Bowl Green's Function Attempt using Toroidals PhL 5.2.10

Notice the early date on this doc. I am wondering about the Green's Function problem for I guess various objects including the bowl. I am using the toroidal n-1/2 P and Q functions which is probably wrong for the bowl. This topic never arises in bowl doc because I don't do Green's functions in that doc, so I will let this rest in peace for now.

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Overview (9.26.10 1 page)

0. In the opening comment, I note that in toroidal coordinates the bowl surface has no "holes" and this ought to make the problem easier to solve. In *spherical* coordinates, the bowl has a "hole" in the sense that the range of the polar angle is partial. The part of the sphere that is not the bowl is then this "hole". In toroidal coordinates the range of the two non-bowl-labeling coordinates are full, not partial, so no holes. That is to say, if you select $\eta = \eta_1$ for a particular bowl, then as you let φ run full range you of course stay on the bowl. Less obviously, if you let μ run its full range you also stay on the bowl.

This is just a "comment" and I don't exactly know how it applies to the Green's Function problem. I think it would mean that you could do a "reduced 1D Green's Function" approach since the 1D boundary conditions are not intertwined between two coordinates. For the charged bowl problem, I think it would mean you can get a solution without dealing with things like "dual integral equations" which arise due to the presence of a "hole".

1. So, I assume a fully general Smythian form in terms of toroidal atoms like $P^{m}_{n-1/2}(ch\mu)$. I *realize* that there must be some SL transform associated with the oscillatory functions $P^{m}_{n-1/2}(ch\mu)$, but at the time of this doc I *don't know* this is the Mehler-Fock transform. I simply assume one could develop the transform from the usual SL mechanism, and then one could solve for the Smythian form coefficients. The resulting Green's function is something having this form

$$V(\mu,\eta,\varphi) = \sqrt{ch\mu - cos\eta}\ \Sigma_{nm}\left[a_n \cos(n\eta) + b_n \sin(n\eta)\right] Q^{m}_{n-1/2}(ch\mu_>)\, P^{m}_{n-1/2}(ch\mu_<)\cos(m\varphi)$$

where $a_n = a_n(\eta_1; \mu_0,\eta_0)$ and similarly for b_n and where the Green's point charge is at $(\mu_0,\eta_0,\varphi_0 = 0)$ and our bowl has the label η_1. As usual, $\mu_> = \max(\mu,\mu_0)$ and so on. I did not take advantage of the known symmetry of the result. Basically, I simply *outline* how the problem might be solved. I then comment that even if you take the limit of the bowl which is a disk ($\eta_1 = \pi$) or an iris ($\eta_1 = 0$), you will still have a messy double sum solution Σ_{nm}. Probably this is the best one can do for a general location of the Green's point charge.

2. I then make a very brief comment about a possible Smythian form for the charged bowl problem.

3. I then ask myself what other authors have to say about toroidal solutions. This I guess led me off to study toroidal coordinates in more depth in my curvilinear folder toroidal doc.

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0. Comment: My problems with the bowl, and it limits the disk and iris, have to do with the fact that you need to use 2 of the 3 variables to define the bowl surface, which means the surface has a "hole" in terms of these variables. For example, sphere has a hole in sphericals, and iris has a hole in cylindricals. But in toroidals, I think you can describe a bowl without the notion of a "hole". Maybe that makes the problem more soluble. [This is also true in oblate spheroidals for disk and iris]

In toroidals, $u = u_0$ in fact describes the surface of a bowl, see doc on toroidals. So one wonders what would happen in a "reduced Green's function method". Or related methods. Or even just the Smythian form method. So maybe I will just blindly try a few things here.

1. How the Smythian Form method might go.

I will switch now to names $\mu = \xi$ for the hyper, and $\eta = u$ for the trig. Then

$$V_1(\mu,\eta,\varphi) = \sqrt{ch\mu - cos\eta}\ \ \Sigma_{m=0}^{\infty}\Sigma_{n=0}^{\infty}\ [\ A_n \cos(n\eta)\ + B_n \sin(n\eta)]\ \ Q^m_{n-1/2}(ch\mu)\cos(m\varphi) \qquad \mu>\mu_0$$
$$V_2(\mu,\eta,\varphi) = \sqrt{ch\mu - cos\eta}\ \ \Sigma_{m=0}^{\infty}\Sigma_{n=0}^{\infty}\ [\ C_n \cos(n\eta)\ + D_n \sin(n\eta)]\ \ P^m_{n-1/2}(ch\mu)\cos(m\varphi) \qquad \mu<\mu_0$$

$V(\mu,\eta_1,\varphi) = 0$ // potential vanishes on the bowl labeled by η_1

I am thinking a Green's function problem for a bowl, where the point charge +q is at $(\mu_0,\eta_0,\varphi_0 = 0)$. This could be inside or outside of the bowl. Convention is V = q/r for point charge. For the moment, I am modeling the entire total potential by the above form -- induced charge plus point charge.

