theta/2)} \mathbf{E} \left(\sqrt{\frac{2\sin(\theta/2)}{1+\sin(\theta/2)}} \right) - \frac{1}{\sqrt{1+\sin(\theta/2)}} \mathbf{K} \left(\sqrt{\frac{2\sin(\theta/2)}{1+\sin(\theta/2)}} \right) \right], \\ P_{3/4}(\cos \theta) &= \frac{2}{3\pi} \left[2\sqrt{1+\sin(\theta/2)} \mathbf{E} \left(\sqrt{\frac{2\sin(\theta/2)}{1+\sin(\theta/2)}} \right) - \frac{1-2\cos \theta}{\sqrt{1+\sin(\theta/2)}} \mathbf{K} \left(\sqrt{\frac{2\sin(\theta/2)}{1+\sin(\theta/2)}} \right) \right]. \end{aligned}$$

1.2 Hansen-Heine Scaling Limits and Hobson Coupling Formula

By taking scaling limits of certain identities for Legendre polynomials, one may extend their validity to Legendre functions of arbitrary complex degree. In this manner, one can compute some multiple elliptic integrals by interpolating Legendre polynomial identities, the latter of which often have clear motivations from the harmonic analysis on spheres.

As we may recall, for positive integers m , the associated Legendre functions of degree ν and order m are defined by

$$\begin{aligned} P_\nu^m(x) &:= (-1)^m (1-x^2)^{m/2} \frac{d^m P_\nu(x)}{dx^m}, & P_\nu^{-m}(x) &:= (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(x); \\ Q_\nu^m(z) &:= (z^2-1)^{m/2} \frac{d^m Q_\nu(z)}{dz^m}, & Q_\nu^{-m}(z) &:= \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} Q_\nu^m(x) \end{aligned}$$

One also writes $P_\nu^0 = P_\nu$ and $Q_\nu^0 = Q_\nu$. The spherical harmonics

$$Y_{\ell m}(\theta, \phi) := \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\phi}, \quad \ell \in \mathbb{Z}_{\geq 0}; m \in \mathbb{Z} \cap [-\ell, \ell]$$

carry the Condon-Shortley phase $\overline{Y_{\ell m}(\theta, \phi)} = (-1)^m Y_{\ell, -m}(\theta, \phi)$.

In the following, we show that the studies of Legendre functions of arbitrary degree can be built on the Mehler-Dirichlet formula for spherical harmonics (see item 8.927 in [13] or §4.5.4 in [14]), which couples two normal vectors $\mathbf{n}_1(\theta_1, \phi_1) = (\sin\theta_1 \cos\phi_1, \sin\theta_1 \sin\phi_1, \cos\theta_1)$ and $\mathbf{n}_2(\theta_2, \phi_2) = (\sin\theta_2 \cos\phi_2, \sin\theta_2 \sin\phi_2, \cos\theta_2)$ on the unit sphere:

$$4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\overline{Y_{\ell m}(\theta_1, \phi_1)} Y_{\ell m}(\theta_2, \phi_2) \cos[(\ell + \frac{1}{2})\beta]}{2\ell+1} = \sum_{\ell=0}^{\infty} P_\ell(\mathbf{n}_1 \cdot \mathbf{n}_2) \cos[(\ell + \frac{1}{2})\beta] = \frac{\theta_H(\cos\beta - \mathbf{n}_1 \cdot \mathbf{n}_2)}{\sqrt{2(\cos\beta - \mathbf{n}_1 \cdot \mathbf{n}_2)}}, \quad |\mathbf{n}_1| = |\mathbf{n}_2| = 1. \quad (10)$$

Here, the Heaviside theta function is defined as $\theta_H(x) = (1 + \frac{x}{|x|})/2, x \in \mathbb{R} \setminus \{0\}$, and the range of the angle β is $(0, \pi)$.

Lemma 1.2 (Hobson Coupling Formula) *For $\theta_1, \theta_2 \in [0, \pi), \nu \in \mathbb{C}$, the Legendre function of the first kind satisfies the Hobson coupling formula*

$$\frac{1}{2\pi} \int_0^{2\pi} P_\nu(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi) d\phi = \begin{cases} P_\nu(\cos\theta_1) P_\nu(\cos\theta_2), & \theta_1 + \theta_2 \leq \pi \\ P_\nu(-\cos\theta_1) P_\nu(-\cos\theta_2), & \theta_1 + \theta_2 \geq \pi \end{cases} \quad (11)$$

which specializes to the following integral identities for $\theta_1, \theta_2 \in [0, \pi)$:

$$\frac{1}{4} \int_0^{2\pi} \mathbf{K}\left(\sin \frac{\theta}{2}\right) d\phi = \begin{cases} \mathbf{K}\left(\sin \frac{\theta_1}{2}\right) \mathbf{K}\left(\sin \frac{\theta_2}{2}\right), & \theta_1 + \theta_2 \leq \pi \\ \mathbf{K}\left(\cos \frac{\theta_1}{2}\right) \mathbf{K}\left(\cos \frac{\theta_2}{2}\right), & \theta_1 + \theta_2 \geq \pi \end{cases} \quad (12)$$

$$\frac{1}{4} \int_0^{2\pi} \mathbf{K}\left(\sqrt{\frac{2\sin(\theta/2)}{1+\sin(\theta/2)}}\right) \frac{d\phi}{\sqrt{1+\sin(\theta/2)}} = \begin{cases} \frac{1}{\sqrt{1+\sin(\theta_1/2)}\sqrt{1+\sin(\theta_2/2)}} \mathbf{K}\left(\sqrt{\frac{2\sin(\theta_1/2)}{1+\sin(\theta_1/2)}}\right) \mathbf{K}\left(\sqrt{\frac{2\sin(\theta_2/2)}{1+\sin(\theta_2/2)}}\right), & \theta_1 + \theta_2 \leq \pi \\ \frac{1}{\sqrt{1+\cos(\theta_1/2)}\sqrt{1+\cos(\theta_2/2)}} \mathbf{K}\left(\sqrt{\frac{2\cos(\theta_1/2)}{1+\cos(\theta_1/2)}}\right) \mathbf{K}\left(\sqrt{\frac{2\cos(\theta_2/2)}{1+\cos(\theta_2/2)}}\right), & \theta_1 + \theta_2 \geq \pi \end{cases} \quad (13)$$

where $\cos\theta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi$. In particular, we have

$$\left[\mathbf{K}\left(\sin \frac{\theta}{2}\right)\right]^2 = \int_0^{\pi/2} \mathbf{K}(\sin\theta \cos\phi) d\phi, \quad \forall \theta \in \left[0, \frac{\pi}{2}\right], \quad (12^\dagger)$$

$$\mathbf{K}\left(\sin \frac{\theta}{2}\right) \mathbf{K}\left(\cos \frac{\theta}{2}\right) = \int_0^{\pi/2} \mathbf{K}\left(\sqrt{1 - \sin^2\theta \cos^2\phi}\right) d\phi, \quad \forall \theta \in \left[0, \frac{\pi}{2}\right], \quad (12^\ddagger)$$

$$\frac{1}{1+\sqrt{u}} \left[\mathbf{K}\left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}}\right)\right]^2 = \int_0^{\pi/2} \mathbf{K}\left(\sqrt{\frac{4\sqrt{u}(1-u)\cos\phi}{1+2\sqrt{u}(1-u)\cos\phi}}\right) \frac{d\phi}{\sqrt{1+2\sqrt{u}(1-u)\cos\phi}}, \quad \forall u \in \left[0, \frac{1}{2}\right], \quad (13^\dagger)$$

$$\frac{\sqrt{2}}{1+\sqrt{u}} \mathbf{K}\left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}}\right) \mathbf{K}\left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right) = \int_0^{\pi/2} \mathbf{K}\left(\sqrt{\frac{2\sqrt{1-4u}(1-u)\cos^2\phi}{1+\sqrt{1-4u}(1-u)\cos^2\phi}}\right) \frac{d\phi}{\sqrt{1+\sqrt{1-4u}(1-u)\cos^2\phi}}, \quad \forall u \in (0, 1). \quad (13^\ddagger)$$

Proof When $\nu = n$ is a non-negative integer, one can verify Eq. 11 by resorting to Eq. 10:

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} P_n(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi) d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{2}{\pi} \int_0^\pi \frac{\theta_H(\cos\beta - \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\phi) \cos[(n + \frac{1}{2})\beta] d\beta}{\sqrt{\cos\beta - \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\phi}} \right] d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{2}{\pi} \int_0^\pi 4\pi \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \frac{Y_{\ell m}(\theta_1, 0) Y_{\ell m}(\theta_2, \phi) \cos[(\ell + \frac{1}{2})\beta] \cos[(n + \frac{1}{2})\beta]}{2\ell + 1} d\beta \right] d\phi = P_n(\cos\theta_1) P_n(\cos\theta_2), \end{aligned}$$

where spherical harmonic modes other than Y_{n0} will have vanishing contribution upon integration in the angular variables β and ϕ . Here, due to the parity relation $P_n(-x) = (-1)^n P_n(x)$ for non-negative integers n , one always has $P_n(\cos\theta_1) P_n(\cos\theta_2) = P_n(-\cos\theta_1) P_n(-\cos\theta_2)$, irrespective of the sum $\theta_1 + \theta_2$.

Fixing two positive parameters s and t , one may consider a neighborhood of the origin in the complex z -plane such that $\operatorname{Re} \cos(sz) > 0, \operatorname{Re} \cos(tz) > 0, \operatorname{Re}[\cos(sz) \cos(tz) + \sin(sz) \sin(tz) \cos\phi] > 0$. Then, given an arbitrary $\nu \in \mathbb{C}$, we can apply the Laplace integral representation for the Legendre functions (Eq. 4) to evaluate the following scaling limit:

$$\begin{aligned} f_{s,t;\nu}(z) &:= P_{\nu/z}(\cos(sz)) P_{\nu/z}(\cos(tz)) - \int_0^{2\pi} P_{\nu/z}(\cos(sz) \cos(tz) + \sin(sz) \sin(tz) \cos\phi) \frac{d\phi}{2\pi} \\ &= \int_0^{2\pi} e^{\frac{\nu}{z} \log(\cos(sz) + i \sin(sz) \cos\phi_1)} \frac{d\phi_1}{2\pi} \int_0^{2\pi} e^{\frac{\nu}{z} \log(\cos(tz) + i \sin(tz) \cos\phi_2)} \frac{d\phi_2}{2\pi} \\ &\quad - \int_0^{2\pi} \left\{ \int_0^{2\pi} e^{\frac{\nu}{z} \log(\cos(sz) \cos(tz) + \sin(sz) \sin(tz) \cos\phi + i \sqrt{1 - [\cos(sz) \cos(tz) + \sin(sz) \sin(tz) \cos\phi]^2} \cos\phi_3)} \frac{d\phi_3}{2\pi} \right\} \frac{d\phi}{2\pi} \\ &\rightarrow \int_0^{2\pi} e^{i\nu s \cos\phi_1} \frac{d\phi_1}{2\pi} \int_0^{2\pi} e^{i\nu t \cos\phi_2} \frac{d\phi_2}{2\pi} - \int_0^{2\pi} \left\{ \int_0^{2\pi} e^{i\nu \sqrt{s^2 + t^2 - 2st \cos\phi} \cos\phi_3} \frac{d\phi_3}{2\pi} \right\} \frac{d\phi}{2\pi} = 0, \quad \text{as } z \rightarrow 0, \end{aligned}$$

where the last equality turns out to be a special case of the Sonine-Gegenbauer formula for Bessel functions [Ref. 15, §12.1]:

$$J_0(\nu s) J_0(\nu t) = \frac{1}{2\pi} \int_0^{2\pi} J_0(\nu \sqrt{s^2 + t^2 - 2st \cos\phi}) d\phi.$$

Thus, with fixed parameters s, t and ν , the function $f_{s,t;\nu}(z)$ is analytic near the origin in the complex z -plane, and $f_{s,t;\nu}(z) = 0$ on the set $\{0\} \cup \{\nu/n | n \in \mathbb{Z} \cap [0, +\infty)\}$, which contains an accumulation point $\{0\}$ in the domain of analyticity for $f_{s,t;\nu}(z)$. Therefore, the function $f_{s,t;\nu}(z)$ must vanish identically in a certain neighborhood of the origin, the spatial extent of which depends on the nature of the positive parameters s and t .

Now suppose that $s + t < \pi$, then $\cos s \cos t + \sin s \sin t \cos\phi \geq \cos(s + t) > -1$. Accordingly, for a complex number z sitting in a certain open neighborhood of the unit interval $[0, 1]$, the expression $\cos(sz) \cos(tz) + \sin(sz) \sin(tz) \cos\phi$ will miss the value -1 , which is the logarithmic branch point of the Legendre functions P_μ of non-integer degree $\mu \notin \mathbb{Z}$. Hence, when $s > 0, t > 0$ and $s + t < \pi$, we have a univalent complex-analytic function $f_{s,t;\nu}(z)$ in an open neighborhood of the unit interval $[0, 1]$, where resides a non-isolated set of zeros. By the principle of analytic continuation, we may conclude that $f_{s,t;\nu}(1) = 0$ for $s > 0, t > 0$ and $s + t < \pi$, which extends, by continuity, to the $s + t = \pi$ scenario as well as those situations where $st = 0$. This proves Eq. 11 for the cases of $\theta_1 + \theta_2 \leq \pi$. The rest of Eq. 11 follows from the invariance of the integral

$$\frac{1}{2\pi} \int_0^{2\pi} P_\nu(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi) d\phi$$

under the equatorial reflections $(\theta_1, \theta_2) \mapsto (\pi - \theta_1, \pi - \theta_2)$.

Setting $\nu = -1/2$ in Eq. 11, we obtain Eq. 12; setting $\nu = -1/4$ or $\nu = -3/4$ in Eq. 11, one verifies Eq. 13. When $\theta_1 = \theta_2$, Eq. 12 (resp. 13) becomes Eq. 12[†] (resp. 13[†]). When $\theta_1 = \pi - \theta_2 = \theta$, one may specialize Eq. 12 into Eq. 12[‡]. A similar procedure on Eq. 13 would bring us the evaluation

$$\begin{aligned} & \int_0^{\pi/2} \mathbf{K} \left(\sqrt{\frac{2\sqrt{1-4u(1-u)\cos^2\phi}}{1 + \sqrt{1-4u(1-u)\cos^2\phi}}} \right) \frac{d\phi}{\sqrt{1 + \sqrt{1-4u(1-u)\cos^2\phi}}} \\ &= \frac{1}{\sqrt{1 + \sqrt{u}} \sqrt{1 + \sqrt{1-u}}} \mathbf{K} \left(\sqrt{\frac{2\sqrt{u}}{1 + \sqrt{u}}} \right) \mathbf{K} \left(\sqrt{\frac{2\sqrt{1-u}}{1 + \sqrt{1-u}}} \right), \quad \forall u \in \left(0, \frac{1}{2}\right], \end{aligned}$$

while the substitution $u = (1-t)^2/(1+t)^2$ and Landen's transformation [Ref. 13, item 8.126] allow us to see that

$$\frac{1}{\sqrt{1+\sqrt{1-u}}} \mathbf{K}\left(\sqrt{\frac{2\sqrt{1-u}}{1+\sqrt{1-u}}}\right) = \frac{\sqrt{1+t}}{1+\sqrt{t}} \mathbf{K}\left(\frac{2\sqrt[4]{t}}{1+\sqrt{t}}\right) = \sqrt{1+t} \mathbf{K}(\sqrt{t}) = \sqrt{\frac{2}{1+\sqrt{u}}} \mathbf{K}\left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right).$$

This verifies Eq. 13[‡] for $u \in (0, 1/2]$. Furthermore, the right-hand side of Eq. 13[‡] is intact under the transformation $u \mapsto 1-u$, whereas the symmetric extension of its left-hand side hinges on the identity

$$\frac{1}{1+\sqrt{u}} \mathbf{K}\left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right) \mathbf{K}\left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}}\right) = \frac{1}{1+\sqrt{1-u}} \mathbf{K}\left(\sqrt{\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}}\right) \mathbf{K}\left(\sqrt{\frac{2\sqrt{1-u}}{1+\sqrt{1-u}}}\right), \quad 0 < u < 1, \quad (14)$$

which is an equivalent formulation for the products of two Landen's transformations:

$$\frac{1+t}{2} \mathbf{K}(\sqrt{1-t}) \mathbf{K}(\sqrt{t}) = \frac{1+t}{(1+\sqrt{t})^2} \mathbf{K}\left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right) \mathbf{K}\left(\frac{2\sqrt[4]{t}}{1+\sqrt{t}}\right), \quad 0 < t < 1, \quad (14')$$

with the correspondence of variables being $u = (1-t)^2/(1+t)^2$. ■

Remark (1) Using Eq. 10, we may readily adapt our proof of the product formula in Eq. 11 to associated Legendre functions:

$$\frac{1}{2\pi} \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} \int_0^{2\pi} P_\nu(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi) \cos m\phi \, d\phi = \begin{cases} P_\nu^m(\cos\theta_1) P_\nu^m(\cos\theta_2), & \theta_1 + \theta_2 \leq \pi \\ P_\nu^m(-\cos\theta_1) P_\nu^m(-\cos\theta_2), & \theta_1 + \theta_2 \geq \pi \end{cases} \quad (11^m)$$

where $\theta_1, \theta_2 \in [0, \pi)$, $\nu \in \mathbb{C}$, $m \in \mathbb{Z}$. Here, the corresponding scaling limit is the Sonine-Gegenbauer formula

$$J_m(\nu s) J_m(\nu t) = \frac{1}{2\pi} \int_0^{2\pi} J_0(\nu \sqrt{s^2 + t^2 - 2st \cos\phi}) \cos m\phi \, d\phi.$$

(Doubtlessly, the cases of

$$P_{-1/2}^1(\cos\theta) = \frac{2}{\pi \sin\theta} \left[\mathbf{E}\left(\sin\frac{\theta}{2}\right) - \mathbf{K}\left(\sin\frac{\theta}{2}\right) \cos^2\frac{\theta}{2} \right],$$

$$P_{-1/4}^1(\cos\theta) = P_{-3/4}^1(\cos\theta) = \frac{\sqrt{1+\sin(\theta/2)}}{\pi \sin\theta} \left[\mathbf{E}\left(\sqrt{\frac{2\sin(\theta/2)}{1+\sin(\theta/2)}}\right) - \mathbf{K}\left(\sqrt{\frac{2\sin(\theta/2)}{1+\sin(\theta/2)}}\right) \left(1 - \sin\frac{\theta}{2}\right) \right]$$

will then lead to generalizations of Eqs. 12 and 13 into integral formulae involving complete elliptic integrals of the second kind.) Effectively, we can combine Eq. 11^m with Eq. 11 to develop a Fourier expansion in ϕ for the function $P_\nu(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi)$ (necessarily under the constraints $0 \leq \theta_1 \leq \pi, 0 \leq \theta_2 \leq \pi, 0 \leq \theta_1 + \theta_2 \leq \pi$), which turns up as a uniformly convergent series formerly known to E. W. Hobson [Ref. 8, §226]:

$$P_\nu(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi) = P_\nu(\cos\theta_1) P_\nu(\cos\theta_2) + 2 \sum_{m=1}^{\infty} \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(\cos\theta_1) P_\nu^m(\cos\theta_2) \cos m\phi.$$

Hence we refer to Eq. 11 as the Hobson coupling formula. Hobson's original approach draws on the Fourier series of $(z + \sqrt{z^2 - 1} \cos\phi)^\nu$ for complex-valued ν , which is not a method based on scaling limits.

(2) The representation of Bessel functions J_m as the scaling limit of (associate) Legendre functions P_ν^m is a special case of Hansen's formula [Ref. 15, §5.7]. Heine extended the limit procedure so that one may connect Bessel functions Y_m and the (associate) Legendre functions Q_ν^m in a similar fashion [Ref. 15, §5.71]. The Hansen-Heine scaling limit procedure ($m = 0$) will be used again during the proof of Eq. 19_(v,v), namely,

$$\int_{-1}^1 [P_\nu(x)]^2 P_\nu(-x) \, dx = \lim_{z \rightarrow \nu} \frac{1+2\cos(\pi z)}{3} \frac{\pi \Gamma\left(\frac{z+1}{2}\right) \Gamma\left(\frac{3z+2}{2}\right)}{[\Gamma\left(\frac{1-z}{2}\right)]^2 [\Gamma\left(\frac{z+2}{2}\right)]^3 \Gamma\left(\frac{3z+3}{2}\right)}, \quad \nu \in \mathbb{C}$$

in Proposition 2.1.

(3) The integral formula in Eq. 12[‡] is usually attributed to Glasser [16], and has been mentioned recently in the work of Bailey *et al.* [2]. We will recover Eq. 12[‡] later in Proposition 3.3 (see §3.2), using a method independent of the Hobson coupling formula for Legendre functions.

