Computation of T(z) at z near 1 PhL 12.12.15

This is all in bowl doc now in Appendix L including the little case studies. 12.31.15

I have probably done all this before somewhere, but today I will do it again.

A realization in bed this AM: App B is concerned with this function,

$$T(z) \equiv \sqrt{z^2 - 1}\ \Sigma_{n=0}^{\infty} \varepsilon_n [Q_{n-1/2}(z) / P_{n-1/2}(z)] \qquad a_n = \varepsilon_n \sqrt{z^2 - 1}\ [Q_{n-1/2}(z) / P_{n-1/2}(z)]$$

This is an example of a series (that is, an finite series in Buck speak) with a parameter here z. What did I learn about such situations in my recent Buck reading? Chapter 4 was on Convergence. We had many tests for convergence such as "the comparison test". But I don't know much about inequalities involving these complicated functions. Doubtless my series is "on the edge". The integral comparison test might be something I could do here, but I think I know it converges, I want to know the limit!

Buck uniform convergence starts with Section 4.2. Write the above as

$$\Sigma_{n=0}^{\infty} f_n(x) \qquad f_n(x) = \varepsilon_n \sqrt{x^2 - 1}\ [Q_{n-1/2}(x) / P_{n-1/2}(x)]$$

then we are in Buck notation which helps me. Buck talks about uniform convergence on a range of x.

Study the convergence of the above series.

From Appendix H I have

$$\frac{Q_{n-1/2}(ch\xi)}{P_{n-1/2}(ch\xi)} \rightarrow \pi e^{-2n\xi} \text{ as } n \rightarrow \infty \qquad \text{(H.5.9)}$$

Now if we want to use ξ instead of x that is fine. Then, incorporating the $\sqrt{x^2 - 1}$ factor,

$$f_n(\xi) = \varepsilon_n\ sh\xi\ \ \pi e^{-2n\xi} = \varepsilon_n (1/2)(e^{\xi} - e^{-\xi})\, \pi e^{-2n\xi} \qquad \text{as } n \rightarrow \infty$$

$$= (\pi\varepsilon_n/2)[\ e^{-n\xi} - e^{-3n\xi}\ ] \approx (\pi\varepsilon_n/2)\, e^{-n\xi} \qquad \text{large n, } \xi > 0$$

Buck page 162 Corollary says that if $r = \lim_{n \leftrightarrow \infty} a_{n+1}/a_n < 1$, the series converges. We have

$$a_{n+1}/a_n = e^{-\xi}$$

Thus the series for sure converges of $\xi > 0$. Notice that $a_n \rightarrow 0$ when $\xi > 0$ as required for any convergent series (Buck Theorem 1 p 161).

What about for $\xi = 0$? We have to go back to

$$f_n(\xi) = \varepsilon_n\ sh\xi\ \ \pi e^{-2n\xi}$$

Notice that $f_n \to 0$ as $\xi \to 0$ so it at least *has a chance* of converging.

The same ratio test says

$$a_{n+1}/a_n = e^{-2\xi}$$

For $\xi = 0$ the ratio is 1, and then we have no conclusion from this test. It could either converge or it could diverge.

How about Raabe's Test Buck p 163? We need ratio < 1-p/n beyond some n. But my ratio is 1, so Raabe does not help!

How about the Root Test of Buck p 162 Theorem 6. We are supposed now to compute

$$r = (a_n)^{1/n}$$

In our case we get,

$$r = (\mathrm{sh}\xi\, e^{-2n\xi})^{1/n} = (\mathrm{sh}\xi)^{1/n}\, e^{-2\xi}$$

Now as $\xi \to 0$ the last factor is 1, and the first is $(0)^{1/n}$. What is the limit of such a thing? Consider

$0^x = y$ $\qquad$ $x > 0$

$x \ln(0) = \ln(y) = -x \infty$ $\qquad$ so $y = 0$.

