P and Q functions for large ν PhL 12.10.15

Now in Appendix H of bowl doc 12.31.15
I was still using the problematic Q function at this point, causing breakups and various problems as you see looking at what is below.

Motivation: I want to know how many terms you should keep in toroidal sums for potential and capacitance. These equations are these:

$$V(\xi,u) = V_0 \frac{\sqrt{2}}{\pi} \sqrt{ch\xi - cosu}\ \Sigma_{n=0}^{\infty} \varepsilon_n P_{n-1/2}(ch\xi) \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \cos(nu) \quad , \qquad (10.1.11)$$

$$C = (2a/\pi) \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \qquad \varepsilon_n = 2-\delta_{n,0} \ . \qquad (10.2.3)$$

So today at least I am only interested in $z \geq 1$.

Tools: I have these home-grown P and Q functions

For P, we select Bateman (24) which says this

$P_\nu(z) = z^\nu F(-\nu/2, 1/2-\nu/2; 1; 1-1/z^2)$

$P(\nu,\xi) \equiv P_\nu(ch\xi) = (ch\xi)^\nu F(-\nu/2, 1/2-\nu/2; 1; th^2\xi)$

For Q, we select Bateman (36) which says this

$Q_\nu(z) = 2^\nu [\Gamma(1+\nu)]^2 (z+1)^{-\nu-1}/\Gamma(2+2\nu) * F(1+\nu,1+\nu,2+2\nu, 2/(1+z)$

$Q(\nu,\xi) = Q_\nu(ch\xi) = 2^\nu [\Gamma(1+\nu)]^2 (ch\xi+1)^{-\nu-1}/\Gamma(2+2\nu) * F(1+\nu,1+\nu,2+2\nu, 2/(1+ch\xi)$

Comment: Since ν is not isolated into the c argument of F(a,b,c;z), these are not natural forms with which to study large ν limits. Still, to the extent that Maple implements F(a,b,c;z) correctly, these functions should be accurate for all values of ν and ξ. It is hard to imagine they suddenly become inaccurate if you take say ν > 10.

Theory Side. I start with this AS result ("as $\nu \to \infty$")

14.15.13
$$P_\nu^{-\mu}(\cosh\xi) = \frac{1}{\nu^\mu}\left(\frac{\xi}{\sinh\xi}\right)^{1/2} I_\mu\left(\left(\nu+\tfrac{1}{2}\right)\xi\right)\left(1+O\left(\frac{1}{\nu}\right)\right),$$

14.15.14
$$Q_\nu^\mu(\cosh\xi) = \frac{\nu^\mu}{\Gamma(\nu+\mu+1)}\left(\frac{\xi}{\sinh\xi}\right)^{1/2} K_\mu\left(\left(\nu+\tfrac{1}{2}\right)\xi\right)\left(1+O\left(\frac{1}{\nu}\right)\right),$$

uniformly for $\xi \in (0,\infty)$.

Supposedly OK for $\mu \geq 0$ and I only care about $\mu = 0$. The I and K are Bessel functions. I think the above come from work of Older.

Experiment #1. Let's test the above claims in Maple.

BesselI(v, x)
BesselK(v, x)

So let's construct the right sides and compare to the left sides for large ν. I will call the right sides of the above equations P1 and Q1, so keeping just the first terms,

$$P1_\nu(ch\xi) = \sqrt{\frac{\xi}{sh\xi}}\ I_0[(\nu+1/2)\xi] \qquad (14.15.13)_{AS}$$

$$Q1_\nu(ch\xi) = \sqrt{\frac{\xi}{sh\xi}}\ K_0[(\nu+1/2)\xi] / \Gamma(\nu+1) \qquad (14.15.14)_{AS}$$