In toroidal coordinates, this appears to be a pure-play Dirichlet problem. [By this I think I mean it is not a mixed BC problem like dual integral equation stuff so you could use my "Dirichlet method of finding a Green's Function."] The "space" is an infinite rectangular solid which has both variables η and φ doing $(0,2\pi)$ and has μ doing $(0,\infty)$. It is a skyscraper sitting on the η,φ plane and going upwards in the vertical μ direction to infinity. The boundary condition V=0 is on a vertical slice of this thing, which is the mapping of the metal bowl. I guess this is not really a "closed surface", and in fact we have to have periodic BC's in the η and φ directions, so not quite a classic Dirichlet problem, but maybe I can make it fly as such.

The point charge lies inside the skyscraper at height μ_0. Above that point, as $\mu \to \infty$ we have to have good behavior, so need $Q_{n-1/2}(ch\mu)$. But below that point, as $\mu \to 0$ so we need $P_{n-1/2}(ch\mu)$. As usual, we require V be continuous at $\mu = \mu_0$ away from the Green's point charge, so write

$$V_1(\mu,\eta,\varphi) = \sqrt{ch\mu - cos\eta}\ \ \Sigma_{nm}\ [\ a_n \cos(n\eta)\ + b_n \sin(n\eta)]\ Q^m_{n-1/2}(ch\mu)\ P^m_{n-1/2}(ch\mu_0)\cos(m\varphi) \qquad \mu>\mu_0$$
$$V_2(\mu,\eta,\varphi) = \sqrt{ch\mu - cos\eta}\ \ \Sigma_{nm}\ [\ a_n \cos(n\eta)\ + b_n \sin(n\eta)]\ P^m_{n-1/2}(ch\mu)\ Q^m_{n-1/2}(ch\mu_0)\cos(m\varphi) \qquad \mu<\mu_0$$

This form causes the two potentials to match at $\mu = \mu_0$ and I think this then takes us down to two coefficient sets which are now called a_n and b_n. We can now examine our one boundary condition:

$$0 = \sqrt{ch\mu - cos\eta_1}\ \ \Sigma_{nm}\,[\,a_n \cos(n\eta_1) + b_n \sin(n\eta_1)]\ \ Q^m_{n-1/2}(ch\mu)\, P^m_{n-1/2}(ch\mu_0)\cos(m\varphi) \qquad \mu>\mu_0$$
$$0 = \sqrt{ch\mu - cos\eta_1}\ \ \Sigma_{nm}\,[\,a_n \cos(n\eta_1) + b_n \sin(n\eta_1)]\ \ P^m_{n-1/2}(ch\mu)\, Q^m_{n-1/2}(ch\mu_0)\cos(m\varphi) \qquad \mu<\mu_0$$

We can do the usual PWA in φ, and remove the overall factor, to reduce this to

$$0 = \ \Sigma_{n}\,[\,a_n \cos(n\eta_1) + b_n \sin(n\eta_1)]\ \ Q^m_{n-1/2}(ch\mu)\, P^m_{n-1/2}(ch\mu_0) \qquad \mu>\mu_0$$
$$0 = \ \Sigma_{n}\,[\,a_n \cos(n\eta_1) + b_n \sin(n\eta_1)]\ \ P^m_{n-1/2}(ch\mu)\, Q^m_{n-1/2}(ch\mu_0) \qquad \mu<\mu_0$$

Comment: At this point, we would like to "invert" to get our coefficients. But I have no idea yet about the orthogonality of the toroidal functions, and doubt they are both in orthogonal sets. The interval is now the range (1,∞) which I have never "worked with" in Legendre world. We must have a SL problem here, and its usual implications. [This is Mehler-Fock!] We might compact the above two conditions this way

$$0 = \ \Sigma_n f_{nm}(\eta_1,\mu_0)\, Q^m_{n-1/2}(ch\mu) \qquad \mu>\mu_0$$
$$0 = \ \Sigma_n g_{nm}(\eta_1,\mu_0)\, P^m_{n-1/2}(ch\mu) \qquad \mu<\mu_0$$

where we know that

$$f_{nm}(\eta_1,\mu_0) = [\,a_n \cos(n\eta_1) + b_n \sin(n\eta_1)]\, P^m_{n-1/2}(ch\mu_0)$$
$$g_{nm}(\eta_1,\mu_0) = [\,a_n \cos(n\eta_1) + b_n \sin(n\eta_1)]\, Q^m_{n-1/2}(ch\mu_0)$$

$$\Rightarrow \qquad f_{nm}/\, g_{nm} = \ P^m_{n-1/2}(ch\mu_0)/\, Q^m_{n-1/2}(ch\mu_0)$$