If $x = 0$, then

$0^0 = y$

$0 \ln 0 = \ln y = 0$ $\qquad$ so $y = 1$

So I think $(0)^{1/n} \to 1$ and we get $r \to 1$ so no conclusion!

How about the Buck test page 163 Theorem 7. We have

$f(n) = \mathrm{sh}\xi\, e^{-2n\xi}$ $\qquad$ $a_n \to 0$ for ξ in $[1,\infty)$

Is this monotone decreasing? It is for $\xi > 0$. For $\xi = 0$ we have $f(n) = 0$, so not monotone. So cannot apply that theorem.

So none of Buck's fancy theorems seems to work on this series.

Interesting Case Study

Consider this series which is simpler than the Q/P series, but perhaps very similar to it:

$$f(\xi) = \Sigma_{n=0}^{\infty} [\ sh\xi\ e^{-2n\xi}\ ]$$

1. Analytically, I can start with $\xi > 0$ and write

$$f(\xi) = sh\xi\ \Sigma_{n=0}^{\infty}\ e^{-2n\xi} = \frac{sh\xi}{1-e^{-2\xi}} = \frac{sh\xi e^{\xi}}{e^{\xi}-e^{-\xi}} = (1/2)\ e^{\xi}$$

Now that I know the sum, I can take the limit to get

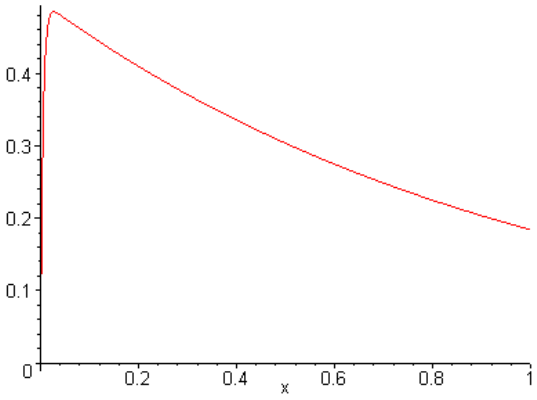
$$f(\xi) = 0.5$$

2. Numerically, here is how this works,

```
restart;
a := n -> sinh(x)*exp(-2*x*n);
```

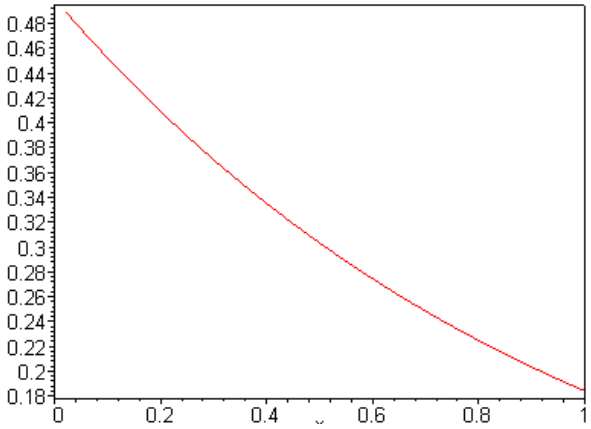
$$a := n \to \sinh(x)\, e^{(-2\,x\,n)}$$

```
sumterms := sum(a(n),n=1..100):
plot(sumterms,x=0..1);
```



If you add up a finite number of terms, you always get f(0) = 0, but if you add up an infinite number of terms you know that f(0) will be 1/2.

```
sumterms := sum(a(n),n=1..10000):
plot(sumterms,x=0..1, axes = boxed);
```



I think this is exactly what is happening in my capacitance case as R → 1 !!!!!

Now what can I say about uniform convergence with this little toy series?