I enter these as follows

```
P1 := (nu,xi) -> sqrt(xi/sinh(xi))*BesselI(0,(nu+1/2)*xi);
```

$$P1 := (\nu, \xi) \to \sqrt{\frac{\xi}{\sinh(\xi)}}\ \text{BesselI}\left(0, \left(\nu + \frac{1}{2}\right)\xi\right)$$

```
Q1 := (nu,xi) -> sqrt(xi/sinh(xi))*BesselK(0,(nu+1/2)*xi)/GAMMA(nu+1);
```

$$Q1 := (\nu, \xi) \to \frac{\sqrt{\frac{\xi}{\sinh(\xi)}}\ \text{BesselK}\left(0, \left(\nu + \frac{1}{2}\right)\xi\right)}{\Gamma(1+\nu)}$$

and verify that things are entered correctly. I then evaluate

```
P(10,0.5);
```

33.44231837

```
P1(10,0.5);
```

33.38357430

```
Q(10,0.5);
```

.002744927469

```
Q1(10,0.5);
```

$.7580347605\ 10^{-9}$

The P look good, the Q look suspicious. I raise ν from 10 to 20

```
P(20,0.5);
```

3500.192513

```
P1(20,0.5);
```

3496.865894

```
Q(20,0.5);
```

.00001338724245

```
Q1(20,0.5);
```

$.5508337527\ 10^{-23}$

So without going any further, I know that something is wrong with $(14.15.14)_{AS}$ for Q. Now I notice that this Q is bolded and in italics. What does this mean? First at the very start of the AS chapter I see this

$\mathbf{F}(a,b;c;z)$ Olver's scaled hypergeometric function:
$F(a,b;c;z)/\Gamma(c)$.

and

14.3.10

$$\boldsymbol{Q}_\nu^\mu(x) = e^{-\mu\pi i}\frac{Q_\nu^\mu(x)}{\Gamma(\nu+\mu+1)}.$$

Well there it is! AS are using

$\boldsymbol{Q}_\nu(z) = Q_\nu(z)/\Gamma(\nu+1)$

I don't care about bolded Q functions, so I will rework things above:

So let's construct the right sides and compare to the left sides for large ν. I will call the right sides of the above equations P1 and Q1, so keeping just the first terms,

$$P1_\nu(ch\xi) = \sqrt{\frac{\xi}{sh\xi}}\ I_0[(\nu+1/2)\xi] \qquad (14.15.13)_{AS}$$

$$Q1_\nu(ch\xi) = \sqrt{\frac{\xi}{sh\xi}}\ K_0[(\nu+1/2)\xi] \qquad (14.15.14)_{AS}$$

I enter these as follows

```
P(20,0.5);
```

3500.192513

```
P1(20,0.5);
```

3496.865894

```
Q(20,0.5);
```

.00001338724245

```
Q1(20,0.5);
```

.00001340124543

and the **train is now back on the track**!!!

I updated H.5 with this fix, and now my claim is this

$$P_{n-1/2}(ch\xi)\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} = \sqrt{\frac{1}{sh\xi}}\ e^{-n(2\xi_0-\xi)} \quad \text{for large n.}$$

Let's test this equation in Maple ! I define

```
F := (n,xi,xi0) -> P(n-1/2,xi)* Q(n-1/2,xi0)/ P(n-1/2,xi0);
```

$$F := (n, \xi, \xi 0) \rightarrow \frac{P\left(n-\frac{1}{2}, \xi\right) Q\left(n-\frac{1}{2}, \xi 0\right)}{P\left(n-\frac{1}{2}, \xi 0\right)}$$

```
G := sqrt(1/sinh(xi))*exp(-n*(2*xi0-xi));
```

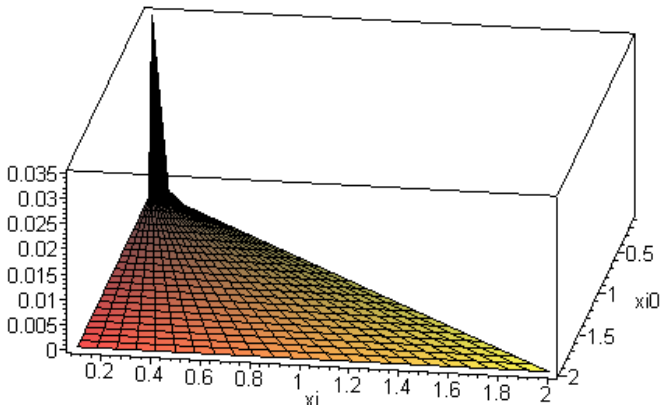
$$G := \sqrt{\frac{1}{\sinh(\xi)}}\, e^{(-n\,(2\,\xi 0-\xi))}$$

I now compare these two expressions for a range of ξ and ξ_0 values but with $\xi < \xi_0$ since that will always be the case for my toroid plot -- only care about region outside the torus.