(4) The Hobson coupling formula is sensitive to the geometric constraint $0 \leq \theta_1 + \theta_2 \leq \pi$. Thus, it is not possible to directly extrapolate the formulae in this lemma to an integral representation for $[P_\nu(x)]^2, -1 < x < 0$ where ν is arbitrary. Practically, this obstacle can be overcome in various ways. For $\nu = -1/2, -1/3, -1/4, -1/6$, the angular restriction in the Hobson coupling formula can be circumvented by using Ramanujan's integral representation for $[\mathbf{K}(k)]^2, 0 \leq k < 1$ (see Proposition 3.3). For generic ν , Hobson's approach yields an integral representation for $P_\nu(x)P_\nu(-x), -1 < x < 1$ and $[P_\nu(x)]^2, 0 \leq x < 1$, and the finite Hilbert transform in Proposition 4.2 (see §4.2) will enable us to express $[P_\nu(x)]^2 - [P_\nu(-x)]^2, 0 \leq x < 1$ in terms of $P_\nu(\xi)P_\nu(-\xi), -1 < \xi < 1$. $\square$

2 Generalized Clebsch-Gordan Integrals

As the $C^2(-1, 1)$ solutions to the Legendre differential equation of arbitrary degree $\nu \in \mathbb{C} \setminus \mathbb{Z}_{<0}$:

$$\frac{d}{dx} \left[(1-x^2) \frac{df(x)}{dx} \right] + \nu(\nu+1)f(x) = 0 \quad (15)$$

are exhausted by linear combinations of $P_\nu(x)$ and $Q_\nu(x)$, a special case in Appell's theory of third order ordinary differential equations [17] then tells us that the $C^3(-1, 1)$ solutions to

$$\frac{d}{dx} \left\{ (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{df(x)}{dx} \right] + 4\nu(\nu+1)(1-x^2)f(x) \right\} + 4\nu(\nu+1)xf(x) = 0, \quad \nu \in \mathbb{C} \setminus \mathbb{Z}_{<0} \quad (16)$$

belong to a three-dimensional space spanned by $[P_\nu(x)]^2$, $P_\nu(x)Q_\nu(x)$ and $[Q_\nu(x)]^2$ (or by an equivalent basis set $\{[P_\nu(x)]^2, P_\nu(x)P_\nu(-x), [P_\nu(-x)]^2\}$ when $\nu \notin \mathbb{Z}$).

The “interpolation” from Legendre polynomials $P_\ell(x), \ell \in \mathbb{Z}_{\geq 0}$ to non-integer degree Legendre functions $P_\nu(x), \nu \notin \mathbb{Z}$ and the “scaling limit” for large degrees will enable us to extend some Clebsch-Gordan integral formulae to generic Legendre functions, as elaborated in the proposition below.

Proposition 2.1 (Generalized Clebsch-Gordan Integrals) *Define*

$$T_{\mu,\nu} := \int_{-1}^1 P_\mu(x)P_\nu(x)P_\nu(-x)dx, \quad \mu, \nu \in \mathbb{C},$$

then we have the symmetry

$$T_{\mu,\nu} = T_{-\mu-1,\nu} = T_{\mu,-\nu-1} = T_{-\mu-1,-\nu-1}, \quad (17)$$

the recursion relation

$$(\mu+1)^2[(\mu+1)^2 - (2\nu+1)^2]T_{\mu+1,\nu} - \mu^2[\mu^2 - (2\nu+1)^2]T_{\mu-1,\nu} = \frac{4(2\mu+1)\sin(\mu\pi)\sin(\nu\pi)}{\pi^2}, \quad (18)$$

the representation in terms of generalized hypergeometric series

$$T_{\mu,\nu} = \frac{2}{\pi^2} \frac{\sin(\mu\pi)\sin(\nu\pi)}{\mu(\mu+1)} \left[\frac{1}{\nu} {}_4F_3 \left(\begin{matrix} 1, \frac{1-\mu}{2}, \frac{\mu+2}{2}, -\nu \\ \frac{2-\mu}{2}, \frac{\mu+3}{2}, 1-\nu \end{matrix} \middle| 1 \right) - \frac{1}{\nu+1} {}_4F_3 \left(\begin{matrix} 1, \frac{1-\mu}{2}, \frac{\mu+2}{2}, \nu+1 \\ \frac{2-\mu}{2}, \frac{\mu+3}{2}, \nu+2 \end{matrix} \middle| 1 \right) \right], \quad \mu, \nu \notin \mathbb{Z} \quad (19)$$

$$T_{\mu,\nu} = \frac{2}{\pi} \frac{\sin(\mu\pi)\cos(\nu\pi)}{2\nu+1} \left[\frac{1}{\mu} {}_5F_4 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, -\frac{\mu}{2}, -\nu, 1+\nu \\ 1, \frac{2-\mu}{2}, \frac{1-2\nu}{2}, \frac{2\nu+3}{2} \end{matrix} \middle| 1 \right) - \frac{1}{\mu+1} {}_5F_4 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1+\mu}{2}, -\nu, 1+\nu \\ 1, \frac{3+\mu}{2}, \frac{1-2\nu}{2}, \frac{3+2\nu}{2} \end{matrix} \middle| 1 \right) \right], \quad \mu, \nu \notin \mathbb{Z}, \nu \neq -\frac{1}{2} \quad (19^*)$$

along with some generalized Clebsch-Gordan integral formulae:

$$T_{\mu,n} = (-1)^n \frac{\pi \Gamma\left(\frac{2n+1-\mu}{2}\right) \Gamma\left(\frac{2n+2+\mu}{2}\right)}{\left[\Gamma\left(\frac{1-\mu}{2}\right)\right]^2 \left[\Gamma\left(\frac{\mu+2}{2}\right)\right]^2 \Gamma\left(\frac{2n+2-\mu}{2}\right) \Gamma\left(\frac{2n+3+\mu}{2}\right)}, \quad \mu \notin \mathbb{Z}, n \in \mathbb{Z}_{\geq 0} \quad (19_{(\mu,n)})$$

$$T_{2m,\nu} = \frac{\pi \cos(\nu\pi) \Gamma\left(\frac{2\nu+1-2m}{2}\right) \Gamma(\nu+1+m)}{\left[\Gamma\left(\frac{1-2m}{2}\right)\right]^2 [\Gamma(m+1)]^2 \Gamma(\nu+1-m) \Gamma\left(\frac{2\nu+3+2m}{2}\right)}, \quad m \in \mathbb{Z}_{\geq 0}, \nu + \frac{1}{2} \notin \mathbb{Z} \quad (19_{(2m,\nu)})$$

$$T_{2m,n+\frac{1}{2}} = \begin{cases} -\frac{(-1)^n}{\pi} \frac{[\Gamma(m+\frac{1}{2})]^2 \Gamma(m-n-\frac{1}{2}) \Gamma(m+n+\frac{3}{2})}{[\Gamma(m+1)]^2 \Gamma(m-n) \Gamma(m+n+2)}, & m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}, m-n \notin \mathbb{Z}_{\leq 0}, m+n+2 \notin \mathbb{Z}_{\leq 0} \\ 0, & m \in \mathbb{Z}_{\geq 0}, (m-n \in \mathbb{Z}_{\leq 0} \text{ or } m+n+2 \in \mathbb{Z}_{\leq 0}) \end{cases} \quad (19_{(2m,n+\frac{1}{2})})$$

$$T_{v,v} = \lim_{z \rightarrow v} \frac{1+2\cos(\pi z)}{3} \frac{\pi \Gamma\left(\frac{z+1}{2}\right) \Gamma\left(\frac{3z+2}{2}\right)}{\left[\Gamma\left(\frac{1-z}{2}\right)\right]^2 \left[\Gamma\left(\frac{z+2}{2}\right)\right]^3 \Gamma\left(\frac{3z+3}{2}\right)}, \quad v \in \mathbb{C} \quad (19_{(v,v)})$$

$$T_{2v-1,v} = \lim_{z \rightarrow v} \frac{\sin(\pi z) \sin(2\pi z)}{\pi^2 z^2}, \quad v \in \mathbb{C} \quad (19_{(2v-1,v)})$$

$$T_{2v+2,v} = -\lim_{z \rightarrow v} \frac{\sin(\pi z) \sin(2\pi z)}{\pi^2 (z+1)^2}, \quad v \in \mathbb{C}. \quad (19_{(2v+2,v)})$$

Proof The symmetry of Legendre functions with respect to degree $P_v(x) = P_{-v-1}(x)$, $\forall v \in \mathbb{C}$ explains the related property of $T_{\mu,v}$ (Eq. 17).

Judging from the familiar recursion relations for Legendre functions

$$(2\mu+1)(1-x^2) \frac{dP_\mu(x)}{dx} = \mu(\mu+1)[P_{\mu-1}(x) - P_{\mu+1}(x)]; \quad (2\mu+1)xP_\mu(x) = (\mu+1)P_{\mu+1}(x) + \mu P_{\mu-1}(x)$$

and the differential equations given in Eqs. 15 and 16, we may carry out the following computations:

$$\begin{aligned} & v(v+1)[(\mu+1)T_{\mu+1,v} + \mu T_{\mu-1,v}] \\ &= v(v+1) \int_{-1}^1 [(\mu+1)P_{\mu+1}(x) + \mu P_{\mu-1}(x)] P_v(x) P_v(-x) dx = (2\mu+1)v(v+1) \int_{-1}^1 x P_\mu(x) P_v(x) P_v(-x) dx \\ &= -\frac{2\mu+1}{4} \int_{-1}^1 P_\mu(x) \frac{d}{dx} \left\{ (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{d(P_v(x)P_v(-x))}{dx} \right] + 4v(v+1)(1-x^2)P_v(x)P_v(-x) \right\} dx \\ &= \frac{2\mu+1}{4} \int_{-1}^1 \left\{ (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{d(P_v(x)P_v(-x))}{dx} \right] + 4v(v+1)(1-x^2)P_v(x)P_v(-x) \right\} \frac{dP_\mu(x)}{dx} dx \\ &= \frac{2\mu+1}{4} \int_{-1}^1 \left\{ (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{d(P_v(x)P_v(-x))}{dx} \right] \right\} \frac{dP_\mu(x)}{dx} dx + \mu(\mu+1)v(v+1)(T_{\mu-1,v} - T_{\mu+1,v}) \\ &= \frac{2\mu+1}{4} \left[\mu(\mu+1) \int_{-1}^1 (1-x^2) P_\mu(x) \frac{d(P_v(x)P_v(-x))}{dx} dx - \frac{4 \sin(\mu\pi) \sin(v\pi)}{\pi^2} \right] + \mu(\mu+1)v(v+1)(T_{\mu-1,v} - T_{\mu+1,v}), \end{aligned}$$

where we have used in the last line the properties $P_v(1) = 1$, $\forall v \in \mathbb{C}$ and

$$\lim_{x \rightarrow -1+0^+} (1-x^2) \frac{dP_v(x)}{dx} = \frac{2 \sin(v\pi)}{\pi}, \quad \forall v \in \mathbb{C}.$$

We may then proceed with the simplification

$$\begin{aligned} & (2\mu+1) \int_{-1}^1 (1-x^2) P_\mu(x) \frac{d(P_v(x)P_v(-x))}{dx} dx \\ &= -(2\mu+1) \int_{-1}^1 (1-x^2) P_v(x) P_v(-x) \frac{dP_\mu(x)}{dx} dx + 2(2\mu+1) \int_{-1}^1 x P_\mu(x) P_v(x) P_v(-x) dx \\ &= \mu(\mu+1)(T_{\mu+1,v} - T_{\mu-1,v}) + 2(\mu+1)T_{\mu+1,v} + 2\mu T_{\mu-1,v}, \end{aligned}$$

so as to confirm the recursion formula of $T_{\mu,v}$ (Eq. 18).

For non-negative integers m and n meeting the requirement $m \leq 2n$, we might recall from the standard Clebsch-Gordan theory that

$$T_{m,n} := \int_{-1}^1 P_m(x) P_n(x) P_n(-x) dx = 2(-1)^n \begin{pmatrix} m & n & n \\ 0 & 0 & 0 \end{pmatrix}^2 = (-1)^n \frac{\pi \Gamma\left(\frac{2n+1-m}{2}\right) \Gamma\left(\frac{2n+2+m}{2}\right)}{\left[\Gamma\left(\frac{1-m}{2}\right)\right]^2 \left[\Gamma\left(\frac{m+2}{2}\right)\right]^2 \Gamma\left(\frac{2n+2-m}{2}\right) \Gamma\left(\frac{2n+3+m}{2}\right)}, \quad (19_{(m,n)})$$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner 3- j symbol. The right-hand side expression in Eq. 19_(m,n) is regarded as zero if m is a positive odd integer, as anticipated from the relations $P_m(x) = -P_m(-x)$ and $\Gamma\left(\frac{1-m}{2}\right) = \infty$ in such circumstances. We have $T_{m,n} = \int_{-1}^1 P_m(x) P_n(x) P_n(-x) dx = 0$ for non-negative integers m and n satisfying $m > 2n$, as indicated by the respective poles of the gamma factors in the denominator on the right-hand side of Eq. 19_(m,n).

Actually, even without the prior knowledge of the standard Clebsch-Gordan theory, one can start from the initial condition $T_{0,n} = \int_{-1}^1 P_n(x) P_n(-x) dx = 2(-1)^n/(2n+1)$, $n \in \mathbb{Z}_{\geq 0}$, and build Eq. 19_(m,n) inductively from the recursion relation given by Eq. 18.

The interpolation formula (known as Dougall's expansion in [Ref. 18, p. 167])

$$\begin{aligned} P_\nu(x) &= 2 \int_0^\pi \frac{\theta_H(\cos \beta - x) \cos \frac{(2\nu+1)\beta}{2}}{\sqrt{2(\cos \beta - x)}} \frac{d\beta}{\pi} = 2 \int_0^\pi \sum_{\ell=0}^\infty P_\ell(x) \cos \frac{(2\ell+1)\beta}{2} \cos \frac{(2\nu+1)\beta}{2} \frac{d\beta}{\pi} \\ &= \sum_{\ell=0}^\infty P_\ell(x) \left[\frac{\sin(\ell-\nu)\pi}{(\ell-\nu)\pi} + \frac{\sin(\ell+\nu+1)\pi}{(\ell+\nu+1)\pi} \right] \end{aligned}$$

allows us to sum Eq. 19_(m,n) as

$$T_{\mu,n} := \int_{-1}^1 P_\mu(x) P_n(x) P_n(-x) dx = \sum_{\ell=0}^\infty T_{\ell,n} \left[\frac{\sin(\ell-\mu)\pi}{(\ell-\mu)\pi} + \frac{\sin(\ell+\mu+1)\pi}{(\ell+\mu+1)\pi} \right]$$

for $\mu \in \mathbb{C} \setminus \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}$. To identify the last term in the equation above with the expression in Eq. 19_(\mu,n), we note that the sum over ℓ in fact truncates after finite terms, which reveals the expression

$$\frac{1}{\sin(\mu\pi)} \left\{ (-1)^n \frac{\pi \Gamma\left(\frac{2n+1-\mu}{2}\right) \Gamma\left(\frac{2n+2+\mu}{2}\right)}{\left[\Gamma\left(\frac{1-\mu}{2}\right)\right]^2 \left[\Gamma\left(\frac{\mu+2}{2}\right)\right]^2 \Gamma\left(\frac{2n+2-\mu}{2}\right) \Gamma\left(\frac{2n+3+\mu}{2}\right)} - \sum_{\ell=0}^\infty T_{\ell,n} \left[\frac{\sin(\ell-\mu)\pi}{(\ell-\mu)\pi} + \frac{\sin(\ell+\mu+1)\pi}{(\ell+\mu+1)\pi} \right] \right\}$$

as a rational function of μ , for whatever integer n . As μ approaches any integer m , it is clear that such a rational function tends to zero with $O(\mu-m)$ convergence rate, so it must be identically vanishing.

By Eq. 11 along with the Mehler-Dirichlet theory, we have the following computation when ν is not an integer:

$$\begin{aligned} P_\nu(x) P_\nu(-x) &= \int_0^{2\pi} P_\nu((1-x^2)\cos\phi - x^2) \frac{d\phi}{2\pi} = 2 \int_0^{2\pi} \left[\int_0^\pi \frac{\theta_H(\cos\beta - (1-x^2)\cos\phi + x^2) \cos \frac{(2\nu+1)\beta}{2}}{\sqrt{2(\cos\beta - (1-x^2)\cos\phi + x^2)}} \frac{d\beta}{\pi} \right] \frac{d\phi}{2\pi} \\ &= 2 \int_0^{2\pi} \left[\int_0^\pi \sum_{\ell=0}^\infty P_\ell((1-x^2)\cos\phi - x^2) \cos \frac{(2\ell+1)\beta}{2} \cos \frac{(2\nu+1)\beta}{2} \frac{d\beta}{\pi} \right] \frac{d\phi}{2\pi} \\ &= \sum_{\ell=0}^\infty P_\ell(x) P_\ell(-x) \left[\frac{\sin(\ell-\nu)\pi}{(\ell-\nu)\pi} + \frac{\sin(\ell+\nu+1)\pi}{(\ell+\nu+1)\pi} \right], \end{aligned}$$

where we have integrated over β and ϕ separately in the last step. Thus, we can perform a decomposition

$$T_{\mu,\nu} = \sum_{\ell=0}^\infty T_{\mu,\ell} \left[\frac{\sin(\ell-\nu)\pi}{(\ell-\nu)\pi} + \frac{\sin(\ell+\nu+1)\pi}{(\ell+\nu+1)\pi} \right] = \frac{\sin(\nu\pi)(\tau_{\mu,\nu} + \tau_{\mu,-\nu-1})}{\left[\Gamma\left(\frac{1-\mu}{2}\right)\right]^2 \left[\Gamma\left(\frac{\mu+2}{2}\right)\right]^2},$$

where

$$\tau_{\mu,\lambda} := \sum_{\ell=0}^\infty \frac{\Gamma\left(\frac{2\ell+1-\mu}{2}\right) \Gamma\left(\frac{2\ell+2+\mu}{2}\right)}{\Gamma\left(\frac{2\ell+2-\mu}{2}\right) \Gamma\left(\frac{2\ell+3+\mu}{2}\right)} \frac{1}{\lambda - \ell} = \frac{2\sin(\mu\pi)}{\lambda\pi^2} \left[\Gamma\left(\frac{1-\mu}{2}\right)\right]^2 \left[\Gamma\left(\frac{\mu+2}{2}\right)\right]^2 {}_4F_3\left(\begin{matrix} 1, \frac{1-\mu}{2}, \frac{\mu+2}{2}, -\lambda \\ \frac{2-\mu}{2}, \frac{\mu+3}{2}, 1-\lambda \end{matrix} \middle| 1\right)$$

follows directly from the definition of ${}_4F_3$. This verifies Eq. 19.

One can prove Eq. 19_(2m,\nu) by directly computing

$$T_{0,\nu} = \frac{\sin(\nu\pi)}{\pi} \sum_{n=0}^\infty (-1)^n T_{0,n} \left(\frac{1}{\nu-n} - \frac{1}{\nu+n+1} \right) = \frac{\sin(\nu\pi)}{\pi} \sum_{n=0}^\infty \frac{2}{2n+1} \left(\frac{1}{\nu-n} - \frac{1}{\nu+n+1} \right) = \frac{2\cos(\nu\pi)}{2\nu+1} \quad (19_{(0,\nu)})$$

and invoking the recursion relation in Eq. 18. Once Eq. 19_(2m,\nu) is given, and $T_{2m+1,\nu} = 0, m \in \mathbb{Z}_{\geq 0}$ is obvious from symmetry considerations, we can interpolate $T_{\mu,\nu}$ as follows:

$$\begin{aligned} T_{\mu,\nu} &= \sum_{\ell=0}^\infty T_{\ell,\nu} \left[\frac{\sin(\ell-\mu)\pi}{(\ell-\mu)\pi} + \frac{\sin(\ell+\mu+1)\pi}{(\ell+\mu+1)\pi} \right] = \frac{\sin(\mu\pi)}{\pi} \sum_{m=0}^\infty T_{2m,\nu} \left(\frac{1}{\mu-2m} - \frac{1}{2m+\mu+1} \right) \\ &= \sin(\mu\pi) \cos(\nu\pi) \sum_{m=0}^\infty \frac{\Gamma\left(\frac{2\nu+1-2m}{2}\right) \Gamma(\nu+1+m)}{\left[\Gamma\left(\frac{1-2m}{2}\right)\right]^2 [\Gamma(m+1)]^2 \Gamma(\nu+1-m) \Gamma\left(\frac{2\nu+3+2m}{2}\right)} \left(\frac{1}{\mu-2m} - \frac{1}{2m+\mu+1} \right), \end{aligned}$$

which in turn, directly accounts for the hypergeometric summation in Eq. 19* concerning ${}_5F_4$. Exploiting again Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, one can rewrite the expression for $T_{2m,\nu}$ as (taking appropriate limits when the resulting fractions assume indeterminate forms)

$$T_{2m,\nu} = -\frac{\sin(\nu\pi)}{\pi} \frac{[\Gamma(m+\frac{1}{2})]^2 \Gamma(m-\nu) \Gamma(m+\nu+1)}{[\Gamma(m+1)]^2 \Gamma(m-\nu+\frac{1}{2}) \Gamma(m+\nu+\frac{3}{2})}, \quad m \in \mathbb{Z}_{\geq 0},$$

which specializes to Eq. 19_(2m, n+1/2).