$f(\xi) = \Sigma_{n=0}^{\infty} [\ sh\xi\ e^{-2n\xi}\ ]$

$a_n = sh\xi\ e^{-2n\xi}$

Does this meet the Cauchy condition (Buck p 183) for $\xi \geq 0$

$|a_n - a_m|_E < \varepsilon$

$|\ sh\xi\ e^{-2n\xi} - sh\xi\ e^{-2m\xi}|_E < \varepsilon$ n,m > N

BUT be careful about that E subscript. If the range is the full $\xi \geq 0$, you are talking about a least upper bound (max) on the interval $[0,\infty)$. This is just a max norm. So where does the function above have its max value, and what is that max?

Let's consider the range $\xi \in (0, .0000001)$. Then

$sh\xi = \xi$ $e^{-n\xi} = 1 - n\xi$

$|\ sh\xi\ e^{-2n\xi} - sh\xi\ e^{-2m\xi}|_E = |\ \xi(1-2n\xi) - \xi(1-2m\xi)|_E$

$= |\xi - 2n\xi^2 - \xi + 2m\xi^2|_E = |\ 2(m-n)\xi^2|_E$

$= |\ m-n\ |\ 2\ \xi_{max}^2$

Now pick some ε for N = 50. Then we are NOT Cauchy in this little model. But I don't like the model, since when m,n increase the expansion is no longer valid.

A similar Case Study

$f(x) = x(e^{-nx} - e^{-mx})$

$f'(x) = e^{-nx}(1-nx) - e^{-mx}(1-mx)$

This is a transcendental equation which has some solution for x where f(x) has its max. So even though this is a simple example, I don't know analytically where the max is, and therefore I don't know the max value, so I am not able to do the Cauchy test because I cannot compute $|f(x)|_E$ on say (0,1).

OK, let's implement the Cauchy program. Select some ε like ε = .001. Then I need to find N such that we will be $< \varepsilon$ for n,m > N, but uniformly on (0,1) say. If x = 0, then $|\ |_{max} = 0$, so $0 < \varepsilon$ and things are OK. So the thing is decided then for x > 0. For some fixed $x = x_0 > 0$ we have $x_0 < x < 1$ for our range. I don't know exactly where the max will be in this range for a given pair n,m. Let's pick $x_0 = .001$. All I can do is fiddle with Maple somehow.

Can I get Maple to numerically locate a max value? It has a bug, but works for my functions.

> We found that the maximum value of $\sin x + \cos x$ is $\sqrt{2}$. For $0 \leq x \leq 3$, the maximum value of $x^2 - 5x + 1$ was found to be 1. However, MAPLE V incorrectly computed the maximum of $\sin x$ (for $0 \leq x \leq 1$) to be 1. The function $\sin x$ is increasing on $[0, 1]$ so the actual maximum value is $\sin 1 \approx 0.841$. We hope this bug will be fixed.

Proof of non-Cauchy. Consider this Maple code,

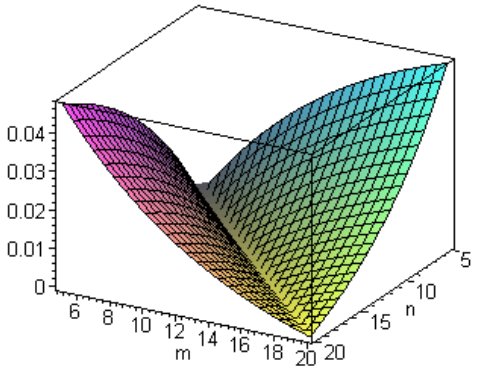
```
restart;
f := x*exp(-n*x) - x*exp(-m*x);
```

$$f := x\,e^{(-n\,x)} - x\,e^{(-m\,x)}$$

```
x := 0.1;
```

$$x := .1$$

```
plot3d(abs(f), n=5..20,m=5..20, axes = boxed);
```



Suppose I have ε = .001 and x is in range (0,1). I can see that for the n,m range I have given, that abs(f) is reaching a value of .04. Thus, whatever max(f) is on (0,1) for these ranges, it is ≥ ,04, and thus does not meet my ε.

****

What about my sum rule?

