For Disk compare Jackson and Bowl result PhL 12.1.15

This topic is in bowl doc in Appendix B.2, all is well. 12.31.15

Take the disk limit of the bowl potential

My bowl potential is this

$$V(\xi,u) = (V_0/\pi)\ \sqrt{ch\xi - cosu}$$
$$\{\ \frac{1}{\sqrt{ch\xi-cosu}}\ \cot^{-1}[\frac{\sqrt{2}\cos(\pi-u/2)}{\sqrt{ch\xi-cosu}}]\ +\ \frac{1}{\sqrt{ch\xi-\cos(2u_0-u)}}\ \cot^{-1}[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0-u)}}]\ \}$$

u_0 in $(0,2\pi)$
u in $(u_0, 2\pi+u_0)$ $ch\xi$ in $(1,\infty)$. // potential of bowl u_0 *(8.7)*

If I set $u_0 \rightarrow \pi$, notice that u has the range $u \in (\pi,3\pi)$. We get

$$\cos(2u_0-u) \rightarrow \cos(2\pi-u) = cosu \qquad \text{for any real u}$$

You can then see that both terms in V are the same! So in this limit we get

$$V(\xi,u) = 2\ (V_0/\pi)\ \sqrt{ch\xi - cosu}\ \ \frac{1}{\sqrt{ch\xi-cosu}}\ \cot^{-1}[\frac{\sqrt{2}\cos(\pi-u/2)}{\sqrt{ch\xi-cosu}}]$$

$$= 2(V_0/\pi)\cot^{-1}[\frac{\sqrt{2}\cos(\pi-u/2)}{\sqrt{ch\xi-cosu}}]$$

Now we know that

$$\cos(\pi-u/2) = \cos(u/2-\pi) = -\cos(u/2)$$

so we then have

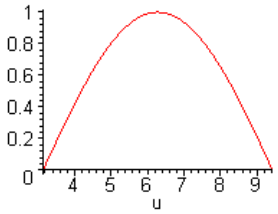
$$V(\xi,u) = (2V_0/\pi)\ \cot^{-1}[-\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-cosu}}]$$

Now I think you can flip this over to get [using $\cot^{-1}(1/x) = \tan(x)$]

$$V(\xi,u) = (2V_0/\pi)\ \tan^{-1}[-\frac{\sqrt{ch\xi-cosu}}{\sqrt{2}\cos(u/2)}]$$