To prove Eq. 19_(v, v), we show that the function

$$\mathfrak{T}(z) := \frac{1}{\sin^2(\pi z)} \left\{ T_{z,z} - \frac{1+2\cos(\pi z)}{3} \frac{\pi \Gamma\left(\frac{z+1}{2}\right) \Gamma\left(\frac{3z+2}{2}\right)}{\left[\Gamma\left(\frac{1-z}{2}\right)\right]^2 \left[\Gamma\left(\frac{z+2}{2}\right)\right]^3 \Gamma\left(\frac{3z+3}{2}\right)} \right\} \equiv \mathfrak{T}(-z-1) \quad (20)$$

is analytic over the whole complex z -plane, with asymptotic behavior $\mathfrak{T}(z) = O(N^{-9/4})$, $z \in C_N, N \rightarrow +\infty$. Here, N is a positive integer and the square contour C_N has vertices $(1-4N)/4 - iN$, $(1+4N)/4 - iN$, $(1+4N)/4 + iN$ and $(1-4N)/4 + iN$.

First, to verify that $\mathfrak{T}(z)$ is an entire function, we only need to check that the expression inside the braces of Eq. 20 has vanishing derivative whenever z is an integer, so that the numerator of Eq. 20 encounters (at least) a second-order zero at every integer $z = n \in \mathbb{Z}$. At the positive odd integers $z = 2n+1, n \in \mathbb{Z}_{\geq 0}$, one may directly compute

$$\left. \frac{\partial T_{z,z}}{\partial z} \right|_{z=2n+1} = 2 \int_{-1}^1 \left. \frac{\partial P_z(x)}{\partial z} \right|_{z=2n+1} P_{2n+1}(x) P_{2n+1}(-x) dx + \int_{-1}^1 [P_{2n+1}(-x)]^2 \left. \frac{\partial P_z(x)}{\partial z} \right|_{z=2n+1} dx = \left. \frac{\partial T_{z,2n+1}}{\partial z} \right|_{z=2n+1} = 0$$

from Eq. 19_(μ, n), which coincides with the behavior

$$\frac{1+2\cos(\pi z)}{3} \frac{\pi \Gamma\left(\frac{z+1}{2}\right) \Gamma\left(\frac{3z+2}{2}\right)}{\left[\Gamma\left(\frac{1-z}{2}\right)\right]^2 \left[\Gamma\left(\frac{z+2}{2}\right)\right]^3 \Gamma\left(\frac{3z+3}{2}\right)} = O((z - (2n+1))^2)$$

attributed to the factor $[\Gamma((1-z)/2)]^2 = O((z - (2n+1))^{-2})$. Meanwhile, at the non-negative even integers $z = 2n, n \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \left. \frac{\partial T_{z,z}}{\partial z} \right|_{z=2n} &= 2 \int_{-1}^1 \left. \frac{\partial P_z(x)}{\partial z} \right|_{z=2n} P_{2n}(x) P_{2n}(-x) dx + \int_{-1}^1 [P_{2n}(-x)]^2 \left. \frac{\partial P_z(x)}{\partial z} \right|_{z=2n} dx = 3 \left. \frac{\partial T_{z,2n}}{\partial z} \right|_{z=2n} \\ &= 3 T_{2n,2n} \left. \frac{\partial}{\partial z} \right|_{z=2n} \log \frac{\Gamma\left(\frac{4n+1-z}{2}\right) \Gamma\left(\frac{4n+2+z}{2}\right)}{\left[\Gamma\left(\frac{1-z}{2}\right)\right]^2 \left[\Gamma\left(\frac{z+2}{2}\right)\right]^2 \Gamma\left(\frac{4n+2-z}{2}\right) \Gamma\left(\frac{4n+3+z}{2}\right)} \end{aligned}$$

is also a result of Eq. 19_(μ, n). As we have

$$3 \left. \frac{\partial}{\partial z} \right|_{z=v} \log \frac{\Gamma\left(\frac{2v+1-z}{2}\right) \Gamma\left(\frac{2v+2+z}{2}\right)}{\left[\Gamma\left(\frac{1-z}{2}\right)\right]^2 \left[\Gamma\left(\frac{z+2}{2}\right)\right]^2 \Gamma\left(\frac{2v+2-z}{2}\right) \Gamma\left(\frac{2v+3+z}{2}\right)} - \left. \frac{\partial}{\partial z} \right|_{z=v} \log \frac{\Gamma\left(\frac{z+1}{2}\right) \Gamma\left(\frac{3z+2}{2}\right)}{\left[\Gamma\left(\frac{1-z}{2}\right)\right]^2 \left[\Gamma\left(\frac{z+2}{2}\right)\right]^3 \Gamma\left(\frac{3z+3}{2}\right)} = -2\pi \tan \frac{v\pi}{2},$$

which vanishes when $v = 2n$ is a non-negative even number, it is clear that the numerator of Eq. 20 has $O((z-2n)^2)$ behavior for $n \in \mathbb{Z}_{\geq 0}$.

Then, we investigate the asymptotic behavior of $\mathfrak{T}(z)$ as $z \rightarrow \infty$. For any point z sitting on the right half of the contour C_N (where N is a large positive integer) satisfying $\operatorname{Re} z > -\frac{1}{2}$, we have a uniform asymptotic formula

$$\begin{aligned} T_{z,z} &= \left[1 + O\left(\frac{1}{N}\right) \right] \int_0^{\pi/2} \left\{ J_0\left(\left(z + \frac{1}{2}\right)\theta\right) [1 + \cos(\pi z)] + Y_0\left(\left(z + \frac{1}{2}\right)\theta\right) \sin(\pi z) \right\} J_0\left(\left(z + \frac{1}{2}\right)\theta\right) \times \\ &\quad \times \left[J_0\left(\left(z + \frac{1}{2}\right)\theta\right) \cos(\pi z) + Y_0\left(\left(z + \frac{1}{2}\right)\theta\right) \sin(\pi z) \right] \sqrt{\frac{\theta^3}{\sin \theta}} d\theta. \end{aligned} \quad (21)$$

Here, we have used the following transformations for $-\pi < \arg v < \pi$:

$$\begin{aligned} T_{v,v} &= \int_0^\pi [P_v(\cos \theta)]^2 P_v(-\cos \theta) \sin \theta d\theta = \int_0^{\pi/2} [P_v(\cos \theta) + P_v(-\cos \theta)] P_v(\cos \theta) P_v(-\cos \theta) \sin \theta d\theta \\ &= \int_0^{\pi/2} \left\{ P_v(\cos \theta) [1 + \cos(v\pi)] - \frac{2}{\pi} Q_v(\cos \theta) \sin(v\pi) \right\} P_v(\cos \theta) \left[P_v(\cos \theta) \cos(v\pi) - \frac{2}{\pi} Q_v(\cos \theta) \sin(v\pi) \right] \sin \theta d\theta, \end{aligned}$$

along with the asymptotic formulae (see [Ref. 19, Chap. 12, Eqs. 12.18 and 12.25] or [Ref. 20, Eqs. 43 and 46]):

$$P_v(\cos \theta) = \sqrt{\frac{\theta}{\sin \theta}} J_0\left(\left(v + \frac{1}{2}\right)\theta\right) + O\left(\frac{1}{2v+1}\right), \quad Q_v(\cos \theta) = -\frac{\pi}{2} \sqrt{\frac{\theta}{\sin \theta}} Y_0\left(\left(v + \frac{1}{2}\right)\theta\right) + O\left(\frac{1}{2v+1}\right)$$

where the error bounds are uniform for the angular variables $\theta \in (0, \pi/2]$, so long as $|\nu| \rightarrow \infty, -\pi < \arg \nu < \pi$ [19, 20]. For the case with degree $\nu = (1 + 4N)/4$ where N is a large positive integer, we split the integral on the right-hand side of Eq. 21 into two parts:

$$\int_0^{\pi/2} (\dots) d\theta = \int_0^{1/\sqrt{2\nu+1}} (\dots) d\theta + \int_{1/\sqrt{2\nu+1}}^{\pi/2} (\dots) d\theta.$$

For the first portion, we may deduce

$$\begin{aligned} & \int_0^{1/\sqrt{2\nu+1}} \left\{ J_0 \left(\left(\nu + \frac{1}{2} \right) \theta \right) [1 + \cos(\nu\pi)] + Y_0 \left(\left(\nu + \frac{1}{2} \right) \theta \right) \sin(\nu\pi) \right\} J_0 \left(\left(\nu + \frac{1}{2} \right) \theta \right) \times \\ & \times \left[J_0 \left(\left(\nu + \frac{1}{2} \right) \theta \right) \cos(\nu\pi) + Y_0 \left(\left(\nu + \frac{1}{2} \right) \theta \right) \sin(\nu\pi) \right] \sqrt{\frac{\theta^3}{\sin \theta}} d\theta = \frac{8[1 + \cos(\nu\pi)][1 + 2\cos(\nu\pi)]}{3\sqrt{3}\pi(1 + 2\nu)^2} \left[1 + O \left(\frac{1}{\sqrt{2\nu+1}} \right) \right] \end{aligned}$$

from three integral formulae valid for $a > 0$ [Ref. 21, items 2.12.42.25, 2.13.22.11, 2.13.25.1]:¹

$$\int_0^\infty [J_0(ax)]^3 x dx = \frac{2}{\sqrt{3}\pi a^2}, \quad \int_0^\infty [J_0(ax)]^2 Y_0(ax) x dx = 0, \quad \int_0^\infty J_0(ax) [Y_0(ax)]^2 x dx = \frac{2}{3\sqrt{3}\pi a^2},$$

as well as the familiar asymptotic behavior of Bessel functions for large arguments $x \gg 1$:

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{\pi}{4} \right), \quad Y_0(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left(x - \frac{\pi}{4} \right),$$

which allows us to verify

$$\begin{aligned} & \int_{1/\sqrt{2\nu+1}}^\infty \left\{ J_0 \left(\left(\nu + \frac{1}{2} \right) x \right) [1 + \cos(\nu\pi)] + Y_0 \left(\left(\nu + \frac{1}{2} \right) x \right) \sin(\nu\pi) \right\} J_0 \left(\left(\nu + \frac{1}{2} \right) x \right) \times \\ & \times \left[J_0 \left(\left(\nu + \frac{1}{2} \right) x \right) \cos(\nu\pi) + Y_0 \left(\left(\nu + \frac{1}{2} \right) x \right) \sin(\nu\pi) \right] x dx = O \left(\frac{1}{(2\nu+1)^{9/4}} \right) \end{aligned}$$

via integration by parts.² Meanwhile, using the asymptotic forms of Bessel functions and integration by parts, one can also verify that³

$$\begin{aligned} & \int_{1/\sqrt{2\nu+1}}^{\pi/2} \left\{ J_0 \left(\left(\nu + \frac{1}{2} \right) \theta \right) [1 + \cos(\nu\pi)] + Y_0 \left(\left(\nu + \frac{1}{2} \right) \theta \right) \sin(\nu\pi) \right\} J_0 \left(\left(\nu + \frac{1}{2} \right) \theta \right) \times \\ & \times \left[J_0 \left(\left(\nu + \frac{1}{2} \right) \theta \right) \cos(\nu\pi) + Y_0 \left(\left(\nu + \frac{1}{2} \right) \theta \right) \sin(\nu\pi) \right] \sqrt{\frac{\theta^3}{\sin \theta}} d\theta \\ & \sim \left[\frac{4}{\pi(2\nu+1)} \right]^{3/2} \int_{1/\sqrt{2\nu+1}}^{\pi-(1/\sqrt{2\nu+1})} \cos^2 \left(\left(\nu + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right) \cos \left(\left(\nu + \frac{1}{2} \right) (\pi - \theta) - \frac{\pi}{4} \right) \frac{d\theta}{\sqrt{\sin \theta}} = O \left(\frac{1}{(2\nu+1)^{9/4}} \right). \end{aligned}$$

¹The first integral here belongs to the generalized Weber-Schafheitlin type [Ref. 15, §13.46]. The proof of the second integral can be found in [22]. The third integral can be derived from the first one, in conjunction with Nicholson's formula [Ref. 15, §13.73] $[J_0(x)]^2 + [Y_0(x)]^2 = \frac{8}{\pi^2} \int_0^\infty K_0(2x \sinh t) dt$ and a special case of the modified Weber-Schafheitlin integral [Ref. 15, §13.45] $\int_0^\infty J_0(x) K_0(2x \sinh t) x dx = (1 + 4 \sinh^2 t)^{-1}$.

²To wit, we need to estimate

$$\begin{aligned} & \int_{1/\sqrt{2\nu+1}}^\infty \frac{\sin(a(2\nu+1)x+b)}{\sqrt{x}} dx = \frac{\sin b}{a\sqrt{2\nu+1}} \int_{\sqrt{2\nu+1}}^\infty \frac{1}{\sqrt{y}} \frac{\partial}{\partial y} \sin(ay) dy - \frac{\cos b}{a\sqrt{2\nu+1}} \int_{\sqrt{2\nu+1}}^\infty \frac{1}{\sqrt{y}} \frac{\partial}{\partial y} \cos(ay) dy \\ & = \frac{\cos(a\sqrt{2\nu+1}+b)}{a(2\nu+1)^{3/4}} + \frac{\sin b}{2a\sqrt{2\nu+1}} \int_{\sqrt{2\nu+1}}^\infty \frac{\sin(ay)}{y^{3/2}} dy - \frac{\cos b}{2a\sqrt{2\nu+1}} \int_{\sqrt{2\nu+1}}^\infty \frac{\cos(ay)}{y^{3/2}} dy = O \left(\frac{1}{(2\nu+1)^{3/4}} \right), \end{aligned}$$

where the coefficients $a, b \in \mathbb{R}$ arise from the product-to-sum formulae of trigonometric functions applied to asymptotic expansions for the products of Bessel functions.

³One might wish to compare this with the direct application of a hypergeometric asymptotic formula for Legendre functions (see [Ref. 8, p. 295] or [Ref. 18, p. 162]):

$$P_\nu(\cos \theta) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})} \sqrt{\frac{2}{\pi \sin \theta}} \left[\cos \left(\left(\nu + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right) + O \left(\frac{1}{\nu} \right) \right], \quad \text{where } \theta \in [\varepsilon, \pi - \varepsilon], \varepsilon > 0.$$

Thus, the asymptotic expansion

$$T_{v,v} = \frac{8[1 + \cos(v\pi)][1 + 2\cos(v\pi)]}{3\sqrt{3}\pi(1+2v)^2} \left[1 + O\left(\frac{1}{\sqrt[4]{2v+1}}\right) \right]$$

is established for $v = (1 + 4N)/4 > 0$. For an arbitrary point z satisfying $\operatorname{Re} z > -1/2$, we may modify the aforementioned procedure by deforming the integral path joining the points 0 and $\pi/2$ into the sum of two line segments:

$$\int_0^{\pi/2} (\dots) d\theta = \int_0^{\sqrt{|2z+1|}/(2z+1)} (\dots) d\theta + \int_{\sqrt{|2z+1|}/(2z+1)}^{\pi/2} (\dots) d\theta,$$

followed by the integrations $\int_0^\infty (\dots) dx$ and $\int_{\sqrt{|2z+1|}/(2z+1)}^\infty (\dots) dx$ along contours that run to infinity in such a manner that $(2z+1)x \rightarrow +\infty$ along the positive real axis. This results in

$$T_{z,z} = \frac{8[1 + \cos(\pi z)][1 + 2\cos(\pi z)]}{3\sqrt{3}\pi(1+2z)^2} \left[1 + O\left(\frac{1}{\sqrt[4]{N}}\right) \right], \quad z \in C_N, \operatorname{Re} z > -\frac{1}{2}, \quad (22a)$$

where the error bound is uniform along the contour C_N , on which both $|\tan(\pi z)|$ and $|\cot(\pi z)|$ are uniformly bounded. After reflection $z \mapsto -z-1$, we obtain

$$T_{z,z} = \frac{8[1 - \cos(\pi z)][1 - 2\cos(\pi z)]}{3\sqrt{3}\pi(1+2z)^2} \left[1 + O\left(\frac{1}{\sqrt[4]{N}}\right) \right], \quad z \in C_N, \operatorname{Re} z < -\frac{1}{2}. \quad (22b)$$

We may combine Eqs. 22a and 22b into a single formula

$$T_{z,z} = \frac{1 + 2\cos(\pi z)}{3} \frac{\pi \Gamma\left(\frac{z+1}{2}\right) \Gamma\left(\frac{3z+2}{2}\right)}{[\Gamma\left(\frac{1-z}{2}\right)]^2 [\Gamma\left(\frac{z+2}{2}\right)]^3 \Gamma\left(\frac{3z+3}{2}\right)} \left[1 + O\left(\frac{1}{\sqrt[4]{N}}\right) \right], \quad z \in C_N,$$

which also implies the bound estimate $\mathfrak{T}(z) = O(N^{-9/4})$ uniformly applicable to all the points z on the rectangular contour C_N , thanks to the inequalities

$$\sup_{z \in C_N} |\cot(\pi z)| \leq \max(1, \coth \pi) = \coth \pi, \quad \sup_{z \in C_N} \frac{1}{|\sin(\pi z)|} \leq \max\left(\sqrt{2}, \frac{1}{\sinh \pi}\right) = \sqrt{2},$$

for $N \in \mathbb{Z}_{>0}$. By Cauchy's integral formula, we have the identity

$$\mathfrak{T}(v) = \lim_{N \rightarrow \infty} \oint_{C_N} \frac{\mathfrak{T}(z) dz}{z - v} = 0, \quad \forall v \in \mathbb{C},$$

which proves Eq. 19_(v,v).

Both Eqs. 19_(2v-1,v) and 19_(2v+2,v) follow directly from the recursion relation (Eq. 18). ■

Remark As in our proof of Eq. 19_(v,v), we can use Bessel functions to establish the following asymptotic behavior for $|z| \rightarrow \infty, \operatorname{Re} z > -1/2$:

$$\begin{aligned} T_{\rho z, z} &\sim \frac{8 \cos(\rho \pi z + \pi z) + \cos(\pi z)}{\pi \rho \sqrt{4 - \rho^2} (1 + 2z)^2} + \frac{16}{\pi^2} \frac{\sin(\rho \pi z) \sin(\pi z)}{\rho \sqrt{4 - \rho^2} (1 + 2z)^2} \left(\pi - \arcsin \sqrt{1 - \frac{\rho^2}{4}} \right), & 0 < \rho < 2; \\ T_{\rho z, z} &\sim \frac{8 \sin(\rho \pi z - \pi z) - \sin(\pi z)}{\pi \rho \sqrt{\rho^2 - 4} (1 + 2z)^2} - \frac{16}{\pi^2} \frac{\sin(\rho \pi z) \sin(\pi z)}{\rho \sqrt{\rho^2 - 4} (1 + 2z)^2} \sinh^{-1} \sqrt{\frac{\rho^2}{4} - 1}, & \rho > 2. \end{aligned}$$

along with an asymptotic reflection formula $T_{\rho(-z-1),-z-1} \sim e^{i\pi(1-\rho)\arg z/|\arg z|} T_{\rho z,z}$, $0 < |\arg z| < \pi$ for $\rho > 0, |z| \rightarrow \infty$. Put differently, for large $|z|$ and $-\pi < \arg z < \pi$, we have an “asymptotic trigonometric modulation” of the standard Clebsch-Gordan integral formulae:

$$T_{\rho z,z} \sim \frac{\pi \Gamma\left(\frac{1+(2-\rho)z}{2}\right) \Gamma\left(\frac{(\rho+2)z+2}{2}\right)}{\left[\Gamma\left(\frac{1-\rho z}{2}\right)\right]^2 \left[\Gamma\left(\frac{2+\rho z}{2}\right)\right]^2 \Gamma\left(\frac{2+(2-\rho)z}{2}\right) \Gamma\left(\frac{(\rho+2)z+3}{2}\right)} \times$$

$$\times \left[\frac{\cos(\rho\pi z + \pi z) + \cos(\pi z)}{1 + \cos(\rho\pi z)} + \frac{2}{\pi} \frac{\sin(\rho\pi z) \sin(\pi z)}{1 + \cos(\rho\pi z)} \left(\pi - \arcsin \sqrt{1 - \frac{\rho^2}{4}} \right) \right], \quad 0 < \rho < 2;$$

$$T_{\rho z,z} \sim \frac{\pi \Gamma\left(\frac{1+(2-\rho)z}{2}\right) \Gamma\left(\frac{(\rho+2)z+2}{2}\right)}{\left[\Gamma\left(\frac{1-\rho z}{2}\right)\right]^2 \left[\Gamma\left(\frac{2+\rho z}{2}\right)\right]^2 \Gamma\left(\frac{2+(2-\rho)z}{2}\right) \Gamma\left(\frac{(\rho+2)z+3}{2}\right)} \times$$

$$\times \left[\frac{\sin(\rho\pi z - \pi z) - \sin(\pi z)}{1 + \cos(\rho\pi z)} - \frac{2}{\pi} \frac{\sin(\rho\pi z) \sin(\pi z)}{1 + \cos(\rho\pi z)} \sinh^{-1} \sqrt{\frac{\rho^2}{4} - 1} \right] \cot \frac{(\rho-2)\pi z}{2}, \quad \rho > 2.$$