$\sqrt{z-1}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ Q_{n-1/2}(z) = (\pi/\sqrt{2}) = 2.221441469$ // sum rule

$\sqrt{z^2-1}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ [Q_{n-1/2}(z) / P_{n-1/2}(z)] = T(z)$

Suppose I say that $P_{n-1/2}(z) \to 1$ as $z \to 1$ and just throw that in. Then

$\sqrt{z-1}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ Q_{n-1/2}(z) = (\pi/\sqrt{2}) = 2.221441469$ for all $z > 1$ certainly

$\sqrt{z^2-1}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ [Q_{n-1/2}(z)] = T(z)$ for $z = 1+\varepsilon$

In the second case write

$$\sqrt{z^2 - 1} = \sqrt{(z+1)(z-1)} \approx \sqrt{2}\sqrt{z - 1}\,.$$

Then we have

$$\sqrt{z - 1}\ \Sigma_{n=0}^{\infty} \varepsilon_n\ Q_{n-1/2}(z) = (\pi/\sqrt{2}) = 2.221441469$$
$$\sqrt{z - 1}\ \Sigma_{n=0}^{\infty} \varepsilon_n\ [Q_{n-1/2}(z)] = T(z)/\sqrt{2} \qquad \text{for } z = 1+\varepsilon$$

Compare the right sides to get

$$T(z) = \pi$$

But I keep getting T(z) = 2.7353. Just take a look at a few combinations for fun:

$$\pi/\sqrt{2} = 2.221441469$$

Wrong answer, but method seems reasonable. Why is the answer wrong? Or is it right? Well, let's review that limit of P from Legendre properties,

the P(+1) finite cases: $P_\nu^\mu(1) = 0$ in these cases : [exception $P_\nu^0(1) = 1$] , else $P_\nu^\mu(1) = \infty$

ν = anything:

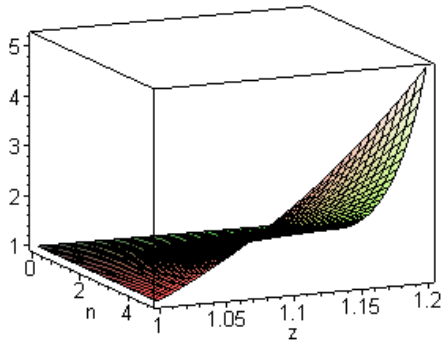
Re(μ) < 0 and $\mu \neq$ integer

μ = integer $P_\nu^\mu(1) = \delta_0^\mu$

In my case $\mu = 0$, so the last line applies: $P_{n-1/2}(1) = 1$ I should have plotted the $P_{n-1/2}$ functions the way I did the $P_{i\tau-1/2}$ functions, I need to add that.

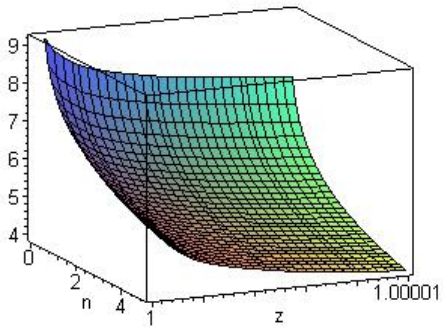
Let's to plot those right now to make sure I have this right. Do in same file where I plot the Mehler ones. Index says this file is sqrt one plus cosx. It is done, here are the results!

```
plot3d(LegendreP(n-1/2, z), n=0..5, z = 1..1.2, axes = boxed, grid = [30,30]);
```



This confirms that at z = 1, P(z) = 1!!! That lower left edge says it all. Also we have

```
plot3d(Q(n-1/2,arccosh(z)), n=0..5, z = 1..1.00001, axes = boxed, grid = [30,30]);
```



Both are "expo" as discussed in the atoms discussion. The P are 0 at the z = 1 boundary, The second one very slowly blows up at z = 1 with ln(z-1). Fine.