Case ξ_0 = 0.1

Here is what I find with this plot command where I start the ξ_0 scan at 0.1 which is fairly small value:

plot3d(G-F, xi0 = 0.1..2, xi = 0.1..xi0, axes = boxed);



With n = 10, the max difference G-F occurs at the corner where xi = xi0 and is about 0.7

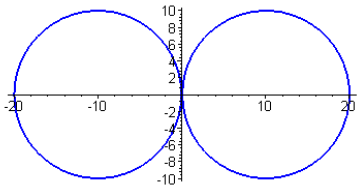
With n = 20, difference drops to 0.3

With n = 30, difference drops to 0.1

With n = 40, difference drops to 0.035

Above n = 40 I start getting anomalies. So the above as about as good as I can do.

Here is the 0.1 case



so the toroid is very close to being "degenerate".

Conclusion: For n = 40 I can get the difference between the exact function G and the asymptotic limit F to be as small as about .04 for the challenging $\xi_0 = 0.1$ value.

Case $\xi_0 = 1$

For a more typical case with $\xi_0 = 1$ here is what happens:

With n = 1, difference drops to - 3 x 10^{-2}
With n = 2, difference drops to 2 x 10^{-2}
With n = 3, difference drops to 10^{-2}
With n = 5, difference drops to 10^{-3}
With n = 10, difference drops to 10^{-5}
With n = 20, difference drops to 10^{-9}
With n = 30, difference drops to 10^{-13}
With n = 40, difference drops to 10^{-18}
With n = 50, difference drops to 10^{-22}
With n = 60, difference drops to 10^{-26}

So in this case, I have unlimited accuracy, and 5 terms is just fine for a plot.

Restatement. I know from doing many plots that that the worst case error in F - G always occurs when $\xi = \xi_0$. But in that case we have this equation

$$P_{n-1/2}(ch\xi)\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} = \sqrt{\frac{1}{sh\xi}}\, e^{-n(2\xi_0-\xi)} \quad \text{for large n.}$$

becomes this equation

$$Q_{n-1/2}(ch\xi_0) = \sqrt{\frac{1}{sh\xi_0}}\, e^{-n\xi_0}$$

So then the question is this: how close are these two functions for values of ξ_0 and n ?

Now simple 1D plots are fine. For example,

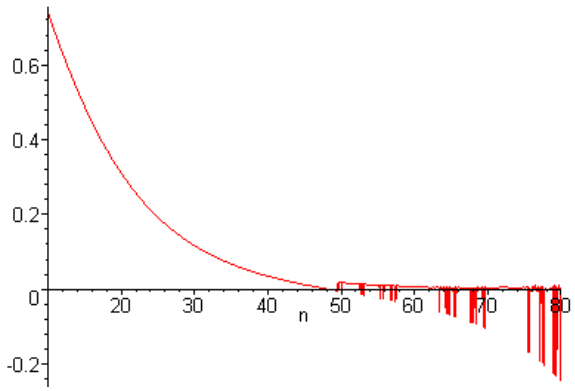
```
F :=  Q(n-1/2,xi0):
G := sqrt(1/sinh(xi0))*exp(-n*xi0);
```

$$G := \sqrt{\frac{1}{\sinh(\xi 0)}}\, e^{(-n\,\xi 0)}$$

```
xi0 := 0.1;
```

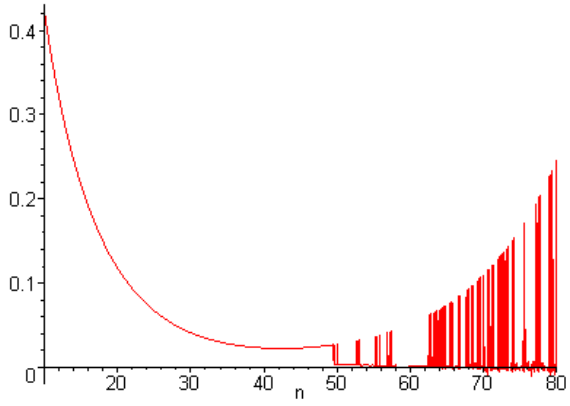
$$\xi 0 := .1$$

```
plot(G-F,n=10..80);
```