For $\rho \neq 1$, the right-hand side of the penultimate (resp. last) asymptotic formula diverges at $z = -2/(2+\rho)$ (resp. $z = -1/\rho$), so the asymptotic analysis does not result in exact forms of $T_{\rho z,z}$ ($\rho \neq 1$) for finitely sized $|z|$. $\square$

Corollary 2.2 (Some Multiple Elliptic Integrals in Clebsch-Gordan Forms) *We have the integral formulae*

$$\frac{\pi^2}{4} T_{0,-1/2} = 2 \int_0^1 \mathbf{K}(\sqrt{t}) \mathbf{K}(\sqrt{1-t}) dt = \frac{\pi^3}{4}, \quad (23)$$

$$\frac{\pi^2}{4} T_{0,1/2} = 2 \int_0^1 [2\mathbf{E}(\sqrt{t}) - \mathbf{K}(\sqrt{t})][2\mathbf{E}(\sqrt{1-t}) - \mathbf{K}(\sqrt{1-t})] dt = 0, \quad (24)$$

$$\frac{\pi^3}{8} T_{-1/2,-1/2} = 2 \int_0^1 [\mathbf{K}(\sqrt{1-t})]^2 \mathbf{K}(\sqrt{t}) dt = \frac{[\Gamma(\frac{1}{4})]^8}{192\pi^2}, \quad (25)$$

$$\frac{\pi^3}{8} T_{-1/3,-1/3} = \frac{27}{2} \int_0^1 \frac{(1-p^2)p(2+p)}{\sqrt{3+6p(1+p+p^2)}} \left[\mathbf{K}\left(\sqrt{\frac{p^3(2+p)}{1+2p}}\right) \right]^2 \mathbf{K}\left(\sqrt{1-\frac{p^3(2+p)}{1+2p}}\right) dp$$

$$= \frac{27}{6} \int_0^1 \frac{(1-p^2)p(2+p)}{\sqrt{1+2p(1+p+p^2)}} \left[\mathbf{K}\left(\sqrt{1-\frac{p^3(2+p)}{1+2p}}\right) \right]^2 \mathbf{K}\left(\sqrt{\frac{p^3(2+p)}{1+2p}}\right) dp = \frac{3\sqrt{3}[\Gamma(\frac{1}{3})]^9}{256\pi^2}, \quad (26)$$

$$\frac{\pi^3}{8} T_{-1/4,-1/4} = \int_0^1 \frac{4}{(1+\sqrt{u})^{3/2}} \left[\mathbf{K}\left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right) \right]^2 \mathbf{K}\left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}}\right) du = 4\sqrt{2} \int_0^1 \frac{(1-t)[\mathbf{K}(\sqrt{t})]^2 \mathbf{K}(\sqrt{1-t})}{(1+t)^{3/2}} dt$$

$$= \int_0^1 \left(\frac{2}{1+\sqrt{u}}\right)^{3/2} \left[\mathbf{K}\left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}}\right) \right]^2 \mathbf{K}\left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right) du = 4 \int_0^1 \frac{(1-t)[\mathbf{K}(\sqrt{1-t})]^2 \mathbf{K}(\sqrt{t})}{(1+t)^{3/2}} dt = \frac{[\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})]^2}{24}, \quad (27)$$

$$\frac{\pi^3}{8} T_{-1/6,-1/6} = \frac{27}{2\sqrt{2}} \int_{-1}^1 \left[\mathbf{K}\left(\sqrt{\frac{1-x}{2}}\right) \right]^2 \mathbf{K}\left(\sqrt{\frac{1+x}{2}}\right) \frac{1-x^2}{(3+x^2)^{7/4}} dx = \frac{27}{4} \int_0^1 \frac{t(1-t)[\mathbf{K}(\sqrt{1-t})]^2 \mathbf{K}(\sqrt{t})}{(1-t+t^2)^{7/4}} dt = \frac{[\Gamma(\frac{1}{4})]^4}{8\sqrt{2}\sqrt{3}}, \quad (28)$$

$$\frac{\pi^3}{8} T_{1/2,-3/4} = 2\sqrt{2} \int_0^1 \frac{2\mathbf{E}(\sqrt{u}) - \mathbf{K}(\sqrt{u})}{1+\sqrt{u}} \mathbf{K}\left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}}\right) \mathbf{K}\left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right) du = \sqrt{2}\pi, \quad (29)$$

$$\frac{\pi^3}{8} T_{3/2,-1/4} = \frac{2\sqrt{2}}{3} \int_0^1 \frac{8(1-2u)\mathbf{E}(\sqrt{u}) - (5-8u)\mathbf{K}(\sqrt{u})}{1+\sqrt{u}} \mathbf{K}\left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}}\right) \mathbf{K}\left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right) du = -\frac{\sqrt{2}\pi}{9}. \quad (30)$$

Proof Special cases of Eqs. 19_(2m,n+1/2), 19_(v,v) and 19_(2v+2,v) lead to Eqs. 23, 24, 25, 26, 27, 28, 29 and 30. $\blacksquare$

3 Spherical Rotations and Multiple Elliptic Integrals

In this section, we shall focus on the transformations of multiple elliptic integrals using variable substitutions motivated by rotations on a unit sphere. In effect, our attention will be momentarily restricted to two special Legendre functions $P_{-1/2}$ and $P_{-1/4}$.

The transformation methods in this section not only provide evaluations of certain Cauchy principal values:

$$\mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2}) d\xi}{\pi(x-\xi)\sqrt{1+\xi}}, \quad -1 < x < 1; \quad \mathcal{P} \int_{-1}^1 \mathbf{K}\left(\sqrt{\frac{1+\xi}{2}}\right) \mathbf{K}\left(\sqrt{\frac{1-\xi}{2}}\right) \frac{2d\xi}{\pi(x-\xi)}, \quad -1 < x < 1 \quad \text{etc.}$$

which will be used in §4, but also pave way for further applications in our subsequent works on elliptic integrals.

3.1 Beltrami Rotations

The technique of spherical rotation, which traces back to Beltrami's work on the Abel integral equations [Ref. 23, p. 328], has found applications in the geometrically-motivated evaluation of certain definite integrals related to Bessel functions (see [Ref. 15, §3.33, §12.12 and §12.14]). The lemma below is a simple realization of the Beltrami rotation in the case of the complete elliptic integrals $\mathbf{K}(k)$ and $\mathbf{E}(k)$.

Lemma 3.1 (Beltrami Transformations) (a) *We have the following integral identities*

$$\mathbf{K}(k) = \frac{2}{\pi} \int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{1-k^2\kappa^2}, \quad 0 < k < 1, \quad (31)$$

$$\mathbf{K}(\xi) = \frac{2}{\pi} \int_0^1 \frac{\sqrt{1-\xi^2} \mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{1-\xi^2(1-\kappa^2)}, \quad 0 < \xi < 1, \quad (31')$$

$$\mathbf{K}(r) = \frac{2}{\pi} \int_0^1 \frac{(1+r) \mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{(1+r)^2 - 4r\kappa^2}, \quad 0 < r < 1, \quad (31_L)$$

$$\mathbf{K}(\eta) = \frac{2}{\pi} \int_0^1 \frac{(1-\eta) \mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{(1-\eta)^2 + 4\eta\kappa^2}, \quad 0 < \eta < 1. \quad (31'_L)$$

(b) *For $0 < k < 1$, we have*

$$\frac{\mathbf{K}(k) - \mathbf{E}(k)}{k^2} = \frac{2}{\pi} \int_0^1 \frac{\mathbf{E}(\sqrt{1-\kappa^2}) d\kappa}{1-k^2\kappa^2}, \quad (32)$$

$$\mathbf{E}(k) = \frac{4(1-k^2)}{\pi} \int_0^1 \frac{\mathbf{E}(\sqrt{1-\kappa^2}) d\kappa}{(1-k^2\kappa^2)^2}, \quad (33)$$

$$\frac{\mathbf{E}(k)}{\sqrt{1-k^2}} = \frac{4(1-k^2)}{\pi} \int_0^1 \frac{\mathbf{E}(\sqrt{1-\kappa^2}) d\kappa}{[1-k^2(1-\kappa^2)]^2}. \quad (33')$$

Proof (a) We start by writing the complete elliptic integral of the first kind as an integral on the unit sphere S^2 :

$$\mathbf{K}(k) = \frac{1}{2} \int_0^\pi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \frac{1}{4\pi} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \frac{1}{\sin \theta (1-k \sin \theta \cos \phi)} = \int_{S^2} \frac{1}{\sqrt{1-Z^2(1-kX)}} \frac{d\sigma}{4\pi}.$$

As we rotate about the Y -axis by a right angle, effectively swapping the rôles of the X - and Z -axes, we may rewrite the integral above as

$$\mathbf{K}(k) = \int_{S^2} \frac{1}{\sqrt{1-X^2(1-kZ)}} \frac{d\sigma}{4\pi} = \int_{S^2} \frac{1}{\sqrt{1-X^2(1-k^2 Z^2)}} \frac{d\sigma}{4\pi} = \frac{1}{4\pi} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \frac{1}{\sqrt{1-\sin^2 \theta \cos^2 \phi} (1-k^2 \cos^2 \theta)},$$

where we have combined the values of the integrand at points $\pm Z$ and restored the spherical coordinates. Integrating over the azimuthal angle ϕ , we obtain

$$\mathbf{K}(k) = \frac{1}{\pi} \int_0^\pi \frac{\mathbf{K}(\sin \theta) \sin \theta d\theta}{1-k^2 \cos^2 \theta},$$

which is equivalent to Eq. 31.

Now that we have proved the relation

$$\mathbf{K}(k) = \frac{2}{\pi} \int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2})}{1-k^2\kappa^2} d\kappa, \quad 0 < k < 1,$$

which analytically continues to $k \in \mathbb{C} \setminus [1, +\infty)$, we may derive Eqs. 31', 31_L and 31'_L, respectively, from the imaginary modulus transformation $\mathbf{K}(\xi) = \mathbf{K}(i\xi/\sqrt{1-\xi^2})/\sqrt{1-\xi^2}$, Landen's transformation $\mathbf{K}(r) = \mathbf{K}(2\sqrt{r}/(1+r))/(1+r)$, and a combination thereof $\mathbf{K}(\eta) = \mathbf{K}(2i\sqrt{\eta}/(1-\eta))/(1-\eta)$.

(b) We adapt our derivation of part (a) to accommodate for the complete elliptic integrals of the second kind $\mathbf{E}(k), 0 < k < 1$:

$$\begin{aligned} \frac{\mathbf{K}(k) - \mathbf{E}(k)}{k^2} &= \frac{1}{2} \int_0^\pi \frac{\sin^2 \theta d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \frac{1}{4\pi} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \frac{\sin \theta}{1-k \sin \theta \cos \phi} = \int_{S^2} \frac{\sqrt{1-Z^2}}{1-kX} \frac{d\sigma}{4\pi} \\ &= \int_{S^2} \frac{\sqrt{1-X^2}}{1-kZ} \frac{d\sigma}{4\pi} = \int_{S^2} \frac{\sqrt{1-X^2}}{1-k^2 Z^2} \frac{d\sigma}{4\pi} = \frac{1}{4\pi} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \frac{\sqrt{1-\sin^2 \theta \cos^2 \phi}}{1-k^2 \cos^2 \theta} = \frac{1}{\pi} \int_0^\pi \frac{\mathbf{E}(\sin \theta) \sin \theta d\theta}{1-k^2 \cos^2 \theta}, \end{aligned}$$

which verifies Eq. 32. As we multiply both sides of Eq. 32 by k^2 , and differentiate in k , we obtain the formula stated in Eq. 33:

$$\frac{k\mathbf{E}(k)}{1-k^2} = \frac{d}{dk} [\mathbf{K}(k) - \mathbf{E}(k)] = \frac{2}{\pi} \frac{d}{dk} \int_0^1 \frac{\mathbf{E}(\sqrt{1-\kappa^2}) k^2 d\kappa}{1-k^2 \kappa^2} = \frac{4k}{\pi} \int_0^1 \frac{\mathbf{E}(\sqrt{1-\kappa^2}) d\kappa}{(1-k^2 \kappa^2)^2}, \quad 0 < k < 1.$$

Now, by analytic continuation, we have

$$\mathbf{E}(k) = \frac{4(1-k^2)}{\pi} \int_0^1 \frac{\mathbf{E}(\sqrt{1-\kappa^2}) d\kappa}{(1-k^2 \kappa^2)^2}, \quad k \in \mathbb{C} \setminus [1, +\infty). \quad (33^*)$$

In particular, by the imaginary modulus transformation for complete elliptic integrals of the second kind [Ref. 24, item 160.02], we obtain

$$\sqrt{1+\xi^2} \mathbf{E}\left(\frac{\xi}{\sqrt{1+\xi^2}}\right) = \mathbf{E}(i\xi) = \frac{4(1+\xi^2)}{\pi} \int_0^1 \frac{\mathbf{E}(\sqrt{1-\kappa^2}) d\kappa}{(1+\xi^2 \kappa^2)^2}, \quad \xi > 0,$$

which is equivalent to Eq. 33'. ■

Remark Admittedly, the spherical rotation is not the only road to any or all of the identities in Eqs. 31, 31', 31_L and 31'_L. For example, one may still verify Eq. 31 via the method of moments and hypergeometric summations [6]. The same can be said for several formulae that we will encounter in Proposition 3.3. However, it is our hope that the spherical rotations provide clearer geometric motivations than a purely combinatorial approach.

The author thanks an anonymous referee for pointing out a recent contribution of V. Anghel [25], which generalizes the Beltrami rotation to hyperspheres embedded in Euclidean spaces of arbitrary dimensions. It is highly probable that the hyperspherical analog of Beltrami rotation can lead to geometric proofs of various integral identities related to elliptic integrals, which we hope to pursue in a separate publication. □

Beltrami transformations allow us to convert one multiple elliptic integral into another. For instance, the following chain of identities

$$\int_0^1 \mathbf{K}(\xi) d\xi = \int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2})}{1+\kappa} d\kappa = \int_0^1 \frac{\mathbf{K}(\xi)\xi}{1+\sqrt{1-\xi^2}} \frac{d\xi}{\sqrt{1-\xi^2}} = \frac{2}{\pi} \int_0^1 \mathbf{K}(\sqrt{1-\kappa^2}) \left[\frac{\kappa \arccos \kappa}{\sqrt{1-\kappa^2}} - \log(2\kappa) \right] d\kappa$$

result from two applications of Eq. 31', and the transformations

$$\int_0^1 \mathbf{E}(k) dk = \frac{2}{\pi} \int_0^1 \frac{\mathbf{E}(\sqrt{1-\kappa^2})}{\kappa^3} [-\kappa + (1+\kappa^2) \tanh^{-1} \kappa] d\kappa = \int_0^1 \frac{(2+\kappa)\mathbf{E}(\sqrt{1-\kappa^2})}{(1+\kappa)^2} d\kappa$$

can be justified by Eqs. 33 and 33'. More examples will be described elsewhere.

In the next corollary, we mention a few consequences of Beltrami transformations that are pertinent to later developments in the current work.

Corollary 3.2 (Some Applications of Beltrami Transformations) (a) For $0 < u < 1$, we have the following duality relations for multiple elliptic integrals:

$$\int_0^{\sqrt{u}} \frac{\mathbf{K}(k) dk}{\sqrt{1-k^2} \sqrt{u-k^2}} = \int_0^1 \frac{k \mathbf{K}(k) dk}{\sqrt{1-k^2} \sqrt{1-k^2 u}}, \quad (34)$$

$$\int_{\sqrt{u}}^1 \frac{k \mathbf{K}(k) dk}{\sqrt{1-k^2} \sqrt{k^2-u}} = \int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{\sqrt{1-\kappa^2} \sqrt{1-\kappa^2 u}}. \quad (35)$$

(b) We have the following Cauchy principal values for $-1 < x < 1$:

$$\mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2})}{\pi(x-\xi)\sqrt{1+\xi}} d\xi = \frac{\mathbf{K}(\sqrt{(1+x)/2})}{\sqrt{1+x}}, \quad (36)$$

$$\mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2}) - 2\mathbf{E}(\sqrt{(1-\xi)/2})}{\pi(x-\xi)\sqrt{1+\xi}} d\xi = \frac{2\mathbf{E}(\sqrt{(1+x)/2}) - \mathbf{K}(\sqrt{(1+x)/2})}{\sqrt{1+x}}. \quad (37)$$

Proof (a) With Eq. 31', we may interchange the order of integrations to compute

$$\int_0^{\sqrt{u}} \frac{\mathbf{K}(k) dk}{\sqrt{1-k^2}\sqrt{u-k^2}} = \int_0^{\sqrt{u}} \left[\frac{2}{\pi} \int_0^1 \frac{\sqrt{1-\xi^2}\mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{1-\xi^2(1-\kappa^2)} \right] \frac{d\xi}{\sqrt{1-\xi^2}\sqrt{u-\xi^2}} = \int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{\sqrt{1-u(1-\kappa^2)}},$$

where the last expression is equivalent to the right-hand side of Eq. 34, by the correspondence $\kappa \mapsto \sqrt{1-k^2}$. Similarly, we may prove Eq. 35 using Eq. 31:

$$\int_{\sqrt{u}}^1 \frac{k\mathbf{K}(k) dk}{\sqrt{1-k^2}\sqrt{k^2-u}} = \int_0^{\sqrt{u}} \left[\frac{2}{\pi} \int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{1-k^2\kappa^2} \right] \frac{k dk}{\sqrt{1-k^2}\sqrt{k^2-u}} = \int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{\sqrt{1-\kappa^2}\sqrt{1-\kappa^2u}}.$$

(b) For $z > 1$, one can readily compute (cf. [Ref. 26, p. 233])

$$\begin{aligned} \mathbf{K}(\sqrt{z}) &= \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-zt^2}} = \int_0^{1/\sqrt{z}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-zt^2}} - i \int_{1/\sqrt{z}}^1 \frac{dt}{\sqrt{1-t^2}\sqrt{zt^2-1}} \\ &= \frac{1}{\sqrt{z}} \int_0^1 \frac{ds}{\sqrt{1-\frac{1}{z}s^2}\sqrt{1-s^2}} - \frac{i}{\sqrt{z}} \int_1^{\sqrt{z}} \frac{ds}{\sqrt{1-\frac{1}{z}s^2}\sqrt{s^2-1}} = \frac{\mathbf{K}(\sqrt{1/z}) - i\mathbf{K}(\sqrt{(z-1)/z})}{\sqrt{z}}. \end{aligned} \quad (38)$$

This is the inverse modulus transformation for complete elliptic integrals of the first kind.

Now, in the identity

$$\mathbf{K}(k) = \frac{2}{\pi} \int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{1-k^2\kappa^2}, \quad \forall k \in \mathbb{C} \setminus [1, +\infty),$$

we approach the limit of $k \pm i0^+ \in [1, +\infty)$, read off the real part, and employ the inverse modulus transformation (Eq. 38), to obtain

$$\mathbf{K}(k) = \frac{k}{\pi} \mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{k^2 - \kappa^2}, \quad 0 < k < 1,$$

which can be reformulated into Eq. 36 after the variable substitutions $k = \sqrt{(1+x)/2}$, $\kappa = \sqrt{(1+\xi)/2}$.