So my P→1 rough method above was not unreasonable!

OK, lets go for models of the P and Q for small argument. Maybe the divergent Q pulls some finite amount out of the P function?? My Leg prop doc does not talk about P(z) near z = 1, I will have to find that elsewhere. Try AS. Here is a battery of information

14.8(ii) $x \to 1+$

14.8.7

$$P_\nu^\mu(x) \sim \frac{1}{\Gamma(1-\mu)}\left(\frac{2}{x-1}\right)^{\mu/2}, \qquad \mu \neq 1,2,3,\ldots,$$

14.8.8

$$P_\nu^m(x) \sim \frac{\Gamma(\nu+m+1)}{m!\,\Gamma(\nu-m+1)}\left(\frac{x-1}{2}\right)^{m/2},$$
$$m = 1,2,3,\ldots,\ \nu \pm m \neq -1,-2,-3,\ldots,$$

14.8.9

$$\boldsymbol{Q}_\nu(x) = -\frac{\ln(x-1)}{2\,\Gamma(\nu+1)} + \frac{\frac{1}{2}\ln 2 - \gamma - \psi(\nu+1)}{\Gamma(\nu+1)} + O(x-1), \qquad \nu \neq -1,-2,-3,\ldots,$$

14.8.10 $\boldsymbol{Q}_{-n}(x) \to (-1)^{n+1}(n-1)!, \quad n = 1,2,3,\ldots,$

14.8.11 $\boldsymbol{Q}_\nu^\mu(x) \sim \frac{\Gamma(\mu)}{2\,\Gamma(\nu+\mu+1)}\left(\frac{2}{x-1}\right)^{\mu/2},$
$\Re\mu > 0,\ \nu+\mu \neq -1,-2,-3,\ldots.$

So I could say

$P_\nu(x) = 1 + \text{order}(x-1)$ I would guess

$Q_\nu(x) = -\ln(x-1)/[2\Gamma(\nu+1)]$ + some other function to be careful about!

If I just set P = 1, I know that I get $T(z) = \pi$ from the perfect sum rule. So let's try to find a correction for P near 1. I have this

$$P_\nu(z) = z^\nu F(-\nu/2, 1/2-\nu/2; 1; 1-1/z^2)$$

So set $z = 1+\varepsilon$ to get

$$1-1/(1+\varepsilon)^2 = [\,(1+\varepsilon)^2 - 1\,] / (1+\varepsilon)^2 = (2\varepsilon\,] / (1+\varepsilon)^2 \approx 2\varepsilon$$

So

$$P_\nu(1+\varepsilon) = z^\nu F(-\nu/2, 1/2-\nu/2; 1; 2\varepsilon)$$

So I can just write out the first term of the series:

$$F(-\nu/2, 1/2-\nu/2; 1; 2\varepsilon) \approx 1 + abx/c = 1 + (-\nu/2)(1/2-\nu/2)2\varepsilon/(1) = 1 + (-\nu/2)(1-\nu)\varepsilon$$

$$= 1 + (1/2)\nu(\nu-1)\varepsilon$$

Now let $\nu = n-1/2$ so that

$$F \approx 1 + (1/2)(n-1/2)(n-3/2)\varepsilon$$

Therefore

$$P_\nu(1+\varepsilon) = (1+\varepsilon)^\nu\,[\,1 + (1/2)\nu(\nu-1)\varepsilon + O(\varepsilon^2)] \qquad \text{as } z \to 1$$