Here you see the error going down nicely to about n = 45, but then Maple has problems of some sort. Here is a plot of $Q_{n-1/2}(ch\xi_0)$ rather than the difference

```
plot(F,n=10..80);
```

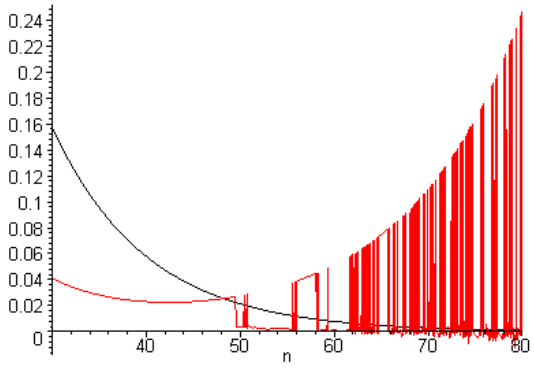


What is this plot telling us?? In theory, we know that

$$Q_{n-1/2}(ch\xi_0) \rightarrow \sqrt{\frac{1}{sh\xi_0}}\; e^{-n\xi_0}$$

Here I plot the two curves separately,

```
plot([F,G],n=30..80, color = [red,black]);
```



The black curve is what we know must happen in theory, whereas the red curve is what happens in Maple.

Explanation #1. Go back to

For Q, we select Bateman (36) which says this

$$Q_\nu(z) = 2^\nu\,[\Gamma(1+\nu)]^2\,(z+1)^{-\nu-1}/\Gamma(2+2\nu) * F(1+\nu,1+\nu,2+2\nu,\ 2/(1+z)$$

$$Q(\nu,\xi) = Q_\nu(ch\xi) = 2^\nu\,[\Gamma(1+\nu)]^2\,(ch\xi+1)^{-\nu-1}/\Gamma(2+2\nu) * F(1+\nu,1+\nu,2+2\nu,\ 2/(1+ch\xi)$$

How does Maple actually compute such a thing? I don't really know, but I can consider the Q function above to be the product of two factors: F, and everything else, which I will call H. Now let's see how Maple does with this H function,

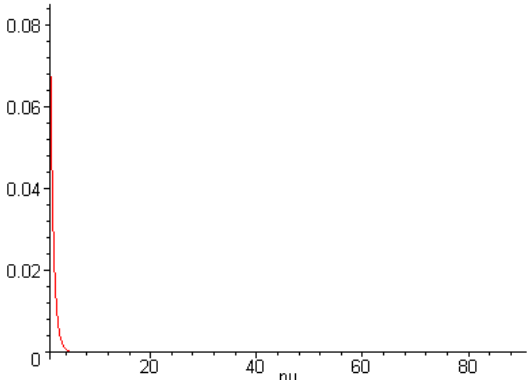
```
H := (nu,xi) -> evalf(2^(nu)*GAMMA(1+nu)^2*(cosh(xi)+1)^(-nu-1)*(1/GAMMA(2+2*nu)));
```

$$H := (\nu, \xi) \rightarrow \mathrm{evalf}\left(\frac{2^\nu\,\Gamma(1+\nu)^2\,(\cosh(\xi)+1)^{(-\nu-1)}}{\Gamma(2+2\nu)}\right)$$

```
xi := 0;
```