From Eq. 36, we may directly compute (cf. [Ref. 27, Eq. 11.215])

$$\begin{aligned} \mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2})\sqrt{1+\xi} d\xi}{\pi(x-\xi)} &= \mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2})[1+(\xi-x)+x] d\xi}{\pi(x-\xi)\sqrt{1+\xi}} \\ &= (1+x) \mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2}) d\xi}{\pi(x-\xi)\sqrt{1+\xi}} - \frac{1}{\pi} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2}) d\xi}{\sqrt{1+\xi}} \\ &= \sqrt{1+x} \mathbf{K}\left(\sqrt{\frac{1+x}{2}}\right) - \frac{1}{\pi} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2}) d\xi}{\sqrt{1+\xi}}, \quad -1 < x < 1. \end{aligned}$$

Here, the last integral can be simplified with the knowledge of $\int_0^1 \mathbf{K}(\sqrt{1-\kappa^2}) d\kappa = \pi^2/4$ [Ref. 13, item 6.141.2], which brings us

$$\mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2})\sqrt{1+\xi} d\xi}{\pi(x-\xi)} = \sqrt{1+x} \mathbf{K}\left(\sqrt{\frac{1+x}{2}}\right) - \frac{\pi}{\sqrt{2}}, \quad -1 < x < 1.$$

Applying integration by parts to the Tricomi transform (cf. [Ref. 27, Eqs. 11.218 and 11.219]), one can use the equation above to deduce that

$$\mathcal{P} \int_{-1}^1 \frac{1}{\pi(x-\xi)} \frac{d}{d\xi} \left[\mathbf{K} \left(\sqrt{\frac{1-\xi}{2}} \right) \sqrt{1+\xi} \right] d\xi = \frac{d}{d\xi} \left[\sqrt{1+x} \mathbf{K} \left(\sqrt{\frac{1+x}{2}} \right) \right] + \frac{1}{\sqrt{2}(x-1)}, \quad -1 < x < 1.$$

In other words, we have

$$\mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2}) - \mathbf{E}(\sqrt{(1-\xi)/2})}{\pi(x-\xi)(1-\xi)\sqrt{1+\xi}} d\xi = \frac{1}{1-x} \left[\frac{\mathbf{E}(\sqrt{(1+x)/2})}{\sqrt{1+x}} - \frac{1}{\sqrt{2}} \right], \quad -1 < x < 1.$$

Carrying this further, we obtain

$$\begin{aligned} \mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2}) - \mathbf{E}(\sqrt{(1-\xi)/2})}{\pi(x-\xi)\sqrt{1+\xi}} d\xi &= \frac{\mathbf{E}(\sqrt{(1+x)/2})}{\sqrt{1+x}} - \frac{1}{\sqrt{2}} + \frac{1}{\pi} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2}) - \mathbf{E}(\sqrt{(1-\xi)/2})}{(1-\xi)\sqrt{1+\xi}} d\xi \\ &= \frac{\mathbf{E}(\sqrt{(1+x)/2})}{\sqrt{1+x}} - \frac{1}{\sqrt{2}} + \frac{1}{\pi} \int_{-1}^1 \frac{d}{d\xi} \left[\mathbf{K} \left(\sqrt{\frac{1-\xi}{2}} \right) \sqrt{1+\xi} \right] d\xi = \frac{\mathbf{E}(\sqrt{(1+x)/2})}{\sqrt{1+x}}, \quad -1 < x < 1. \end{aligned}$$

We may combine the relation above with Eq. 36 into a symmetric form

$$\mathcal{P} \int_{-1}^1 \frac{\mathbf{K}(\sqrt{(1-\xi)/2}) - 2\mathbf{E}(\sqrt{(1-\xi)/2})}{\pi(x-\xi)\sqrt{1+\xi}} d\xi = \frac{2\mathbf{E}(\sqrt{(1+x)/2}) - \mathbf{K}(\sqrt{(1+x)/2})}{\sqrt{1+x}}, \quad -1 < x < 1,$$

as stated in Eq. 37. ■

3.2 Ramanujan Rotations

The next proposition is based on reverse engineering of some formulae in Ramanujan's notebooks that were stated without proofs.

Proposition 3.3 (Ramanujan Transformations) (a) For $s, t \in [0, 1)$, we may represent $\mathbf{K}(\sqrt{s})\mathbf{K}(\sqrt{t})$ and $[\mathbf{K}(\sqrt{t})]^2$ as integrals on the unit sphere S^2 :

$$\mathbf{K}(\sqrt{s})\mathbf{K}(\sqrt{t}) = \frac{1}{8} \int_{S^2} \frac{d\sigma}{\sqrt{(1-sX^2)(1-tY^2)-Z^2}}, \quad (39)$$

$$[\mathbf{K}(\sqrt{t})]^2 = \int_{S^2} \frac{2(2-t)\mathbf{K}(\sqrt{X^2+Y^2})}{(2-t)^2-t^2X^2} \frac{d\sigma}{4\pi} \quad (40)$$

$$= \int_{S^2} \frac{(1+t)\mathbf{K}(\sqrt{X^2+Y^2})}{(1+t)^2-4tX^2} \frac{d\sigma}{4\pi}, \quad (40_L)$$

where $X = \sin\theta\cos\phi, Y = \sin\theta\sin\phi, Z = \cos\theta$ are the Cartesian coordinates on the unit sphere S^2 . Consequently, the following identities hold for $0 < u < 1$:

$$\int_0^{\sqrt{u}} \frac{\mathbf{K}(k)dk}{\sqrt{1-k^2}\sqrt{u-k^2}} = \int_0^1 \frac{k\mathbf{K}(k)dk}{\sqrt{1-k^2}\sqrt{1-k^2u}} = \frac{1}{1+\sqrt{u}} \left[\mathbf{K} \left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}} \right) \right]^2, \quad (41)$$

$$\int_0^{\sqrt{1-u}} \frac{\mathbf{K}(k)dk}{\sqrt{1-k^2}\sqrt{1-u-k^2}} = \int_0^1 \frac{k\mathbf{K}(k)dk}{\sqrt{1-k^2}\sqrt{1-k^2(1-u)}} = \frac{2}{1+\sqrt{u}} \left[\mathbf{K} \left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}} \right) \right]^2. \quad (42)$$

(b) We have the integral identity

$$\int_{S^2} \mathbf{K}(\sqrt{X^2+Y^2}) f(|X|) \frac{d\sigma}{4\pi} = \frac{2}{\pi} \int_0^1 f(k) \mathbf{K} \left(\sqrt{\frac{1+k}{2}} \right) \mathbf{K} \left(\sqrt{\frac{1-k}{2}} \right) dk \quad (43)$$

so long as the function $f(|X|), 0 \leq |X| \leq 1$ assumes non-negative values and the surface integral is convergent. This allows us to convert Eq. 40 into the following identities for $0 < t < 1$:

$$[\mathbf{K}(\sqrt{t})]^2 = \frac{2}{\pi} \int_0^1 \frac{\mathbf{K}(\sqrt{\mu})\mathbf{K}(\sqrt{1-\mu})d\mu}{1-\mu t}, \quad (40^*)$$

$$[\mathbf{K}(\sqrt{1-t})]^2 = \frac{8}{\pi} \int_0^1 \frac{\mathbf{K}(\sqrt{\mu})\mathbf{K}(\sqrt{1-\mu})d\mu}{(1+\sqrt{t})^2 - \mu(1-\sqrt{t})^2}, \quad (40_L^*)$$

and to deduce the following integral formulae:⁴

$$\int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2})}{\sqrt{(2t-1)^2 - (1-\kappa^2)}} d\kappa = \frac{[\mathbf{K}(\sqrt{1-t}) - i\mathbf{K}(\sqrt{t})]^2}{2}, \quad 0 < t < \frac{1}{2}; \quad (44)$$

$$\int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2})}{\sqrt{(2t-1)^2 - (1-\kappa^2)}} d\kappa = \frac{[\mathbf{K}(\sqrt{1-t}) + i\mathbf{K}(\sqrt{t})]^2}{2}, \quad t < 0. \quad (44')$$

Moreover, we have the identity below for $0 < \lambda \leq 1/2$:

$$\int_{1-2\lambda}^1 \frac{k\mathbf{K}(k)dk}{\sqrt{1-k^2}\sqrt{k^2 - (1-2\lambda)^2}} = \mathbf{K}(\sqrt{\lambda})\mathbf{K}(\sqrt{1-\lambda}) = \int_{(1-\lambda)/(1+\lambda)}^1 \frac{\mathbf{K}(k)dk}{\sqrt{1-k^2}\sqrt{(1+\lambda)^2 k^2 - (1-\lambda)^2}}, \quad (45)$$

or equivalently,

$$\int_0^{\pi/2} \frac{\mathbf{K}(\sin\theta)d\theta}{\sqrt{1-(1-2\lambda)^2\cos^2\theta}} = \mathbf{K}(\sqrt{\lambda})\mathbf{K}(\sqrt{1-\lambda}) = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1+\lambda)^2 - (1-\lambda)^2\sin^2\theta - 4\lambda\sin^2\phi}}. \quad (45')$$

Proof (a) Writing $d\sigma = \sin\theta d\theta d\phi$ for the surface element on the unit sphere, we have

$$\mathbf{K}(\sqrt{s})\mathbf{K}(\sqrt{t}) = \frac{1}{8} \int_0^\pi \frac{d\theta}{\sqrt{1-s\cos^2\theta}} \int_0^{2\pi} \frac{d\phi}{\sqrt{1-t\cos^2\phi}} = \frac{1}{8} \int_{S^2} \frac{d\sigma}{\sqrt{1-sZ^2}\sqrt{1-Z^2-tX^2}}$$

where the Cartesian coordinates $X = \sin\theta\cos\phi, Z = \cos\theta$ were used in the last step. As in Lemma 3.1, we interchange the rôles of the X - and Z -axes, to deduce

$$\begin{aligned} \mathbf{K}(\sqrt{s})\mathbf{K}(\sqrt{t}) &= \frac{1}{8} \int_{S^2} \frac{d\sigma}{\sqrt{1-sX^2}\sqrt{1-X^2-tZ^2}} = \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \frac{\sin\theta}{\sqrt{1-s\sin^2\theta\cos^2\phi}\sqrt{1-\sin^2\theta\cos^2\phi-t\cos^2\theta}} \\ &= \int_0^{\pi/2} \frac{\sin\theta d\theta}{\sqrt{1-t\cos^2\theta}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-s\sin^2\theta\cos^2\phi}\sqrt{1-\frac{\sin^2\theta}{1-t\cos^2\theta}\cos^2\phi}}. \end{aligned}$$

For any two numbers $a, b \in (0, 1)$, we can verify the identity

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-a\cos^2\phi}\sqrt{1-b\cos^2\phi}} = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-a-(b-a)\cos^2\psi}} \quad (46)$$

with a simple substitution $\phi = \arctan(\sqrt{1-a}\tan\psi)$. Now, setting $a = s\sin^2\theta, b = \frac{\sin^2\theta}{1-t\cos^2\theta}$, we are led to the formula

$$\begin{aligned} \mathbf{K}(\sqrt{s})\mathbf{K}(\sqrt{t}) &= \int_0^{\pi/2} \sin\theta d\theta \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1-s)(1-\sin^2\theta\cos^2\phi) + (s-t)\cos^2\theta + st\cos^2\theta\sin^2\theta\sin^2\phi}} \\ &= \frac{1}{8} \int_{S^2} \frac{d\sigma}{\sqrt{(1-s)(1-X^2) + (s-t)Z^2 + stY^2Z^2}} = \frac{1}{8} \int_{S^2} \frac{d\sigma}{\sqrt{(1-sY^2)(1-tZ^2) - X^2}} \end{aligned} \quad (39')$$

⁴Here, in Eq. 44, we have chosen the univalent branches of the square roots such that $\text{Im} \sqrt{(2t-1)^2 - (1-\kappa^2)} \geq 0$, and $\mathbf{K}(\sqrt{1-t}) \geq 0, \mathbf{K}(\sqrt{t}) \geq 0$; in Eq. 44', it is understood that $\sqrt{(2t-1)^2 - (1-\kappa^2)} \geq 0$ on the left-hand side and

$$\mathbf{K}(\sqrt{t}) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+|t|\cos^2\theta}} \geq 0, \quad \text{Im} \mathbf{K}(\sqrt{1-t}) = \text{Im} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-(1-t)\cos^2\theta}} \leq 0, \quad \forall t < 0.$$

after a literal transformation from the spherical coordinates to Cartesian coordinates, and an algebraic simplification according to the spherical constraint $X^2 + Y^2 + Z^2 = 1$. Clearly, the last term in Eq. 39' is equivalent to the right-hand side of Eq. 39, as the integral in question remains invariant under a cyclic shift of variables $(X, Y, Z) \mapsto (Z, X, Y)$.

When $s = t$, we may swap the X - and Z -axes in the penultimate term of Eq. 39', to derive

$$\begin{aligned} [\mathbf{K}(\sqrt{t})]^2 &= \frac{1}{8} \int_{S^2} \frac{d\sigma}{\sqrt{(1-t)(1-Z^2) + t^2 Y^2 X^2}} = \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \frac{1}{\sqrt{1-t+t^2 \sin^2 \theta \sin^2 \phi \cos^2 \phi}} \\ &= \frac{1}{2} \int_0^{\pi/2} d\theta \int_0^\pi d\phi \frac{1}{\sqrt{1-t+\frac{t^2}{4} \sin^2 \theta \sin^2 \phi}} = \frac{1}{8} \int_{S^2} \frac{d\sigma}{\sqrt{(1-Z^2)\left(1-t+\frac{t^2}{4} Y^2\right)}}. \end{aligned}$$

Switching the Y - and Z -axes, we then arrive at

$$\begin{aligned} [\mathbf{K}(\sqrt{t})]^2 &= \frac{1}{8} \int_{S^2} \frac{d\sigma}{\sqrt{(1-Y^2)\left(1-t+\frac{t^2}{4} Z^2\right)}} = \int_0^{\pi/2} \sin \theta d\theta \int_0^{\pi/2} d\phi \frac{1}{\sqrt{1-\sin^2 \theta \sin^2 \phi} \sqrt{1-t+\frac{t^2}{4} \cos^2 \theta}} \\ &= \int_0^{\pi/2} \frac{\mathbf{K}(\sin \theta) \sin \theta d\theta}{\sqrt{1-t+\frac{t^2}{4} \cos^2 \theta}} = \int_0^\pi \frac{\mathbf{K}(\sin \theta) \sin \theta d\theta}{\sqrt{(2-t)^2 - t^2 \sin^2 \theta}}. \end{aligned}$$

Judging from the familiar integral formula

$$\int_0^{2\pi} \frac{d\phi}{2-t+t \sin \theta \cos \phi} = \frac{2\pi}{\sqrt{(2-t)^2 - t^2 \sin^2 \theta}},$$

it is evident that

$$[\mathbf{K}(\sqrt{t})]^2 = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi \frac{2\mathbf{K}(\sin \theta) \sin \theta d\theta}{2-t+t \sin \theta \cos \phi} = \frac{1}{4\pi} \int_{S^2} \frac{2\mathbf{K}(\sqrt{X^2 + Y^2}) d\sigma}{2-t+tX}.$$

Pairing up the values of the integrand at $\pm X$, we obtain

$$[\mathbf{K}(\sqrt{t})]^2 = \frac{1}{4\pi} \int_{S^2} \frac{2\mathbf{K}(\sqrt{X^2 + Y^2}) d\sigma}{2-t+tX} = \frac{2-t}{4\pi} \int_{S^2} \frac{2\mathbf{K}(\sqrt{X^2 + Y^2}) d\sigma}{(2-t)^2 - t^2 X^2},$$

as claimed in Eq. 40. Then, Landen's transformation

$$\mathbf{K}(\sqrt{t}) = \frac{1}{1+\sqrt{t}} \mathbf{K}\left(\frac{2\sqrt[4]{t}}{1+\sqrt{t}}\right), \quad 0 \leq t < 1 \quad (47)$$

brings us to Eq. 40_L.

On account of Eqs. 40 and 40_L, we may put down the relations

$$\begin{aligned} \left[\mathbf{K}\left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}}\right) \right]^2 &= (1+\sqrt{u}) \int_{S^2} \frac{\mathbf{K}(\sqrt{X^2 + Y^2}) d\sigma}{1-uX^2} \frac{d\sigma}{4\pi} = (1+\sqrt{u}) \int_0^1 \frac{k\mathbf{K}(k) dk}{\sqrt{1-k^2} \sqrt{1-k^2 u}}, \\ \left[\mathbf{K}\left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right) \right]^2 &= \frac{1+\sqrt{u}}{2} \int_{S^2} \frac{\mathbf{K}(\sqrt{X^2 + Y^2}) d\sigma}{1-(1-u)X^2} \frac{d\sigma}{4\pi} = \frac{1+\sqrt{u}}{2} \int_0^1 \frac{k\mathbf{K}(k) dk}{\sqrt{1-k^2} \sqrt{1-k^2(1-u)}}, \end{aligned}$$

which confirm, respectively, the right halves of Eqs. 41 and 42. The left halves of Eqs. 41 and 42 follow from the duality relation in Eq. 34.

One may wish to compare Eqs. 40 and 40_L (applicable to $0 < u < 1$) to the Hobson coupling formula in Eq. 13[†] (confined to $0 \leq u \leq 1/2$).

(b) We start with a spherical rotation

$$\begin{aligned} \int_{S^2} \mathbf{K}(\sqrt{X^2 + Y^2}) f(|X|) \frac{d\sigma}{4\pi} &= \int_{S^2} \mathbf{K}(\sqrt{Z^2 + Y^2}) f(|Z|) \frac{d\sigma}{4\pi} \\ &= \frac{2}{\pi} \int_0^1 dZ \int_0^{\pi/2} d\phi \mathbf{K}\left(\sqrt{Z^2 + (1-Z^2)\sin^2 \phi}\right) f(|Z|) = \frac{2}{\pi} \int_0^1 dk \int_0^{\pi/2} d\phi \mathbf{K}\left(\sqrt{k^2 \cos^2 \phi + \sin^2 \phi}\right) f(k) \\ &= \frac{2}{\pi} \int_0^1 f(k) \left[\int_0^{\sqrt{1-k^2}} \frac{\mathbf{K}(\sqrt{1-\kappa^2})}{\sqrt{1-k^2-\kappa^2}} d\kappa \right] dk = \frac{2}{\pi} \operatorname{Re} \int_0^1 f(k) \left[\int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2})}{\sqrt{1-k^2-\kappa^2}} d\kappa \right] dk, \end{aligned}$$

where we have made the substitution $\kappa = \sqrt{1-k^2} \cos \phi$ in the last line. Then, we may evaluate

$$\operatorname{Re} \left[\int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2})}{\sqrt{1-k^2-\kappa^2}} d\kappa \right] = -\operatorname{Im} \left[\int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2})}{\sqrt{k^2-1+\kappa^2}} d\kappa \right]$$

by analytic continuation. From Eq. 40, we have

$$\int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2})}{\sqrt{1-z+\frac{z^2}{4}\kappa^2}} d\kappa = \left[\int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-zt^2}} \right]^2, \quad 0 \leq z < 1.$$

After setting $z = 2/(1+k) \geq 1$ in the inverse modulus transformation formula (Eq. 38), we obtain

$$(1+k) \int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2})}{\sqrt{k^2-1+\kappa^2}} d\kappa = \frac{1+k}{2} \left[\mathbf{K} \left(\sqrt{\frac{1+k}{2}} \right) - i \mathbf{K} \left(\sqrt{\frac{1-k}{2}} \right) \right]^2, \quad 0 \leq k < 1. \quad (48)$$

Reading off the imaginary parts on both sides of the formula above, we arrive at the following equation for $\lambda = (1-k)/2 \in (0, 1/2]$:

$$\int_{1-2\lambda}^1 \frac{k \mathbf{K}(k) dk}{\sqrt{1-k^2} \sqrt{k^2-(1-2\lambda)^2}} = \mathbf{K}(\sqrt{\lambda}) \mathbf{K}(\sqrt{1-\lambda}), \quad (49)$$

as well as the identity claimed in Eq. 43.

Applying Eq. 43 to Eq. 40, we obtain the result stated in Eq. 40*:

$$\begin{aligned} [\mathbf{K}(\sqrt{t})]^2 &= \int_{S^2} \frac{2(2-t)\mathbf{K}(\sqrt{X^2+Y^2})}{(2-t)^2-t^2X^2} \frac{d\sigma}{4\pi} \\ &= \frac{2}{\pi} \left[\int_0^1 \frac{1}{2-t+kt} \mathbf{K} \left(\sqrt{\frac{1+k}{2}} \right) \mathbf{K} \left(\sqrt{\frac{1-k}{2}} \right) dk + \int_0^1 \frac{1}{2-t-k't} \mathbf{K} \left(\sqrt{\frac{1+k'}{2}} \right) \mathbf{K} \left(\sqrt{\frac{1-k'}{2}} \right) dk' \right] \\ &= \frac{2}{\pi} \left[\int_0^{1/\sqrt{2}} \frac{2\mathbf{K}(\sqrt{\mu})\mathbf{K}(\sqrt{1-\mu})}{2-t+(1-2\mu)t} d\mu + \int_{1/\sqrt{2}}^1 \frac{2\mathbf{K}(\sqrt{\mu'})\mathbf{K}(\sqrt{1-\mu'})}{2-t-(2\mu'-1)t} d\mu' \right] = \frac{2}{\pi} \int_0^1 \frac{\mathbf{K}(\sqrt{\mu})\mathbf{K}(\sqrt{1-\mu})}{1-\mu t} d\mu. \end{aligned}$$

(An equivalent form of Eq. 40* has appeared as Eq. 26 in [6], which was derived combinatorially using generalized hypergeometric series, instead of the geometric interpretation given here.) One may derive Eq. 40*_L from Eq. 40* and Landen's transformation

$$\mathbf{K}(\sqrt{1-t}) = \frac{2}{1+\sqrt{t}} \mathbf{K} \left(\frac{1-\sqrt{t}}{1+\sqrt{t}} \right).$$

Writing $t = (1-k)/2 \in (0, 1/2)$, we may recast Eq. 48 into Eq. 44. By analytically continuing Eq. 44 to negative valued $t < 0$ with the aid of the integral representation of the elliptic integral $\mathbf{K}$ for complex-valued modulus, we arrive at Eq. 44'. Here, the change in the sign from $-i\mathbf{K}(\sqrt{t})$ in Eq. 44 to $+i\mathbf{K}(\sqrt{t})$ in Eq. 44' is worthy of special attention. It is ready to appreciate that such a sign change ensures compatibility with the natural choices of the univalent branches of the square roots occurring in all the places.