Now

$$(1+\varepsilon)^\nu = 1 + \nu\varepsilon$$

so then

$$P_\nu(1+\varepsilon) \approx [1 + \nu\varepsilon]\,[\,1 + (1/2)\nu(\nu-1)\varepsilon + O(\varepsilon^2)]$$

$$\approx 1 + \{\,1 + (1/2)(\nu-1)\}\nu\varepsilon\,\} = 1 + \{\,1 + \nu/2 - 1/2\}\nu\varepsilon\,\} = 1 + \{\,\nu/2 + 1/2\}\nu\varepsilon\,\}$$

$$\approx 1 + (1/2)\nu(\nu+1)\varepsilon + O(\varepsilon^2)$$

a perfectly clean result. Then

$$1/P_\nu(1+\varepsilon) = 1/[\,1 + (1/2)\nu(\nu+1)\varepsilon + O(\varepsilon^2)] \approx [\,1 - (1/2)\nu(\nu+1)\varepsilon + O(\varepsilon^2)]$$

Meanwhile, look at $Q_\nu(z)$ near $z = 1$:

$$Q_\nu(1+\varepsilon) \approx \{ -\ln\varepsilon/[2\Gamma(\nu+1)] + [(1/2)(\ln 2) - \gamma - \psi(\nu+1)]/\Gamma(\nu+1) + k(\nu)\,\varepsilon \}$$

As $\varepsilon \to 0$, the first term blows up positively as a log thing, the second term is fixed for ν. I just presume there is some $O(\varepsilon)$ term added onto this which I write as unknown $k(\nu)\,\varepsilon$.

Footnote. AS says this

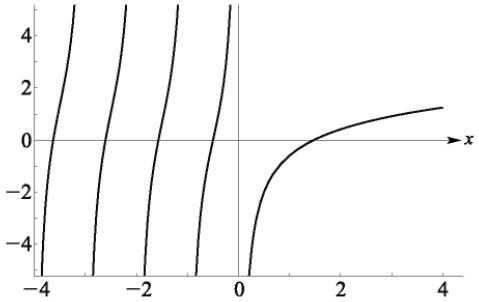


Figure 5.3.3: $\psi(x)$.

and I am interested in $\nu = n - 1/2$ so $\nu+1 = n+1/2$

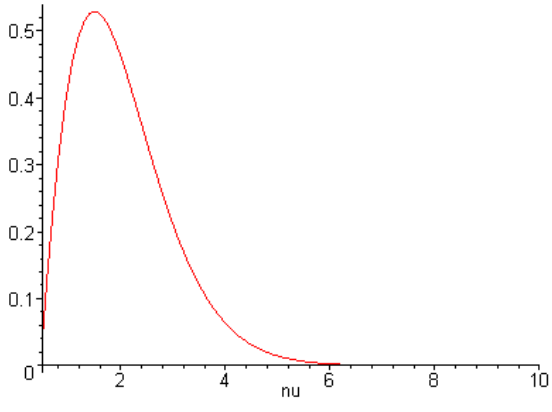
$$[(1/2)\ln 2 - \gamma - \psi(\nu+1)]/\Gamma(\nu+1) = (1/2)\ln 2 - \gamma - \psi(n+1/2)]/\Gamma(n+1/2)$$

So first of all, ψ is not singular for any value of n in my range $n = 0..\infty$. Here is a plot

```
f := Psi(nu+1)/GAMMA(nu+1);

plot(f,nu=1/2..10);
```

$$f := \frac{\Psi(\nu+1)}{\Gamma(\nu+1)}$$



So as $n \to \infty$, this second term varies as shown, fading away eventually. But it is there!

So back to our modeling: We now have

$$1/P_\nu(1+\varepsilon) = [\,1 - (1/2)\nu(\nu+1)\varepsilon + O(\varepsilon^2)]$$

$$Q_\nu(1+\varepsilon) \approx \{ - \ln\varepsilon /[2\Gamma(\nu+1)] + [(1/2)(\ln 2)-\gamma - \psi(\nu+1)]/\Gamma(\nu+1) + k(\nu)\,\varepsilon \}$$

$$Q_\nu(1+\varepsilon) /\!/ P_\nu(1+\varepsilon)$$

$$\approx \{ - \ln\varepsilon /[2\Gamma(\nu+1)] + [(1/2)(\ln 2)-\gamma - \psi(\nu+1)]/\Gamma(\nu+1) + k(\nu)\,\varepsilon \}\,\{\,1 - (1/2)\nu(\nu+1)\varepsilon + O(\varepsilon^2)\}$$