This suggests that we only have to invert one of our two equations, and maybe then we will pick the one which has orthogonality [P] . If we can do this, we will end up with a condition of the form:

$$[\,a_n \cos(n\eta_1) + b_n \sin(n\eta_1)] \ = \text{something}$$

Suppose it is the second we invert and suppose $P^m_{n-1/2}(ch\mu)$ for fixed m is a complete set on (1,∞). Then we know we can get a solution where $g_{nm} \neq 0$ (an issue I have worried about before in general). We would then have one condition on our two coefficients.

The other condition would no doubt come from our "pillbox condition", which I postpone for the moment.

Suppose we are successful and we come up with expressions for the solution coefficients a_n and b_n. We then end up with this as our solution:

$$V_1(\mu,\eta,\varphi) = \sqrt{ch\mu - cos\eta}\ \ \Sigma_{nm}\,[\,a_n \cos(n\eta) + b_n \sin(n\eta)]\, Q^m_{n-1/2}(ch\mu)\, P^m_{n-1/2}(ch\mu_0)\cos(m\varphi) \qquad \mu>\mu_0$$
$$V_2(\mu,\eta,\varphi) = \sqrt{ch\mu - cos\eta}\ \ \Sigma_{nm}\,[\,a_n \cos(n\eta) + b_n \sin(n\eta)]\, P^m_{n-1/2}(ch\mu)\, Q^m_{n-1/2}(ch\mu_0)\cos(m\varphi) \qquad \mu<\mu_0$$

This is for a spherical bowl described by $\eta = \eta_1$, so we will get some $a_n = a_n(\eta_1; \mu_0,\eta_0)$ etc. So, as always happens we get a solution of this form:

$$V(\mu,\eta,\varphi) = \sqrt{ch\mu - cos\eta}\ \Sigma_{nm}\,[\,a_n \cos(n\eta) + b_n \sin(n\eta)]\, Q^m_{n-1/2}(ch\mu_>)\, P^m_{n-1/2}(ch\mu_<)\cos(m\varphi)$$

which is a double sum on n and m of some "strange functions". We can take the disk and iris limits of this result by adjusting η_1 inside $a_n(\eta_1; \mu_0,\eta_0)$ and b_n as well. We know $\eta_1 = \pi$ is the disk, and $\eta_1 = 0$ is the iris. But in all these cases, our solution is expressed as a double sum of obscure special functions like $P^m_{n-1/2}(ch\mu)$, so in that sense our disk or iris solution will be similar to the solution we got in oblate spheroidal coordinates. If we want to compute the charge density on the bowl, we will need ∂_η of the above, and that will still involve $P^m_{n-1/2}(ch\mu)$ functions. As we move around on the bowl's surface, the parameter μ is what varies (as well as φ).

So the point is *the same as it was in oblates*: how are you going to arrive at the known "elementary function" solutions for things like the charge density on the bowl?

I could imagine having to redo the above problem using a separate second-term expression for the point charge written in terms of toroidal harmonics, but the "point" just mentioned won't be any different.

2. What about the charged bowl? // 7.9.10

We can certainly take our Smythian atomic form and set m = 0 to get

$$V_1(\mu,\eta,\varphi) = \sqrt{ch\mu - cos\eta}\ \ \Sigma_{n=0}^{\infty}\,[\,A_n \cos(n\eta) + B_n \sin(n\eta)]\ \ Q_{n-1/2}(ch\mu) \qquad \eta > \eta_1$$
$$V_2(\mu,\eta,\varphi) = \sqrt{ch\mu - cos\eta}\ \ \Sigma_{n=0}^{\infty}\,[\,C_n \cos(n\eta) + D_n \sin(n\eta)]\ \ P_{n-1/2}(ch\mu) \qquad \eta < \eta_1$$

In the oblate spheroid case, once we had m = 0, that forced n = 0. But here that does not seem to happen. But this form is no good, we really want to see $e^{-n\eta}$ or some such expo so as we move on constant bowls out away from our bowl, we get decay. Then I am not sure what happens with the P/Q functions. This was just a quick comment, I have not really tried anything here. [But η has a finite range $0,\pi$ so this last remark makes no sense.]

3. What do my various authors have to say on this subject.

See doc on toroidals.

Note added 2.2.11: I have come a long way since this doc!