The leftmost equality in Eq. 45 has been already proved in Eq. 49, and its counterpart in Eq. 45' follows from Eq. 35. To demonstrate the rightmost equality in Eq. 45, we perform the following computations for $0 < \lambda \leq 1/2$:

$$\int_{(1-\lambda)/(1+\lambda)}^1 \frac{\mathbf{K}(k) dk}{\sqrt{1-k^2} \sqrt{(1+\lambda)^2 k^2 - (1-\lambda)^2}} = \operatorname{Re} \int_0^{\pi/2} \frac{\mathbf{K}(\sin \theta) d\theta}{\sqrt{4\lambda - (1+\lambda)^2 \cos^2 \theta}} = \frac{1}{2\sqrt{\lambda}} \operatorname{Re} \left[\mathbf{K} \left(\frac{i(1-\sqrt{\lambda})}{2\sqrt[4]{\lambda}} \right) \mathbf{K} \left(\frac{(1+\sqrt{\lambda})}{2\sqrt[4]{\lambda}} \right) \right]$$

by using an analytic continuation of the leftmost term of Eq. 45' in the last step. According to the imaginary modulus transformation and Landen's transformation, we have

$$\mathbf{K} \left(\frac{i(1-\sqrt{\lambda})}{2\sqrt[4]{\lambda}} \right) = \frac{2\sqrt[4]{\lambda}}{1+\sqrt{\lambda}} \mathbf{K} \left(\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}} \right) = \sqrt[4]{\lambda} \mathbf{K}(\sqrt{1-\lambda});$$

whereas the inverse modulus transformation (Eq. 38) and Landen's transformation lead us to

$$\operatorname{Re} \left[\mathbf{K} \left(\frac{(1+\sqrt{\lambda})}{2\sqrt[4]{\lambda}} \right) \right] = \frac{2\sqrt[4]{\lambda}}{1+\sqrt{\lambda}} \mathbf{K} \left(\frac{2\sqrt[4]{\lambda}}{1+\sqrt{\lambda}} \right) = 2\sqrt[4]{\lambda} \mathbf{K}(\sqrt{\lambda}).$$

Thus, the rightmost equality in Eq. 45 is verified. The connection to its counterpart in Eq. 45' can be proved as follows:

$$\begin{aligned} \int_{\sqrt{u}}^1 \frac{\mathbf{K}(k) dk}{\sqrt{1-k^2}\sqrt{k^2-u}} &= \int_0^{\pi/2} \frac{1}{\sqrt{1-(1-u)\sin^2\theta}} \mathbf{K} \left(\sqrt{\frac{u}{1-(1-u)\sin^2\theta}} \right) d\theta \\ &= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-(1-u)\sin^2\theta-u\sin^2\phi}}. \end{aligned}$$

Here, we have used the variable substitution $k = \sqrt{u/[1-(1-u)\sin^2\theta]}$ to complete the first line in the equation above, before spelling out $\mathbf{K}(\sqrt{u/[1-(1-u)\sin^2\theta]})$ as an integral in ϕ in the second line.

The last equality in Eq. 45' implies the following evaluation for $0 < u < 1$:

$$\int_0^{\pi/2} d\theta \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-u\sin^2\theta-(1-u)\sin^2\phi}} = \frac{2}{1+\sqrt{u}} \mathbf{K} \left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}} \right) \mathbf{K} \left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}} \right), \quad (45'^*)$$

so an interchange of the angular variables θ and ϕ brings us back to the $u \leftrightarrow 1-u$ symmetry displayed in Eq. 14.

One may also wish to compare the formula

$$\frac{2}{1+\sqrt{u}} \mathbf{K} \left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}} \right) \mathbf{K} \left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}} \right) = \int_{\sqrt{u}}^1 \frac{\mathbf{K}(k) dk}{\sqrt{1-k^2}\sqrt{k^2-u}}, \quad 0 < u < 1 \quad (50)$$

to the corresponding result from Hobson coupling (Eq. 13[‡]). ■

Remark (1) Up to a variable substitution, our intermediate result in part (a)

$$[\mathbf{K}(\sqrt{t})]^2 = \int_0^{\pi/2} \frac{\mathbf{K}(\sin\theta)\sin\theta d\theta}{\sqrt{1-t+\frac{t^2}{4}\cos^2\theta}} = \int_0^1 \frac{\mathbf{K}(\sqrt{1-\kappa^2}) d\kappa}{\sqrt{1-t+\frac{t^2}{4}\kappa^2}}, \quad 0 \leq t < 1 \quad (40')$$

is related to formula 13.0.18 in [28], and item 2.16.5.7 of [29]. However, it is highly probable that the ultimate source of this formula is Ramanujan.

Entry 7(x) in Chapter 17 of Ramanujan's second notebook essentially reads

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{d\theta d\phi}{\sqrt{(1-u\sin^2\theta)(1-u\sin^2\theta\sin^2\phi)}} = \left(\int_0^{\pi/2} \frac{d\psi}{\sqrt{1-u\sin^4\psi}} \right)^2,$$

which was verified by B. C. Berndt with some highly technical transformations of the generalized hypergeometric series ${}_4F_3$ [Ref. 11, pp. 110-111]. In fact, the left-hand side of Ramanujan's formula equals

$$\int_0^{\pi/2} \frac{\mathbf{K}(\sqrt{u}\sin\theta) d\theta}{\sqrt{1-u\sin^2\theta}} = \int_0^{\pi/2} \frac{\mathbf{K}(\sin\theta)\sin\theta d\theta}{\sqrt{1-u\sin^2\theta}}$$

after integration in ϕ and reference to Eq. 34, while its right-hand side equals $\pi^2[P_{-1/4}(1-2u)]^2/4$, according to the integral identity in Eq. 6. In this manner, we see that Ramanujan's formula is actually equivalent to Eq. 40', a consequence of spherical rotations.

In [Ref. 11, pp. 111-112], Berndt followed up with a combinatorial proof of Entry 7(xi) in Ramanujan's notebook:

$$\begin{aligned} \int_0^{\pi/2} \int_0^{\pi/2} \frac{k \sin\theta d\theta d\phi}{\sqrt{(1-k^2\sin^2\theta)(1-k^2\sin^2\theta\sin^2\phi)}} &= \int_0^{\pi/2} \int_0^{\arcsin k} \frac{d\theta d\phi}{\sqrt{1-k^2\sin^2\theta-\sin^2\theta\sin^2\phi}} \\ &= \frac{1}{2} \left\{ \left[\mathbf{K} \left(\sqrt{\frac{1+k}{2}} \right) \right]^2 - \left[\mathbf{K} \left(\sqrt{\frac{1-k}{2}} \right) \right]^2 \right\}, \end{aligned}$$

which turns out to be exactly the real part of our Eq. 44, a geometrically motivated result.

Berndt has described his proofs of these two entries as “undoubtedly not those found by Ramanujan”. Judging from the use of spherical coordinates in Ramanujan’s presentation, we take leave to think that the technique of spherical rotations given in the proof of this proposition might be closer to a rediscovery of the pathway that Ramanujan has originally undertaken.

(2) Combining our analysis in part (b) with Eq. 46, we have effectively shown that spherical geometry entails the following integral formula

$$\int_0^{\pi/2} \mathbf{K}\left(\sqrt{k^2 \cos^2 \phi + \sin^2 \phi}\right) d\phi = \int_0^{\pi/2} \frac{\mathbf{K}(\sin \theta) d\theta}{\sqrt{1-k^2 \cos^2 \theta}} = \mathbf{K}\left(\sqrt{\frac{1+k}{2}}\right) \mathbf{K}\left(\sqrt{\frac{1-k}{2}}\right),$$

which is a relation that has nearly 80 years of history. In 2008, Bailey *et al.* used the method of Bessel moments to discover the following integral identity (Eq. 49 in [2])

$$\int_0^{\pi/2} \frac{\mathbf{K}(\sin \theta) d\theta}{\sqrt{1-\cos^2 \alpha \cos^2 \theta}} = \mathbf{K}\left(\sin \frac{\alpha}{2}\right) \mathbf{K}\left(\cos \frac{\alpha}{2}\right),$$

and remarked on its equivalence to a formula derived by Glasser in 1976:

$$\int_0^{\pi/2} \mathbf{K}\left(\sqrt{1-\sin^2 \alpha \cos^2 \phi}\right) d\phi = \mathbf{K}\left(\sin \frac{\alpha}{2}\right) \mathbf{K}\left(\cos \frac{\alpha}{2}\right).$$

As pointed out by Zucker [3], the left-hand side in the formula of Glasser, being a type of generalized Watson integral, can be expressed in terms of $\mathfrak{F}_4$, an Appell hypergeometric function, which in turn, reduces to the product of two complete elliptic integrals of the first kind, according to a result by Bailey in 1933 [30]. Needless to say, all these displayed formulae are equivalent to each other, and echo back to Eq. 12[‡], which was derived earlier from the Hobson coupling formula. $\square$

Corollary 3.4 (Some Cauchy Principal Values) *For $-1 < x < 1$, we have*

$$\mathcal{P} \int_{-1}^1 \mathbf{K}\left(\sqrt{\frac{1+\xi}{2}}\right) \mathbf{K}\left(\sqrt{\frac{1-\xi}{2}}\right) \frac{2d\xi}{\pi(x-\xi)} = \left[\mathbf{K}\left(\sqrt{\frac{1+x}{2}}\right) \right]^2 - \left[\mathbf{K}\left(\sqrt{\frac{1-x}{2}}\right) \right]^2 \quad (51)$$

and

$$\begin{aligned} & \mathcal{P} \int_{-1}^1 \left[\mathbf{K}\left(\sqrt{\frac{1+\xi}{2}}\right) \mathbf{K}\left(\sqrt{\frac{1-\xi}{2}}\right) + \mathbf{K}\left(\sqrt{\frac{1-\xi}{2}}\right) \mathbf{E}\left(\sqrt{\frac{1+\xi}{2}}\right) - \mathbf{K}\left(\sqrt{\frac{1+\xi}{2}}\right) \mathbf{E}\left(\sqrt{\frac{1-\xi}{2}}\right) \right] \frac{d\xi}{\pi(x-\xi)} \\ &= - \left[\mathbf{K}\left(\sqrt{\frac{1-x}{2}}\right) \right]^2 + \mathbf{K}\left(\sqrt{\frac{1-x}{2}}\right) \mathbf{E}\left(\sqrt{\frac{1-x}{2}}\right) + \mathbf{K}\left(\sqrt{\frac{1+x}{2}}\right) \mathbf{E}\left(\sqrt{\frac{1+x}{2}}\right). \end{aligned} \quad (52)$$

Proof From Eq. 40* and the inverse modulus transformation $\text{Re}\{\mathbf{K}(1/\sqrt{t})\}^2/t = [\mathbf{K}(\sqrt{t})]^2 - [\mathbf{K}(\sqrt{1-t})]^2$ for $t \in (0, 1)$ (cf. Eq. 38), we may deduce

$$[\mathbf{K}(\sqrt{t})]^2 - [\mathbf{K}(\sqrt{1-t})]^2 = \frac{2}{\pi} \mathcal{P} \int_0^1 \frac{\mathbf{K}(\sqrt{\mu}) \mathbf{K}(\sqrt{1-\mu}) d\mu}{t-\mu}, \quad 0 < t < 1,$$

which is equivalent to Eq. 51.

Now, we proceed as in the proof of Corollary 3.2(b), and compute

$$\begin{aligned} \mathcal{P} \int_{-1}^1 \mathbf{K}\left(\sqrt{\frac{1+\xi}{2}}\right) \mathbf{K}\left(\sqrt{\frac{1-\xi}{2}}\right) \frac{2(1+\xi)d\xi}{\pi(x-\xi)} &= (1+x) \left\{ \left[\mathbf{K}\left(\sqrt{\frac{1+x}{2}}\right) \right]^2 - \left[\mathbf{K}\left(\sqrt{\frac{1-x}{2}}\right) \right]^2 \right\} - \frac{2}{\pi} \int_{-1}^1 \mathbf{K}\left(\sqrt{\frac{1+\xi}{2}}\right) \mathbf{K}\left(\sqrt{\frac{1-\xi}{2}}\right) d\xi \\ &= (1+x) \left\{ \left[\mathbf{K}\left(\sqrt{\frac{1+x}{2}}\right) \right]^2 - \left[\mathbf{K}\left(\sqrt{\frac{1-x}{2}}\right) \right]^2 \right\} - \frac{\pi^2}{2}, \end{aligned} \quad (53)$$

where we have quoted the result $T_{0,-1/2} = \int_{-1}^1 P_{-1/2}(\xi) P_{-1/2}(-\xi) d\xi = \pi$ from a limit scenario of Eq. 19_(0,v). Likewise, we may deduce from the equation above another Cauchy principal value:

$$\mathcal{P} \int_{-1}^1 \mathbf{K} \left(\sqrt{\frac{1+\xi}{2}} \right) \mathbf{K} \left(\sqrt{\frac{1-\xi}{2}} \right) \frac{2(1-\xi^2) d\xi}{\pi(x-\xi)} = (1-x^2) \left\{ \left[\mathbf{K} \left(\sqrt{\frac{1+x}{2}} \right) \right]^2 - \left[\mathbf{K} \left(\sqrt{\frac{1-x}{2}} \right) \right]^2 \right\} + \frac{\pi^2 x}{2}.$$

Like what we did in the proof of Corollary 3.2(b), we can use the last equation to derive

$$\mathcal{P} \int_{-1}^1 \frac{d}{d\xi} \left[(1-\xi^2) \mathbf{K} \left(\sqrt{\frac{1+\xi}{2}} \right) \mathbf{K} \left(\sqrt{\frac{1-\xi}{2}} \right) \right] \frac{2 d\xi}{\pi(x-\xi)} = \frac{d}{dx} \left((1-x^2) \left\{ \left[\mathbf{K} \left(\sqrt{\frac{1+x}{2}} \right) \right]^2 - \left[\mathbf{K} \left(\sqrt{\frac{1-x}{2}} \right) \right]^2 \right\} + \frac{\pi^2 x}{2} \right). \quad (54)$$

after integration by parts. A little algebra then reveals that Eqs. 53 and 54 together entail Eq. 52. $\blacksquare$

4 Finite Hilbert Transform and Tricomi Pairing

4.1 Parseval Identity for Tricomi Transforms

The finite Hilbert transform on the interval $(-1, 1)$, also known as the Tricomi transform, is defined via a Cauchy principal value (see [Ref. 31, Chap. 4] or [Ref. 27, Chap. 11]):

$$(\widehat{\mathcal{T}}f)(x) := \mathcal{P} \int_{-1}^1 \frac{f(\xi) d\xi}{\pi(x-\xi)}, \quad \text{a.e. } x \in (-1, 1).$$

This induces a continuous linear operator $\widehat{\mathcal{T}} : L^p(-1, 1) \rightarrow L^p(-1, 1)$ for $1 < p < +\infty$ [Ref. 32, p. 188].

We can derive some new integral relations from a Parseval-type identity for the Tricomi transform (see [Ref. 31, §4.3, Eq. 2] or [Ref. 27, Eq. 11.237]):

$$\int_{-1}^1 f(x) (\widehat{\mathcal{T}}g)(x) dx + \int_{-1}^1 g(x) (\widehat{\mathcal{T}}f)(x) dx = 0, \quad (55)$$

where $f \in L^p(-1, 1)$, $p > 1$; $g \in L^q(-1, 1)$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} < 1$. We call such a procedure “Tricomi pairing”, for which an example in the proposition below puts finishing touches on a proof for the conjectural identity stated in the introduction.

Proposition 4.1 (An Application of Tricomi Pairing to Multiple Elliptic Integrals) *There are several integrals that evaluate to the same number $[\Gamma(\frac{1}{4})]^8/(128\pi^2)$:*

$$\begin{aligned} \int_0^1 \left[\mathbf{K}(\sqrt{1-k^2}) \right]^3 dk &= \frac{10}{3} \int_0^1 [\mathbf{K}(k)]^3 dk = 5 \int_0^1 [\mathbf{K}(k)]^3 k dk \\ &= 3 \int_0^1 [\mathbf{K}(k)]^2 \mathbf{K}(\sqrt{1-k^2}) dk = 2 \int_0^1 \mathbf{K}(k) \left[\mathbf{K}(\sqrt{1-k^2}) \right]^2 dk = 6 \int_0^1 [\mathbf{K}(k)]^2 \mathbf{K}(\sqrt{1-k^2}) k dk. \end{aligned} \quad (56)$$

Proof If we set $f(x) = \mathbf{K}(\sqrt{(1-x)/2})/\sqrt{1+x}$, $-1 < x < 1$ and $g(x) = 2\mathbf{K}(\sqrt{(1+x)/2})\mathbf{K}(\sqrt{(1-x)/2})$, $-1 < x < 1$ in Eq. 55, while recalling the Cauchy principal values given in Eqs. 36 and 51, then we arrive at

$$2 \int_{-1}^1 \left[\mathbf{K} \left(\sqrt{\frac{1+x}{2}} \right) \right]^2 \mathbf{K} \left(\sqrt{\frac{1-x}{2}} \right) \frac{dx}{\sqrt{1+x}} = - \int_{-1}^1 \mathbf{K} \left(\sqrt{\frac{1-\xi}{2}} \right) \left\{ \left[\mathbf{K} \left(\sqrt{\frac{1+\xi}{2}} \right) \right]^2 - \left[\mathbf{K} \left(\sqrt{\frac{1-\xi}{2}} \right) \right]^2 \right\} \frac{d\xi}{\sqrt{1+\xi}}.$$

This instantly rearranges into $\int_0^1 [\mathbf{K}(\sqrt{1-k^2})]^3 dk = 3 \int_0^1 [\mathbf{K}(k)]^2 \mathbf{K}(\sqrt{1-k^2}) dk$ (an identity that Wan conjectured numerically in [6] without an analytic proof), which reveals the equivalence between the leading items of the first two lines in Eq. 56.

Writing $k = (1-\xi)/(1+\xi)$ and employing Landen’s transformation $\mathbf{K}(2\sqrt{\xi/(1+\xi)}) = (1+\xi)\mathbf{K}(\xi)$, one has

$$\int_0^1 \left[\mathbf{K}(\sqrt{1-k^2}) \right]^3 dk = \int_0^1 \left[\mathbf{K} \left(\sqrt{1 - \left(\frac{1-\xi}{1+\xi} \right)^2} \right) \right]^3 d \frac{1-\xi}{1+\xi} = 2 \int_0^1 [\mathbf{K}(\xi)]^3 (1+\xi) d\xi;$$

writing $k = (1 - \xi)/(1 + \xi)$ and employing Landen's transformation $2\mathbf{K}((1 - \xi)/(1 + \xi)) = (1 + \xi)\mathbf{K}(\sqrt{1 - \xi^2})$, one has

$$\int_0^1 [\mathbf{K}(k)]^3 dk = \int_0^1 \left[\mathbf{K}\left(\frac{1 - \xi}{1 + \xi}\right) \right]^3 d\frac{1 - \xi}{1 + \xi} = \frac{1}{4} \int_0^1 \left[\mathbf{K}\left(\sqrt{1 - \xi^2}\right) \right]^3 (1 + \xi) d\xi = \frac{1}{4} \int_0^1 [\mathbf{K}(\sqrt{1 - k^2})]^3 dk + \frac{1}{4} \int_0^1 [\mathbf{K}(\eta)]^3 \eta d\eta,$$

where the last step is a trivial substitution $\xi \mapsto \sqrt{1 - \eta^2}$. The two simultaneous equations displayed above make it possible to eliminate any one among the three quantities $\int_0^1 [\mathbf{K}(\sqrt{1 - k^2})]^3 dk$, $\int_0^1 [\mathbf{K}(k)]^3 dk$, $\int_0^1 [\mathbf{K}(k)]^3 dk$, and determine the ratio between the two remaining numbers. Thus, we have verified the chain of identities in the first line of Eq. 56 (cf. [Ref. 6, Eq. 29]). The relations *within* the second line of Eq. 56 can be likewise established by successive Landen's transformations, as detailed in the first paragraph of [Ref. 6, p. 139].

As we recall from Eq. 25 that

$$6 \int_0^1 [\mathbf{K}(k)]^2 \mathbf{K}(\sqrt{1 - k^2}) k dk = \frac{[\Gamma(\frac{1}{4})]^8}{128\pi^2},$$

the verification is complete. ■

Remark Sometimes, the output of Tricomi pairing can also be immediately recovered by more straightforward means. For example, upon setting

$$\begin{aligned} f(\xi) &= 2\mathbf{K}\left(\sqrt{\frac{1 + \xi}{2}}\right) \mathbf{K}\left(\sqrt{\frac{1 - \xi}{2}}\right), \\ g(\xi) &= \mathbf{K}\left(\sqrt{\frac{1 + \xi}{2}}\right) \mathbf{K}\left(\sqrt{\frac{1 - \xi}{2}}\right) + \mathbf{K}\left(\sqrt{\frac{1 - \xi}{2}}\right) \mathbf{E}\left(\sqrt{\frac{1 + \xi}{2}}\right) - \mathbf{K}\left(\sqrt{\frac{1 + \xi}{2}}\right) \mathbf{E}\left(\sqrt{\frac{1 - \xi}{2}}\right) \end{aligned}$$

in Eq. 55, one arrives at a vanishing identity:

$$0 = \int_0^1 \left[\mathbf{K}(\sqrt{1 - k^2}) \right]^2 \left[\mathbf{K}(\sqrt{1 - k^2}) \mathbf{K}(k) + \mathbf{K}(\sqrt{1 - k^2}) \mathbf{E}(k) - 3\mathbf{K}(k) \mathbf{E}(\sqrt{1 - k^2}) \right] k dk.$$

As pointed out by an anonymous referee, the equation above is anticipated from the fact that the integrand is precisely the derivative of $k^2(1 - k^2)\mathbf{K}(k)[\mathbf{K}(\sqrt{1 - k^2})]^3$, which makes it less surprising than Eq. 56. It might be still interesting to ask if Eq. 56 can be likewise reduced into finite steps of algebraic manipulations on elliptic integrals and applications of the Newton-Leibniz formula (cf. §5). □

4.2 Tricomi Transform of $P_v(x)P_v(-x)$

The formula in Eq. 51 can be rewritten as

$$\mathcal{P} \int_{-1}^1 \frac{2P_{-1/2}(\xi)P_{-1/2}(-\xi)}{\pi(x - \xi)} d\xi = -\{[P_{-1/2}(x)]^2 - [P_{-1/2}(-x)]^2\}, \quad -1 < x < 1.$$

This is not accidental. In the next proposition, we will generalize such a Tricomi transform relation to Legendre functions of arbitrary degree v .