This is the first time I think I have done this little analysis, by the way. Now let's drop everything of order ε or higher in this product, so then we just get

$$Q_\nu(1+\varepsilon) /P_\nu(1+\varepsilon) = Q_\nu(1+\varepsilon) = - \ln\varepsilon /[2\Gamma(\nu+1)] + [(1/2)(\ln 2)-\gamma - \psi(\nu+1)]/\Gamma(\nu+1)$$

Now my sum rule applies again and we get $T(z) = \pi$. Rewrite the above as

$$Q_\nu(z) /P_\nu(z) = Q_\nu(z) = - \ln(1-z) /[2\Gamma(\nu+1)] + [(1/2)(\ln 2)-\gamma - \psi(\nu+1)]/\Gamma(\nu+1)$$

So how exactly does the sum work?

$$\Sigma_{n=0}^{\infty} \varepsilon_n [\sqrt{z^2 - 1}\; Q_{n-1/2}(z) / P_{n-1/2}(z)] = T(z)$$

Be careful, don't forget the $\sqrt{z^2 - 1}$ factor!!

$$\sqrt{z^2 - 1} = \sqrt{2\varepsilon}$$

so then we have

$$T(1+\varepsilon) = \Sigma_{n=0}^{\infty} \varepsilon_n \sqrt{2\varepsilon}\; \{- \ln(\varepsilon) /[2\Gamma(\nu+1)] + [(1/2)(\ln 2)-\gamma - \psi(\nu+1)]/\Gamma(\nu+1) \}$$

So "how it works" is that somehow the $\sqrt{2\varepsilon}$ rescues the divergent $\ln(\varepsilon)$. The second term must go to 0 just from the $\sqrt{2\varepsilon}$ since everything there is finite. So I conclude that, since $\nu+1 = n+1/2$

$$T(1+\varepsilon) = \Sigma_{n=0}^{\infty} \varepsilon_n \sqrt{2\varepsilon}\; \{- \ln(\varepsilon) /[2\Gamma(n+1/2)]$$

or

$$T(1+\varepsilon) = - (1/\sqrt{2}\,) \Sigma_{n=0}^{\infty} \varepsilon_n\; [\Gamma(n+1/2)]^{-1}\; \varepsilon^{1/2} \ln(\varepsilon)$$

$$= - (1/\sqrt{2}\,)\varepsilon^{1/2} \ln(\varepsilon)\; \Sigma_{n=0}^{\infty} \varepsilon_n\; [\Gamma(n+1/2)]^{-1}$$

But this is a problem because consider

$$f(\varepsilon) \equiv \varepsilon^{1/2} \ln(\varepsilon) = \varepsilon^{1/2} /\; [1/\ln\varepsilon)] = 0/0$$

Apply l'Hopital

$$\partial_\varepsilon [\ln\varepsilon]^{-1} = -[\ln\varepsilon]^{-2}\,\partial_\varepsilon(\ln\varepsilon) = -[\ln\varepsilon]^{-2}(1/\varepsilon)$$

$$\partial_\varepsilon\, \varepsilon^{1/2} = (1/2)\varepsilon^{-1/2}$$

Then ratio of these is

$$f(\varepsilon) \rightarrow \quad (1/2)\varepsilon^{-1/2} \;/\; \{ -[\ln\varepsilon]^{-2}(1/\varepsilon)\} = -(1/2)\,(\ln\varepsilon)^2\,\varepsilon^{1/2}$$

and I haven't learned a thing. I know that $\varepsilon^{1/2}\ln(\varepsilon) \rightarrow 0$ which saysT(z)→ 0 but we know that is wrong.