This then is our result where u lies in $(\pi,3\pi)$. I confirm with my test of get(u) in the Maple code that Maple really is using u in this range during its plots. Notice this plot

```
plot(-cos(u/2), u = Pi..3*Pi);
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Thus, the argument of $\tan^{-1}$ is always positive and so V(ξ,u) is always positive. Right at the disk surface we would have $u = \pi+\varepsilon$ and there -cos(u/2) is positive and very small, so $\tan^{-1}$ argument is $+\infty$ and then the $\tan^{-1} = \pi/2$ and we get $V = (2V_0/\pi)(\pi/2) = V_0$, as expected.

Now Jackson's disk potential is this (green page 92 with my correction factor a added)

$$V_{disk}(\rho,z) = (2V_0/\pi)\sin^{-1}\left[\,2a/\left[\sqrt{(\rho-a)^2+z^2}+\sqrt{(\rho+a)^2+z^2}\,\right]\right] \qquad (C.1)$$

and my task is to show that

$$\tan^{-1}\left[-\frac{\sqrt{ch\xi-cosu}}{\cos(u/2)}\right] = \sin^{-1}\left[\,2a/\left[\sqrt{(\rho-a)^2+z^2}+\sqrt{(\rho+a)^2+z^2}\,\right]\right]$$

I am armed with all the equations in box (1.3.10). The tough question is how to make contact with those two Jackson radicals! I know that

$$th\xi = 2a\rho/(\rho^2+z^2+a^2) \quad \Rightarrow \quad \rho^2+z^2+a^2 = 2a\rho/th\xi$$

$$tanu = 2az/(\rho^2+z^2-a^2) \quad \Rightarrow \quad \rho^2+z^2-a^2 = 2az/tanu$$

But Jackson has for example

$$(\rho-a)^2+z^2 = -2a\rho+\rho^2+a^2+z^2 = -2a\rho+2a\rho/th\xi = 2a\rho(coth\xi-1)$$

$$(\rho+a)^2+z^2 = +2a\rho+\rho^2+a^2+z^2 = +2a\rho+2a\rho/th\xi = 2a\rho(coth\xi+1)$$

We know that $\rho = ash\xi/(ch\xi - cosu)$ so then we have

$$(\rho-a)^2+z^2 = 2a^2sh\xi(coth\xi-1)/(ch\xi-cosu)$$

$$(\rho+a)^2+z^2 = 2a^2sh\xi(coth\xi+1)/(ch\xi-cosu)$$

and then at least I have expressed Jackson's radical arguments entirely in toroidals!

Now consider

$$sh\xi(coth\xi-1) = ch\xi-sh\xi = e^{-\xi}$$
$$sh\xi(coth\xi+1) = ch\xi+sh\xi = e^{\xi}$$

Then we have

$$(\rho-a)^2+z^2 = 2a^2e^{-\xi}/(ch\xi - cosu)$$

$$(\rho+a)^2+z^2 = 2a^2e^{\xi}/(ch\xi - cosu)$$

and then

$$\sqrt{(\rho-a)^2+z^2} + \sqrt{(\rho+a)^2+z^2} = \sqrt{2}\, a \sqrt{1/(ch\xi-cosu)}\ [\, e^{-\xi/2} + e^{\xi/2}]$$

$$= \sqrt{2}\, a \sqrt{1/(ch\xi-cosu)}\ \ 2ch(\xi/2)$$

$$= 2\sqrt{2}\, a\, ch(\xi/2) / \sqrt{ch\xi-cosu}$$

So now my task is to show that

$$\tan^{-1}[-\frac{\sqrt{ch\xi-cosu}}{\cos(u/2)}\,] = \sin^{-1}[\, 2a/ [2\sqrt{2}\, a\, ch(\xi/2) / \sqrt{ch\xi-cosu}\,]$$

or

$$\tan^{-1}[-\frac{\sqrt{ch\xi-cosu}}{\cos(u/2)}\,] = \sin^{-1}[1\ / [\sqrt{2}\ \ ch(\xi/2) / \sqrt{ch\xi-cosu}\,]$$

for u in the range $\pi,3\pi$ where both sides are positive. Now we draw the triangle with angle θ and x,y,r. Then identify

$$\sin\theta = y/r \quad \theta = \sin^{-1}(y/r) \qquad y = 1 \qquad r = \sqrt{2}\ \ ch(\xi/2) / \sqrt{ch\xi-cosu}$$

Then

$$x^2 = r^2-y^2 = 2ch^2(\xi/2)/(ch\xi-cosu) - 1 = (2ch^2(\xi/2) - ch\xi + cosu\,)/(ch\xi-cosu)$$

$$= (\, ch\xi + 1 - ch\xi + cosu\,)/(ch\xi-cosu) = (\, 1 + cosu\,)/(ch\xi-cosu)$$

Therefore

$$\theta = \tan^{-1}(y/x) = \tan^{-1}(yx^{-1}) = \tan^{-1}[\sqrt{ch\xi-cosu} / \sqrt{1+cosu}\,]$$

So we then have

$$\sin^{-1}[\, 2a/ [\, \sqrt{(\rho-a)^2+z^2} + \sqrt{(\rho+a)^2+z^2}\ ] = \theta = \tan^{-1}[\sqrt{ch\xi-cosu}\, /\sqrt{1+cosu}\,]$$

Now for u in the range π to 2π where - cos(u/2) is positive, we replace $\sqrt{1+\cos u} = -\sqrt{2}\cos(u/2)$ to get

$$\sin^{-1}\left[\, 2a/\left[\sqrt{(\rho-a)^2+z^2} + \sqrt{(\rho+a)^2+z^2}\,\right]\right] = \tan^{-1}\left[\sqrt{ch\xi-\cos u}\,/-\sqrt{2}\cos(u/2)\right]$$

But recall our result

$$V(\xi,u) = (2V_0/\pi)\tan^{-1}\left[-\frac{\sqrt{ch\xi-\cos u}}{\sqrt{2}\cos(u/2)}\right]$$

So things agree!!!

The limit $u_0 \rightarrow 0$? Spherical bowl.

$$V(\xi,u) = (V_0/\pi)\ \sqrt{ch\xi - \cos u}$$

$$\left\{\ \frac{1}{\sqrt{ch\xi-\cos u}}\ \cot^{-1}\left[\frac{\sqrt{2}\cos(\pi-u/2)}{\sqrt{ch\xi-\cos u}}\right] + \frac{1}{\sqrt{ch\xi-\cos u}}\ \cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}}\right]\ \right\}$$

$$= (V_0/\pi)\ \left\{\cot^{-1}\left[\frac{\sqrt{2}\cos(\pi-u/2)}{\sqrt{ch\xi-\cos u}}\right] + \cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}}\right]\right\}$$

$$= (V_0/\pi)\ \left\{\cot^{-1}\left[-\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}}\right] + \cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}}\right]\right\}$$

$$= (V_0/\pi)\ \left\{\cot^{-1}(-x) + \cot^{-1}(x)\right\} = (V_0/\pi)\ \{\pi\} = V_0$$