Proposition 4.2 (Tricomi Transform of $P_v(\xi)P_v(-\xi)$) *For any $v \in \mathbb{C} \setminus \mathbb{Z}$, we have*

$$\mathcal{P} \int_{-1}^1 \frac{2P_v(\xi)P_v(-\xi)}{\pi(x - \xi)} d\xi = \frac{[P_v(x)]^2 - [P_v(-x)]^2}{\sin(v\pi)}, \quad -1 < x < 1. \quad (57)$$

For $n \in \mathbb{Z}_{\geq 0}$, there is an identity

$$\mathcal{P} \int_{-1}^1 \frac{[P_n(\xi)]^2}{2(x - \xi)} d\xi = \mathcal{P} \int_{-1}^1 \frac{[P_{-n-1}(\xi)]^2}{2(x - \xi)} d\xi = P_n(x)Q_n(x), \quad -1 < x < 1. \quad (58)$$

Proof To verify Eq. 57, it would suffice to demonstrate that

$$\begin{aligned} 0 &= \int_{-1}^1 P_\ell(x) \left\{ \frac{[P_v(x)]^2 - [P_v(-x)]^2}{\sin(v\pi)} - \mathcal{P} \int_{-1}^1 \frac{2P_v(\xi)P_v(-\xi)}{\pi(x - \xi)} d\xi \right\} dx \\ &= \frac{1}{\sin(v\pi)} \int_{-1}^1 P_\ell(x) \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx + \frac{4}{\pi} \int_{-1}^1 Q_\ell(x) P_v(x) P_v(-x) dx, \quad \forall \ell \in \mathbb{Z}_{\geq 0}. \end{aligned} \quad (59)$$

Here, in the last line, we have used the Parseval identity of Tricomi pairing (Eq. 55), along with the Neumann integral representation for Legendre functions of the second kind (cf. [Ref. 27, Eq. 11.269] or [Ref. 33, Table 1.12A, Eq. 12A.26]):

$$Q_\ell(x) = \mathcal{P} \int_{-1}^1 \frac{P_\ell(\xi) d\xi}{2(x-\xi)}, \quad \forall x \in (-1, 1), \forall \ell \in \mathbb{Z}_{\geq 0}. \quad (60)$$

For a non-negative even number ℓ , both addends in the last line of Eq. 59 vanish because the integrands are odd functions. We may now settle Eq. 59 for odd numbers ℓ by induction. For $\ell = 1$, we use Eq. 16 to compute

$$\begin{aligned} & \frac{1}{\sin(v\pi)} \int_{-1}^1 P_1(x) \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx = \frac{1}{\sin(v\pi)} \int_{-1}^1 x \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx \\ &= -\frac{1}{4v(v+1)\sin(v\pi)} \lim_{x \rightarrow -1+0^+} (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{d\{ [P_v(x)]^2 - [P_v(-x)]^2 \}}{dx} \right] \\ &+ \frac{1}{4v(v+1)\sin(v\pi)} \lim_{x \rightarrow -1+0^+} (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{d\{ [P_v(x)]^2 - [P_v(-x)]^2 \}}{dx} \right] = \frac{4\sin(v\pi)}{v(v+1)\pi^2}. \end{aligned}$$

Meanwhile, we may check that

$$\begin{aligned} & \frac{4}{\pi} \int_{-1}^1 Q_1(x) P_v(x) P_v(-x) dx = \frac{4}{\pi} \int_{-1}^1 \left(-1 + \frac{x}{2} \log \frac{1+x}{1-x} \right) P_v(x) P_v(-x) dx \\ &= -\frac{4}{\pi} \int_{-1}^1 P_v(x) P_v(-x) dx - \frac{2}{\pi} \int_{-1}^1 \frac{\log \frac{1+x}{1-x}}{4v(v+1)} \frac{d}{dx} \left\{ (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{d(P_v(x)P_v(-x))}{dx} \right] + 4v(v+1)(1-x^2)P_v(x)P_v(-x) \right\} dx \\ &= \frac{4}{\pi} \int_{-1}^1 \frac{1}{4v(v+1)} \frac{d}{dx} \left[(1-x^2) \frac{d(P_v(x)P_v(-x))}{dx} \right] dx = -\frac{4\sin(v\pi)}{v(v+1)\pi^2}. \end{aligned}$$

Thus, Eq. 59 holds for $\ell = 1$. Noting that the Legendre functions of the first and second kinds satisfy similar recursion relations, namely,

$$\begin{aligned} (2\mu+1)(1-x^2) \frac{dP_\mu(x)}{dx} &= \mu(\mu+1)[P_{\mu-1}(x) - P_{\mu+1}(x)]; & (2\mu+1)xP_\mu(x) &= (\mu+1)P_{\mu+1}(x) + \mu P_{\mu-1}(x); \\ (2\mu+1)(1-x^2) \frac{dQ_\mu(x)}{dx} &= \mu(\mu+1)[Q_{\mu-1}(x) - Q_{\mu+1}(x)]; & (2\mu+1)xQ_\mu(x) &= (\mu+1)Q_{\mu+1}(x) + \mu Q_{\mu-1}(x), \end{aligned}$$

we can deduce

$$\begin{aligned} & \frac{v(v+1)(\ell+1)}{\sin(v\pi)} \int_{-1}^1 P_{\ell+1}(x) \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx + \frac{4v(v+1)(\ell+1)}{\pi} \int_{-1}^1 Q_{\ell+1}(x) P_v(x) P_v(-x) dx \\ &+ \frac{v(v+1)\ell}{\sin(v\pi)} \int_{-1}^1 P_{\ell-1}(x) \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx + \frac{4v(v+1)\ell}{\pi} \int_{-1}^1 Q_{\ell-1}(x) P_v(x) P_v(-x) dx \\ &= \frac{v(v+1)(2\ell+1)}{\sin(v\pi)} \int_{-1}^1 x P_\ell(x) \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx + \frac{4v(v+1)(2\ell+1)}{\pi} \int_{-1}^1 x Q_\ell(x) P_v(x) P_v(-x) dx, \end{aligned}$$

as well as

$$\begin{aligned} & \frac{4v(v+1)}{2\ell+1} \int_{-1}^1 [(\ell+1)P_{\ell+1}(x) + \ell P_{\ell-1}(x)] \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx = 4v(v+1) \int_{-1}^1 x P_\ell(x) \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx \\ &= -\int_{-1}^1 P_\ell(x) \frac{d}{dx} \left\{ (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{d\{ [P_v(x)]^2 - [P_v(-x)]^2 \}}{dx} \right] + 4v(v+1)(1-x^2)[P_v(x)]^2 - 4v(v+1)(1-x^2)[P_v(-x)]^2 \right\} dx \\ &= \int_{-1}^1 \left\{ (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{d\{ [P_v(x)]^2 - [P_v(-x)]^2 \}}{dx} \right] + 4v(v+1)(1-x^2)[P_v(x)]^2 - 4v(v+1)(1-x^2)[P_v(-x)]^2 \right\} \frac{dP_\ell(x)}{dx} dx \\ &+ \frac{8[1+(-1)^\ell]\sin^2(v\pi)}{\pi^2} \\ &= \ell(\ell+1) \int_{-1}^1 (1-x^2) P_\ell(x) \frac{d\{ [P_v(x)]^2 - [P_v(-x)]^2 \}}{dx} dx + \frac{4\ell(\ell+1)v(v+1)}{2\ell+1} \int_{-1}^1 [P_{\ell-1}(x) - P_{\ell+1}(x)] \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx \\ &+ \frac{8[1+(-1)^\ell]\sin^2(v\pi)}{\pi^2}, \end{aligned}$$

which leads to a recursion relation

$$(\ell+1)^2[(\ell+1)^2-(2\nu+1)^2] \int_{-1}^1 P_{\ell+1}(x)[P_\nu(x)]^2 - [P_\nu(-x)]^2 dx \\ - \ell^2[\ell^2-(2\nu+1)^2] \int_{-1}^1 P_{\ell-1}(x)[P_\nu(x)]^2 - [P_\nu(-x)]^2 dx = -\frac{8[1+(-1)^\ell](2\ell+1)\sin^2(\nu\pi)}{\pi^2},$$

and

$$\frac{4\nu(\nu+1)}{2\ell+1} \int_{-1}^1 [(\ell+1)Q_{\ell+1}(x) + \ell Q_{\ell-1}(x)]P_\nu(x)P_\nu(-x) dx = 4\nu(\nu+1) \int_{-1}^1 xQ_\ell(x)P_\nu(x)P_\nu(-x) dx \\ = - \int_{-1}^1 Q_\ell(x) \frac{d}{dx} \left\{ (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{d(P_\nu(x)P_\nu(-x))}{dx} \right] + 4\nu(\nu+1)(1-x^2)P_\nu(x)P_\nu(-x) \right\} dx \\ = \int_{-1}^1 \left\{ (1-x^2) \frac{d}{dx} \left[(1-x^2) \frac{d(P_\nu(x)P_\nu(-x))}{dx} \right] + 4\nu(\nu+1)(1-x^2)P_\nu(x)P_\nu(-x) \right\} \frac{dQ_\ell(x)}{dx} dx \\ = \ell(\ell+1) \int_{-1}^1 (1-x^2)Q_\ell(x) \frac{d(P_\nu(x)P_\nu(-x))}{dx} dx + \frac{4\ell(\ell+1)\nu(\nu+1)}{2\ell+1} \int_{-1}^1 [Q_{\ell-1}(x) - Q_{\ell+1}(x)]P_\nu(x)P_\nu(-x) dx \\ - \frac{2[1+(-1)^\ell]\sin(\nu\pi)}{\pi},$$

which brings us another recursion relation

$$(\ell+1)^2[(\ell+1)^2-(2\nu+1)^2] \int_{-1}^1 Q_{\ell+1}(x)P_\nu(x)P_\nu(-x) dx \\ - \ell^2[\ell^2-(2\nu+1)^2] \int_{-1}^1 Q_{\ell-1}(x)P_\nu(x)P_\nu(-x) dx = \frac{2[1+(-1)^\ell](2\ell+1)\sin(\nu\pi)}{\pi}.$$

Here, we have used the identities

$$\lim_{x \rightarrow -1+0^+} (1-x^2) \frac{d}{dx} \left[2(1-x^2)P_\nu(x) \frac{dP_\nu(x)}{dx} \right] = \frac{8\sin^2(\nu\pi)}{\pi^2}; \quad P_\ell(1) = (-1)^\ell P_\ell(-1) = 1, \forall \ell \in \mathbb{Z}_{\geq 0}$$

to take care of boundary contributions to the integral concerning $P_\ell(x)$, and resorted to the limit behavior

$$\lim_{x \rightarrow -1+0^+} (1-x^2)^2 \frac{dQ_\mu(x)}{dx} \frac{d(P_\nu(x)P_\nu(-x))}{dx} = -\frac{2\sin(\nu\pi)}{\pi} \\ \lim_{x \rightarrow -1+0^+} (1-x^2)^2 \frac{dQ_\mu(x)}{dx} \frac{d(P_\nu(x)P_\nu(-x))}{dx} = \frac{2\cos(\mu\pi)\sin(\nu\pi)}{\pi}$$

for the integration by parts involving $Q_\ell(x)$. Therefore, the last line of Eq. 59 satisfies a homogeneous recursion

$$(\ell+1)^2[(\ell+1)^2-(2\nu+1)^2] \left\{ \frac{1}{\sin(\nu\pi)} \int_{-1}^1 P_{\ell+1}(x)[P_\nu(x)]^2 - [P_\nu(-x)]^2 dx + \frac{4}{\pi} \int_{-1}^1 Q_{\ell+1}(x)P_\nu(x)P_\nu(-x) dx \right\} \\ = \ell^2[\ell^2-(2\nu+1)^2] \left\{ \frac{1}{\sin(\nu\pi)} \int_{-1}^1 P_{\ell-1}(x)[P_\nu(x)]^2 - [P_\nu(-x)]^2 dx + \frac{4}{\pi} \int_{-1}^1 Q_{\ell-1}(x)P_\nu(x)P_\nu(-x) dx \right\}, \quad \ell \in \mathbb{Z}_{\geq 0}$$

and the truthfulness of Eq. 59 for $\ell=1$ entails any scenario with a larger odd number ℓ . This completes the verification of Eq. 57 for $\nu \in \mathbb{C} \setminus \mathbb{Z}$.

For a fixed $x \in (-1, 1)$, both sides of Eq. 57 represent continuous functions of ν , so we may handle Eq. 58 by investigating the $\nu \rightarrow n$ limit where $n \in \mathbb{Z}_{\geq 0}$. Suppose that n is even and non-negative, then we have

$$\lim_{\nu \rightarrow n} \frac{[P_\nu(x)]^2 - [P_\nu(-x)]^2}{\sin(\nu\pi)} = 2P_n(x) \lim_{\nu \rightarrow n} \frac{P_\nu(x) - P_\nu(-x)}{\sin(\nu\pi)} = 2P_n(x) \lim_{\nu \rightarrow n} \frac{\cos(\nu\pi)P_\nu(x) - P_\nu(-x)}{\sin(\nu\pi)} \\ = \frac{4}{\pi} P_n(x) \lim_{\nu \rightarrow n} Q_\nu(x) = \frac{4}{\pi} P_n(x) Q_n(x) = \frac{4}{\pi} P_n(-x) Q_n(x).$$

If we start from an odd and positive n instead, we will end up with

$$\lim_{\nu \rightarrow n} \frac{[P_\nu(x)]^2 - [P_\nu(-x)]^2}{\sin(\nu\pi)} = 2P_n(x) \lim_{\nu \rightarrow n} \frac{P_\nu(x) + P_\nu(-x)}{\sin(\nu\pi)} = -2P_n(x) \lim_{\nu \rightarrow n} \frac{\cos(\nu\pi)P_\nu(x) - P_\nu(-x)}{\sin(\nu\pi)} \\ = -\frac{4}{\pi} P_n(x) \lim_{\nu \rightarrow n} Q_\nu(x) = -\frac{4}{\pi} P_n(x) Q_n(x) = \frac{4}{\pi} P_n(-x) Q_n(x).$$

Hence, we have proved

$$\mathcal{D} \int_{-1}^1 \frac{2P_n(\xi)P_n(-\xi)}{\pi(x-\xi)} d\xi = \mathcal{D} \int_{-1}^1 \frac{2P_{-n-1}(\xi)P_{-n-1}(-\xi)}{\pi(x-\xi)} d\xi = (-1)^n \frac{4}{\pi} P_n(x)Q_n(x) = \frac{4}{\pi} P_n(-x)Q_n(x), \quad n \in \mathbb{Z}_{\geq 0},$$

which is equivalent to Eq. 58. In fact, Eq. 58 is not particularly surprising. It is just a special case of the stronger statement that (cf. [Ref. 27, Eqs. 11.280 and 11.281] or [Ref. 33, Table 1.12A, Eq. 12A.27])

$$\mathcal{D} \int_{-1}^1 \frac{P_n(\xi)p(\xi)}{2(x-\xi)} d\xi = Q_n(x)p(x), \quad \deg p(x) \leq n, \quad (61)$$

where $p(x)$ is any polynomial whose degree does not exceed n . ■

We follow up with some additional examples involving the product of four elliptic integrals in the integrands.

Corollary 4.3 (Another Application of Tricomi Pairing) *We have an integral identity*

$$\int_{-1}^1 x[P_\nu(x)]^3 P_\nu(-x) dx = \frac{\sin(2\nu\pi)\cos(\nu\pi)}{(2\nu+1)^2\pi}, \quad \nu \in \mathbb{C} \setminus \{-1/2\}, \quad (62)$$

which leads to

$$\begin{aligned} -\frac{\pi}{2} &= \int_{-1}^1 x[P_{-1/2}(x)]^3 P_{-1/2}(-x) dx \\ &= \frac{32}{\pi^4} \int_0^1 (1-2t)[\mathbf{K}(\sqrt{t})]^3 \mathbf{K}(\sqrt{1-t}) dt = -\frac{32}{\pi^4} \int_0^1 (1-2t)[\mathbf{K}(\sqrt{1-t})]^3 \mathbf{K}(\sqrt{t}) dt, \end{aligned} \quad (63)$$

$$\begin{aligned} -\frac{9\sqrt{3}}{4\pi} &= \int_{-1}^1 x[P_{-1/3}(x)]^3 P_{-1/3}(-x) dx \\ &= \frac{216}{\sqrt{3}\pi^4} \int_0^1 \frac{(1-p^2)p(2+p)}{1+2p} \left[1 - 2 \frac{27p^2(1+p)^2}{4(1+p+p^2)^3} \right] \left[\mathbf{K}\left(\sqrt{\frac{p^3(2+p)}{1+2p}}\right) \right]^3 \mathbf{K}\left(\sqrt{1 - \frac{p^3(2+p)}{1+2p}}\right) dp \\ &= -\frac{72}{\sqrt{3}\pi^4} \int_0^1 \frac{(1-p^2)p(2+p)}{1+2p} \left[1 - 2 \frac{27p^2(1+p)^2}{4(1+p+p^2)^3} \right] \left[\mathbf{K}\left(\sqrt{1 - \frac{p^3(2+p)}{1+2p}}\right) \right]^3 \mathbf{K}\left(\sqrt{\frac{p^3(2+p)}{1+2p}}\right) dp, \end{aligned} \quad (64)$$

$$\begin{aligned} -\frac{2\sqrt{2}}{\pi} &= \int_{-1}^1 x[P_{-1/4}(x)]^3 P_{-1/4}(-x) dx \\ &= \frac{32\sqrt{2}}{\pi^4} \int_0^1 \frac{1-2u}{(1+\sqrt{u})^2} \left[\mathbf{K}\left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}}\right) \right]^3 \mathbf{K}\left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right) du \\ &= -\frac{64\sqrt{2}}{\pi^4} \int_0^1 \frac{1-2u}{(1+\sqrt{u})^2} \left[\mathbf{K}\left(\sqrt{\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right) \right]^3 \mathbf{K}\left(\sqrt{\frac{2\sqrt{u}}{1+\sqrt{u}}}\right) du, \end{aligned} \quad (65)$$

$$\begin{aligned} -\frac{27}{16\pi} &= \int_{-1}^1 x[P_{-1/6}(x)]^3 P_{-1/6}(-x) dx \\ &= \frac{54}{\pi^4} \int_0^1 \frac{t(1-t)(1+t)(2-t)(1-2t)}{(1-t+t^2)^3} [\mathbf{K}(\sqrt{t})]^3 \mathbf{K}(\sqrt{1-t}) dt \\ &= -\frac{54}{\pi^4} \int_0^1 \frac{t(1-t)(1+t)(2-t)(1-2t)}{(1-t+t^2)^3} [\mathbf{K}(\sqrt{1-t})]^3 \mathbf{K}(\sqrt{t}) dt. \end{aligned} \quad (66)$$

Proof For integer degrees $\nu \in \mathbb{Z}$, the left-hand side of Eq. 62 represents the integration of an odd function over the interval $[-1, 1]$, hence vanishing. This is consistent with the corresponding behavior on the right-hand side.