Summary of what was done above:

I considered this series

$$T(z) \equiv \Sigma_{n=0}^{\infty}\, \varepsilon_n\, [\sqrt{z^2 - 1}\; Q_{n-1/2}(z) / P_{n-1/2}(z)]$$

I made the assumption that this is true,

$$\lim_{z\rightarrow 1} T(z) = \Sigma_{n=0}^{\infty}\, \varepsilon_n\, \lim_{z\rightarrow 1}\, [\sqrt{z^2 - 1}\; Q_{n-1/2}(z) / P_{n-1/2}(z)]$$

But then I calculated that

$$\lim_{z\rightarrow 1}\, [\sqrt{z^2 - 1}\; Q_{n-1/2}(z) / P_{n-1/2}(z)] = \lim_{z\rightarrow 1}\, [\sqrt{z^2 - 1}\; Q_{n-1/2}(z)$$

$$= \sqrt{2\varepsilon}\, [\, -\ln\varepsilon\, /[2\Gamma(\nu+1)] + q(\nu)\,]$$

$$= 0$$

and then I concluded that

$$\lim_{z\rightarrow 1} T(z) = 0$$

which I know is the wrong result. Therefore, the "order interchange" must not be allowed.

Now go back to

$$F(x) = \Sigma_{k=0}^{\infty}\, f_k(x) \qquad\qquad f_k(x) = \varepsilon_k\sqrt{x^2 - 1}\; [Q_{k-1/2}(x) / P_{k-1/2}(x)]$$

$$F_n(x) = \Sigma_{k=0}^{n}\, f_k(x) \quad = \text{the } n^{th} \text{ partial sum}$$

I now roll out some Buck info:

Def: [183A] A *series* $\Sigma f_n(x)$ **converges uniformly** iff the *sequence* of partial sums $F_n(x)$ converges uniformly to F(x) as per the following definition:

Def: [182A] A *sequence* $\{f_n(x)\}$ **converges uniformly** to F(x) on domain D if it converges for all x in D "with the same N". That is, there is an N such that $\| f_n(x) - F(x) \|_D < \varepsilon$ for n> N *and* for all x in D. The point is that the same large integer N works for all x in D, you don't need N(x).

The partial sums $F_n(x)$ form a sequence. Let the domain D $1 \le x \le 2$ say. Consider

$$\| \Sigma_{k=0}^{n} f_k(x) - F(x) \|_D < \varepsilon \quad \text{for n> N } \textit{and} \text{ for all x in D}$$

I think this is not true for my series. As you get closer and closer to x = 1, you need more and more terms, so there is no N that works for all x in $1 \le x \le 2$. Thus, my series does not converge uniformly on [1,2].

Probably this means you are not allowed to do the order interchange!!! And that is why my result above which says $T(z) \to 0$ is not valid.

Buck does not tell us how to compute the limiting function F(x) at x = 1.

What about my sum rule?

$$\sqrt{z-1}\ \Sigma_{n=0}^{\infty} \varepsilon_n\ Q_{n-1/2}(z) = (\pi/\sqrt{2}) = 2.221441469 \qquad \text{// sum rule}$$

If I could interchange order, I would conclude that this sum was 0!

Plan B: Does any source list sums involving ring functions or Legendre functions in general?

AS has nothing to say about such sums. AS is really about the functions themselves, not what you can do with them.

$$T(z) \equiv \sqrt{z^2-1}\ \Sigma_{n=0}^{\infty} \varepsilon_n\ [Q_{n-1/2}(z) / P_{n-1/2}(z)]$$

GR7 page 1021 has this

$$\frac{1}{z-t} = \sum_{k=0}^{\infty} (2k+1)\, P_k(t)\, Q_k(z)$$

There is a tiny section on "toroidal functions" in GR7, but no sums are given.

Web search on toroidal functions? I did some, then I reread my previous search on this.

I should add that $T(z) < \pi$. !

OK, once again I give up on doing this limit analytically, just wanted to give it another shot.