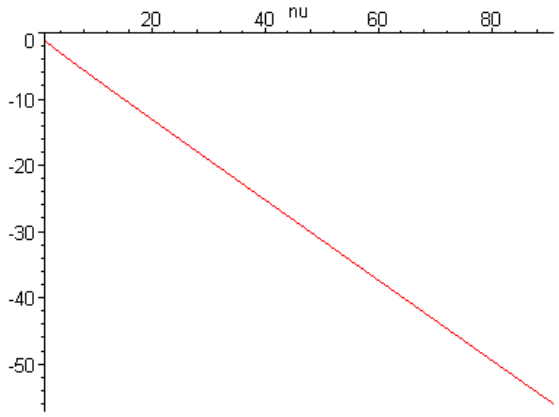
$$\xi := 0$$

```
plot(H(nu,xi),nu = 1..91);
```



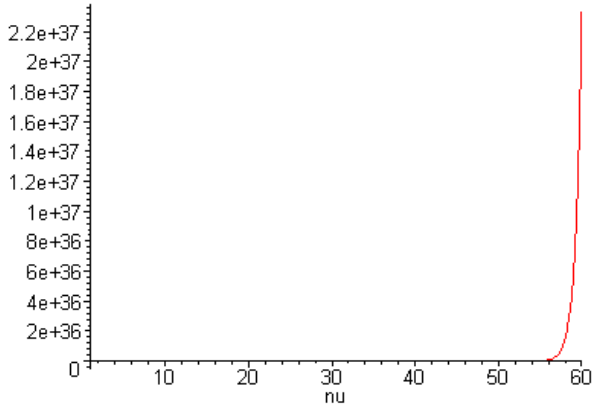
The value of ξ does not matter for H, since ch(ξ) → 1.What we find is that for n up to 91, Maple seems able to compute the H function. Here is a log plot

```
plot(log10(H(nu,xi)),nu = 1..91);
```



If cannot plot beyond n = 91! How consider this situation,

```
plot(1/H(nu,xi),nu = 1..60);
```



I think Maple has a max and min FP number which it can handle, and we are getting close here. I forget where this is discussed in Maple help. I do have this

```
1/H(1000,0);
```

$$.8196702808\ 10^{604}$$

I remember now. This came up in one of my docs where it is a plotting problem only.

"Maple is uncomfortable plotting numbers larger than about 10^{40} so we pre-scale Ω down by 10^{1519} to make the scaled Ω be Maple-digestible. This scaling process in turn creates numbers smaller than 10^{-40} which are also rejected by the Maple plotter, so we use a Heaviside function to pin small numbers to 0 "

from Lagrange doc.

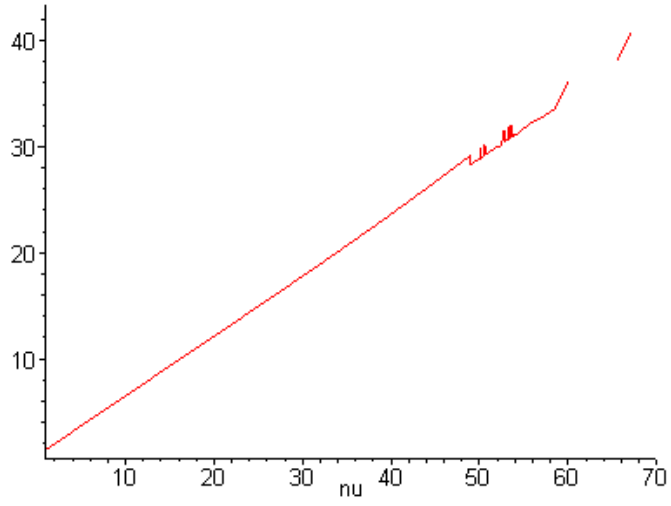
Conclusion: I don't think Maple has any problem computing the H part of the Q function, even up to n = 1000 as you see. So the problem is, not surprisingly, with the F part of the function.