We now turn our attention to the scenarios where $\nu \in \mathbb{C} \setminus (\mathbb{Z} \cup \{-1/2\})$. From the Tricomi transform formula in Eq. 57, one may readily deduce the following relation

$$\mathcal{D} \int_{-1}^1 \frac{2\xi P_\nu(\xi)P_\nu(-\xi)}{\pi(x-\xi)} d\xi = \mathcal{D} \int_{-1}^1 \frac{2[x-(x-\xi)]P_\nu(\xi)P_\nu(-\xi)}{\pi(x-\xi)} d\xi = x \frac{[P_\nu(x)]^2 - [P_\nu(-x)]^2}{\sin(\nu\pi)} - \frac{2}{\pi} \int_{-1}^1 P_\nu(\xi)P_\nu(-\xi) d\xi.$$

Here, according to Eq. 19_(0,\nu), we have

$$T_{0,\nu} = \int_{-1}^1 P_\nu(\xi)P_\nu(-\xi) d\xi = \frac{2\cos(\nu\pi)}{2\nu+1},$$

so we may combine the formula

$$\mathcal{P} \int_{-1}^1 \frac{2\xi P_v(\xi)P_v(-\xi)}{\pi(x-\xi)} d\xi = x \frac{[P_v(x)]^2 - [P_v(-x)]^2}{\sin(v\pi)} - \frac{4\cos(v\pi)}{(2v+1)\pi}$$

with Eq. 57 for the implementation of the following Tricomi pairing:

$$\begin{aligned} \int_{-1}^1 x P_v(x) P_v(-x) \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx &= \int_{-1}^1 x P_v(x) P_v(-x) \left[\mathcal{P} \int_{-1}^1 \frac{2P_v(\xi)P_v(-\xi)\sin(v\pi)}{\pi(x-\xi)} d\xi \right] dx \\ &= - \int_{-1}^1 \left[\mathcal{P} \int_{-1}^1 \frac{2\xi P_v(\xi)P_v(-\xi)\sin(v\pi)}{\pi(x-\xi)} d\xi \right] P_v(x) P_v(-x) dx \\ &= - \int_{-1}^1 x P_v(x) P_v(-x) \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx + \frac{2\sin(2v\pi)}{(2v+1)\pi} \int_{-1}^1 P_v(x) P_v(-x) dx. \end{aligned}$$

After rearrangement, we obtain

$$4 \int_{-1}^1 x [P_v(x)]^3 P_v(-x) dx = 2 \int_{-1}^1 x P_v(x) P_v(-x) \{ [P_v(x)]^2 - [P_v(-x)]^2 \} dx = \frac{2\sin(2v\pi)}{(2v+1)\pi} \int_{-1}^1 P_v(x) P_v(-x) dx,$$

which entails the claimed identity in Eq. 62.

The left-hand side of Eq. 62 extends to be a continuous function in $v \in \mathbb{C}$. Taking the $v \rightarrow -1/2$ limit, one arrives at Eq. 63. The special cases $v = -1/3, -1/4, -1/6$ correspond to Eqs. 64, 65 and 66. $\blacksquare$

A key step in the proof of Proposition 4.1 hinges on the identity

$$\int_0^1 [\mathbf{K}(\sqrt{1-k^2})]^3 dk = 3 \int_0^1 [\mathbf{K}(k)]^2 \mathbf{K}(\sqrt{1-k^2}) dk, \quad i.e. \quad \int_{-1}^1 \frac{[P_{-1/2}(x)]^3}{\sqrt{1+x}} dx = 3 \int_{-1}^1 \frac{P_{-1/2}(x)[P_{-1/2}(-x)]^2}{\sqrt{1+x}} dx.$$

This result is actually just a special case within a family of identities satisfied by multiple elliptic integrals involving the product of three complete elliptic integrals (of the first and second kinds). To flesh out, for any integer $n \in \mathbb{Z}$, we have

$$\mathcal{P} \int_{-1}^1 \frac{P_{(2n+1)/2}(\xi)}{\sqrt{1+\xi}} \frac{d\xi}{2(x-\xi)} = \frac{Q_{(2n+1)/2}(x)}{\sqrt{1+x}} \equiv (-1)^{n+1} \frac{\pi}{2} \frac{P_{(2n+1)/2}(-x)}{\sqrt{1+x}}, \quad \forall x \in (-1, 1), \quad (67)$$

which generalizes Corollary 3.2(b), and Eq. 67 entails

$$\int_{-1}^1 \frac{[P_{(2n+1)/2}(x)]^3}{\sqrt{1+x}} dx = 3 \int_{-1}^1 \frac{P_{(2n+1)/2}(x)[P_{(2n+1)/2}(-x)]^2}{\sqrt{1+x}} dx \quad (68)$$

after Tricomi pairing with Eq. 57. In particular, for $n = 0$, Eq. 68 specializes to

$$\int_0^1 [2\mathbf{E}(\sqrt{1-k^2}) - \mathbf{K}(\sqrt{1-k^2})]^3 dk = 3 \int_0^1 [2\mathbf{E}(k) - \mathbf{K}(k)]^2 [2\mathbf{E}(\sqrt{1-k^2}) - \mathbf{K}(\sqrt{1-k^2})] dk.$$

The identity stated in Eq. 67 is the Tricomi transform of $P_{(2n+1)/2}(x)/\sqrt{1+x} \equiv P_{-(2n+3)/2}(x)/\sqrt{1+x}, n \in \mathbb{Z}$, which extends the Neumann integral representation of Legendre functions Q_ℓ for $\ell \in \mathbb{Z}_{\geq 0}$ (Eq. 60). In the next proposition, we shall prove Eq. 67 in an even broader context, which in turn, also gives rise to an independent verification of Eq. 57 in Proposition 4.2.

Proposition 4.4 (Generalized Neumann Integrals) For $n \in \mathbb{Z}_{\geq 0}$ and $\text{Re}(v-n) > -1$, one has

$$\mathcal{P} \int_{-1}^1 (1+\xi)^{v-n} P_v(\xi) \frac{d\xi}{2(x-\xi)} = (1+x)^{v-n} Q_v(x), \quad -1 < x < 1. \quad (69)$$

Proof We note that there is a standard moment formula for Legendre functions [Ref. 13, item 7.127]:

$$\int_{-1}^1 (1+x)^\sigma P_v(x) dx = \frac{2^{\sigma+1} [\Gamma(\sigma+1)]^2}{\Gamma(\sigma+v+2) \Gamma(1+\sigma-v)}, \quad \text{Re } \sigma > -1. \quad (70)$$

One can verify Eq. 70 by termwise integration over the Taylor series expansion for $P_v(x) = {}_2F_1\left(-v, v+1 \mid \frac{1-x}{2}\right)$ with respect to $(1-x)/2$ and a reduction of the generalized hypergeometric series ${}_3F_2$.

We now consider the Mellin inversion of Eq. 70:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{[\Gamma(s)]^2 2^s}{\Gamma(s+\nu+1)\Gamma(s-\nu)} \frac{ds}{(1+\xi)^s} = \begin{cases} P_\nu(\xi), & -1 < \xi < 1 \\ 0, & \xi > 1 \end{cases} \quad (71)$$

where $c > 0$ and the integration $\int_{c-i\infty}^{c+i\infty} := \lim_{T \rightarrow +\infty} \int_{c-iT}^{c+iT}$ is carried out along a vertical line $\text{Re } s = c$.

To facilitate analysis, we momentarily assume that $\text{Re}(\nu - n) > -\frac{1}{2}$ and pick $c = \text{Re}(\nu - n) + \frac{1}{2}$, so that $c > 0$ and $\text{Re}(\nu - n - c) = -\frac{1}{2}$. For $-1 < x < 1$, one may enlist Eq. 71 to compute

$$\begin{aligned} \mathcal{P} \int_{-1}^1 \frac{(1+\xi)^{\nu-n} P_\nu(\xi) d\xi}{2(x-\xi)} &= \frac{1}{2\pi i} \int_{\text{Re}(\nu-n)+\frac{1}{2}-i\infty}^{\text{Re}(\nu-n)+\frac{1}{2}+i\infty} \frac{[\Gamma(s)]^2 2^s}{\Gamma(s+\nu+1)\Gamma(s-\nu)} \left[\lim_{\varepsilon \rightarrow 0^+} \int_{-1}^{\infty} \left(\frac{1}{x-\xi+i\varepsilon} + \frac{1}{x-\xi-i\varepsilon} \right) \frac{(1+\xi)^{\nu-n-s} d\xi}{4} \right] ds \\ &= \frac{(1+x)^{\nu-n}}{4i} \int_{\text{Re}(\nu-n)+\frac{1}{2}-i\infty}^{\text{Re}(\nu-n)+\frac{1}{2}+i\infty} \frac{[\Gamma(s)]^2 2^s \cot(\nu\pi - s\pi)}{\Gamma(s+\nu+1)\Gamma(s-\nu)} \frac{ds}{(1+x)^s}, \end{aligned} \quad (72)$$

where the integration over $\xi \in (-1, \infty)$ is an elementary exercise in complex analysis. From Eq. 72, it is straightforward to verify the Legendre differential equation:

$$\begin{aligned} &\frac{d}{dx} \left[(1-x^2) \frac{d}{dx} \mathcal{P} \int_{-1}^1 \frac{(1+\xi)^{\nu-n} P_\nu(\xi) d\xi}{2(1+x)^{\nu-n}(x-\xi)} \right] + \nu(\nu+1) \mathcal{P} \int_{-1}^1 \frac{(1+\xi)^{\nu-n} P_\nu(\xi) d\xi}{2(1+x)^{\nu-n}(x-\xi)} \\ &= \frac{1}{4i} \left\{ \int_{\text{Re}(\nu-n)+\frac{1}{2}-i\infty}^{\text{Re}(\nu-n)+\frac{1}{2}+i\infty} \frac{[\Gamma(s)]^2 2^s \cot(\nu\pi - s\pi)}{\Gamma(s+\nu+1)\Gamma(s-\nu)} \frac{2s^2 ds}{(1+x)^{s+1}} - \int_{\text{Re}(\nu-n)+\frac{1}{2}-i\infty}^{\text{Re}(\nu-n)+\frac{1}{2}+i\infty} \frac{[\Gamma(s)]^2 2^s \cot(\nu\pi - s\pi)}{\Gamma(s+\nu+1)\Gamma(s-\nu)} \frac{(s+\nu)(s-\nu-1) ds}{(1+x)^s} \right\} \\ &= \frac{1}{4i} \left(\int_{\text{Re}(\nu-n)+\frac{1}{2}-i\infty}^{\text{Re}(\nu-n)+\frac{1}{2}+i\infty} - \int_{\text{Re}(\nu-n)-\frac{1}{2}-i\infty}^{\text{Re}(\nu-n)-\frac{1}{2}+i\infty} \right) \frac{[\Gamma(s)]^2 2^s \cot(\nu\pi - s\pi)}{\Gamma(s+\nu+1)\Gamma(s-\nu)} \frac{2s^2 ds}{(1+x)^{s+1}} = 0 \end{aligned}$$

by noting that $s = 0$ and $s = \nu - n$ are both removable singularities for the integrand in the last line. Thus, we are sure that the left-hand side of Eq. 69 can be written as $(1+x)^{\nu-n} [a_{\nu,n} P_\nu(x) + b_{\nu,n} Q_\nu(x)]$ for some coefficients $a_{\nu,n}$ and $b_{\nu,n}$, provided that $\text{Re}(\nu - n) > -\frac{1}{2}$.

Instead of directly coping with $a_{\nu,n}$ and $b_{\nu,n}$ for generic ν and n , we impose a temporary constraint that $n = 0, -\frac{1}{2} < \nu < 0$. Again, by Eq. 72, we can compute two moment integrals

$$\begin{aligned} \int_{-1}^1 [a_{\nu,0} P_\nu(x) + b_{\nu,0} Q_\nu(x)] dx &= \frac{1}{2\pi i} \int_{\nu+\frac{1}{2}-i\infty}^{\nu+\frac{1}{2}+i\infty} \frac{[\Gamma(s)]^2 \pi \cot(\nu\pi - s\pi)}{\Gamma(s+\nu+1)\Gamma(s-\nu)} \frac{ds}{1-s} \\ &= \frac{\cos(\nu\pi)}{\nu(\nu+1)} + \sum_{m=1}^{\infty} \frac{[\Gamma(m+\nu)]^2}{(1-m-\nu)\Gamma(m)\Gamma(m+2\nu+1)} = -\frac{2}{\nu(\nu+1)} \sin^2 \frac{\nu\pi}{2}; \\ \int_{-1}^1 [a_{\nu,0} P_\nu(x) + b_{\nu,0} Q_\nu(x)] (1+x) dx &= \frac{1}{2\pi i} \int_{\nu+\frac{1}{2}-i\infty}^{\nu+\frac{1}{2}+i\infty} \frac{[\Gamma(s)]^2 \pi \cot(\nu\pi - s\pi)}{\Gamma(s+\nu+1)\Gamma(s-\nu)} \frac{2ds}{2-s} \\ &= -\frac{2\cos(\nu\pi)}{(\nu-1)\nu(\nu+1)(\nu+2)} + \sum_{m=1}^{\infty} \frac{2[\Gamma(m+\nu)]^2}{(2-m-\nu)\Gamma(m)\Gamma(m+2\nu+1)} = -\frac{2[\nu^2 + \nu + \cos(\pi\nu) - 1]}{(\nu-1)\nu(\nu+1)(\nu+2)} \end{aligned}$$

by closing the contours to the right. These two moment integrals reveal that $a_{\nu,0} = 0, b_{\nu,0} = 1$ for $-\frac{1}{2} < \nu < 0$. For any fixed $x \in (-1, 1)$, the integral

$$\int_{\text{Re } \nu + \frac{1}{2} - i\infty}^{\text{Re } \nu + \frac{1}{2} + i\infty} \frac{[\Gamma(s)]^2 2^s \cot(\nu\pi - s\pi)}{\Gamma(s+\nu+1)\Gamma(s-\nu)} \frac{ds}{4i(1+x)^s} = \frac{1}{(1+x)^\nu} \mathcal{P} \int_{-1}^1 \frac{(1+\xi)^\nu P_\nu(\xi) d\xi}{2(x-\xi)}$$

is analytic for $\text{Re } \nu > -\frac{1}{2}$ and is equal to $Q_\nu(x)$ for $-\frac{1}{2} < \nu < 0$, so it can be identified with $Q_\nu(x)$ for $\text{Re } \nu > -\frac{1}{2}$. Moreover, so long as $\text{Re}(\nu - n) > -\frac{1}{2}$ for some $n \in \mathbb{Z}_{\geq 0}$, we have

$$\left(\int_{\text{Re}(\nu-n)+\frac{1}{2}-i\infty}^{\text{Re}(\nu-n)+\frac{1}{2}+i\infty} - \int_{\text{Re } \nu + \frac{1}{2} - i\infty}^{\text{Re } \nu + \frac{1}{2} + i\infty} \right) \frac{[\Gamma(s)]^2 2^s \cot(\nu\pi - s\pi)}{\Gamma(s+\nu+1)\Gamma(s-\nu)} \frac{ds}{(1+x)^s} = 0,$$

as the ratio $\cot(\nu\pi - s\pi)/\Gamma(s-\nu)$ remains bounded at all the removable singularities enclosed in the contour. This allows us to simplify Eq. 72 into Eq. 69 for $\text{Re}(\nu - n) > -\frac{1}{2}, n \in \mathbb{Z}_{\geq 0}$. By analytic continuation in ν , one can extend the applicability to $\text{Re}(\nu - n) > -1, n \in \mathbb{Z}_{\geq 0}$. ■

Remark One may well recognize that Eq. 69 incorporates the classical result in Eq. 61 as a special case:

$$\mathcal{P} \int_{-1}^1 \frac{P_n(\xi)p(\xi)}{2(x-\xi)} d\xi = Q_n(x)p(x), \quad \deg p(x) \leq n \in \mathbb{Z}_{\geq 0}.$$

To deduce Eq. 57 from Eq. 69, we need the Hardy-Poincaré-Bertrand formula (see [Ref. 31, §4.3, Eq. 4] or [Ref. 27, Eq. 11.52]):

$$\widehat{\mathcal{T}}[f(\widehat{\mathcal{T}}g) + g(\widehat{\mathcal{T}}f)] = (\widehat{\mathcal{T}}f)(\widehat{\mathcal{T}}g) - fg, \quad (73)$$

which applies to the scenarios where $f \in L^p(-1, 1)$, $p > 1$; $g \in L^q(-1, 1)$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} < 1$. By Eqs. 69 and 70, we have the following identities valid for $-1 < \nu < 0$:

$$(1+x)^{-\nu-1}Q_{-\nu-1}(x) = \frac{\pi}{2} \mathcal{P} \int_{-1}^1 (1+\xi)^{-\nu-1}P_{-\nu-1}(\xi) \frac{d\xi}{\pi(x-\xi)}, \quad (74)$$

$$(1+x)^{\nu+1}Q_{\nu}(x) - \frac{2^{\nu}[\Gamma(\nu+1)]^2}{\Gamma(2\nu+2)} = \frac{\pi}{2} \mathcal{P} \int_{-1}^1 (1+\xi)^{\nu+1}P_{\nu}(\xi) \frac{d\xi}{\pi(x-\xi)}. \quad (75)$$

Taking the last pair of equations (Eqs. 74 and 75) as inputs for the Hardy-Poincaré-Bertrand formula (Eq. 73), we obtain the following identity for $-1 < \nu < 0$ and $-1 < x < 1$:

$$\frac{\pi}{2} \mathcal{P} \int_{-1}^1 P_{\nu}(\xi)[Q_{\nu}(\xi) + Q_{-\nu-1}(\xi)] \frac{d\xi}{\pi(x-\xi)} = Q_{\nu}(x)Q_{-\nu-1}(x).$$

According to the relation between Q_{ν} and P_{ν} (Eq. 3), the last equation reduces to the Tricomi transform of $P_{\nu}(\xi)P_{\nu}(-\xi)$ (Eq. 57) upon analytic continuation in ν . $\square$

5 Discussion and Outlook

In our evaluations of some generalized Clebsch-Gordan integrals (Eqs. 25-28), the expressions involving Euler's gamma function actually correspond to certain special values of complete elliptic integrals as well. For example, the knowledge of $\mathbf{K}(1/\sqrt{2}) = [\Gamma(\frac{1}{4})]^2/(4\sqrt{\pi})$ [Ref. 13, item 8.129.1] allows us to recast the formula $[\Gamma(\frac{1}{4})]^8/(128\pi^2) = \int_0^1 [\mathbf{K}(\sqrt{1-k^2})]^3 dk$ into the following form:

$$2 \left[\int_0^1 \frac{ds}{\sqrt{1-s^2}\sqrt{1-(s^2/2)}} \right]^4 = \int_0^1 \left[\int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-(1-k^2)t^2}} \right]^3 dk. \quad (76)$$

Here, both sides are integrations of algebraic functions over algebraic domains, which qualify them as members in the “ring of periods” defined by Kontsevich and Zagier [1]. It is generally believed that identities for periods can be proved by “algebraic means” [1], namely, relying on nothing else than additivity of the integral, algebraic change of variables, and the Newton-Leibniz-Stokes formula. However, the analytic proof we produced for the generalized Clebsch-Gordan integrals does not fall into such a category: we have invoked Bessel functions, which are “exponential periods” [1]. We would like to see purely algebraic evaluations of the generalized Clebsch-Gordan integrals when the choice of degree ν leads to special values of complete elliptic integrals. In particular, it could be interesting to search for potential connections to high degree modular equations and the Chowla-Selberg theory.

After communicating the first version of this manuscript to Prof. Jonathan M. Borwein in January 2013, I was sent a preview of a forthcoming book [34] that contains a sketched proof for Eq. 76 using lattice sums and modular forms, totally independent of the methods presented in my work. Later this January, James G. Wan also wrote me about his plan to give a fuller account for the proofs of his own conjectures in a joint work with Rogers and Zucker, currently available as [35]. Some new integrals in [35] have inspired me to compose a short sequel [36] to the current work, in which there are further applications of the Hardy-Poincaré-Bertrand formula (Eq. 73) and the Tricomi transform of $(1+\xi)^{\nu-n}P_{\nu}(\xi)$, $n \in \mathbb{Z}_{\geq 0}$, $\text{Re}(\nu-n) > -1$ (Eq. 69), as well as an extension of the Hansen-Heine scaling analysis in Proposition 2.1 to the computations of other multiple elliptic integrals.

The spherical methods in this work (Legendre functions and spherical rotations) enable us to handle a large variety of multiple elliptic integrals, but they are by no means a cure-all. The methods of Bessel moments [2, 4, 5], geometric transformations of Watson type [3], hypergeometric summations [6] still play fundamental rôles in our quantitative understandings for integrals over elliptic integrals.

With a synthesis of various techniques, it is sometimes possible to handle integrals over the product of more than three Legendre functions of the same degree $v \in \mathbb{C}$, such as

$$\int_{-1}^1 x[P_v(x)]^4 dx = \lim_{z \rightarrow v} \frac{2 \sin^4(\pi z) [\psi^{(2)}(z+1) + \psi^{(2)}(-z) + 28\zeta(3)]}{(2z+1)^2 \pi^4}, \quad \text{where } \psi^{(2)}(z) := \frac{d^3}{dz^3} \log \Gamma(z). \quad (77)$$

One may wish to compare Eq. 77 to the integrals $\int_{-1}^1 x[P_v(x)]^3 P_v(-x) dx, v \in \mathbb{C}$ (Eq. 62) that reduce to elementary functions. Clearly, special cases of Eq. 77 bring us some interesting evaluations of multiple elliptic integrals. For example, we have the following integral representations for Apéry's constant $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$:

$$\zeta(3) = -\frac{\pi^4}{243} \int_{-1}^1 x[P_{-1/3}(x)]^4 dx = -\frac{\pi^4}{168} \int_{-1}^1 x[P_{-1/4}(x)]^4 dx = -\frac{2\pi^4}{189} \int_{-1}^1 x[P_{-1/6}(x)]^4 dx$$

as well as a critical scenario involving $\zeta(5) = \sum_{n=1}^{\infty} n^{-5}$:

$$\zeta(5) = -\frac{\pi^4}{372} \int_{-1}^1 x[P_{-1/2}(x)]^4 dx = \frac{8}{93} \int_0^1 (2t-1)[\mathbf{K}(\sqrt{t})]^4 dt.$$

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