Here is a plot

```
F := (nu,xi) -> hypergeom([1+nu,1+nu],[2+2*nu],2/(1+cosh(xi)));
```

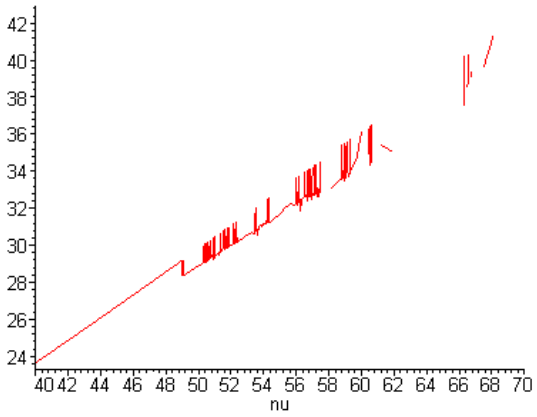
$$F := (\nu, \xi) \rightarrow \text{hypergeom}\left([1+\nu, 1+\nu], [2+2\nu], 2\,\frac{1}{\cosh(\xi)+1}\right)$$

```
plot(log10(F(nu,0.1)),nu=1..70);
```



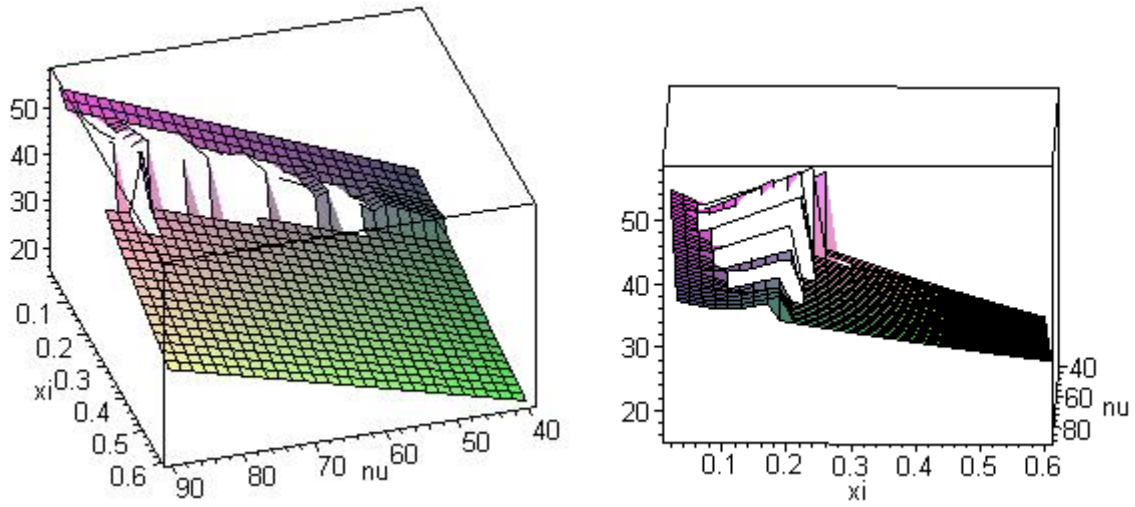
You see that trouble starts brewing around n = 50 when $\xi = 0.1$. When $\xi = 1$ the above plot is clean. It seems clean all the way with $\xi = .01$ up to $\xi = .08$, but then trouble again at $\xi = .09$. Has trouble at $\xi = 0.2$. So something is "spotty" in the way Maple computes this F function!

```
plot(log10(F(nu,0.1)),nu=40..70);
```



Here is another way to look at the problem.

```
plot3d(log10(F(nu,xi)),nu = 40..90, xi = 0.02..0.6, axes = boxed, grid = [30,30])
```



Here you see the breakup in the region above n = 40 and in the small ξ region.

Raising Digits to 30 has no effect on this breakup.

There is nothing I can do about this.

Fact: As you increase n in the high range further, the problems with ξ move higher.

For example, with n = 40 you are OK with $\xi > .25$, but at n = 90 you break up there.

New Question: Does our series

$$\Sigma_{n=0}^{\infty} \varepsilon_n P_{n-1/2}(ch\xi) \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \cos(nu)$$

have the property that if you truncate somewhere, the error is less in magnitude than the last term.! I recall a theorem from Buck on this which I will go and find right now and quote here: Oops, I see that rule only applies to an alternating series.

Is our series monotonically decreasing for any compatible pair ξ and ξ_0?