Bug found with σ on the bowl. PhL 1.27.15

I had trouble here with the disk limit of σ_{in} and this problem is explored more in v2 of this doc with the same title. I failed to realize that you have to handle R→∞ at the same time. I then went off pondering whether my whole understanding of σ is correct! All resolved now.

Here is the problem. The first question is OK, but not the second.

Question 1: I say that σ is infinite at the bowl edge. How does that appear in my result? I never checked on that.

$$\sigma_+ = \sigma_{in} = (V_0/4\pi^2 R)\,[\quad (B/A) - \tan^{-1}(B/A)]$$

$$A = \sqrt{2}\cos(u_0/2) \text{ and } B = \sqrt{ch\xi - cosu_0}$$

At the bowl edge, we are at the focal point so we have $\xi \to +\infty$. Then $B \to \infty$ and

$$\sigma_{in}(\text{near edge}) = (V_0/4\pi^2 R)\,[\quad (B/A) - \pi/2\,] \approx \quad (V_0/4\pi^2 R)\,(B/A)$$

Now $B/A = (ch\xi)^{1/2}/[\sqrt{2}\cos(u_0/2)]$ so the answer close to the edge is

$$\sigma_{in}(\text{near edge}) = (V_0/4\pi^2 R)\,(ch\xi)^{1/2}/\,[\sqrt{2}\cos(u_0/2)]$$

which does indeed blow up as claimed as $\xi \to \infty$.

Question 2. How about at the bowl center? There $\xi \to 0$ and we get $B = \sqrt{ch\xi - cosu_0} = \sqrt{1 - cosu_0}$. Then

$$B/A = \sqrt{1 - cosu_0}\,/\,[\sqrt{2}\cos(u_0/2)\,]$$

Now $\sqrt{1 - cosu_0} = \sqrt{2}\,\sin(u_0/2)$

$$B/A = \sqrt{2}\,\sin(u_0/2)/\,[\sqrt{2}\cos(u_0/2)\,] = \tan(u_0/2)$$

and then we have

$$\sigma_{in}(\text{at center}) = (V_0/4\pi^2 R)\,[\quad \tan(u_0/2) - \tan^{-1}(\tan(u_0/2))]$$

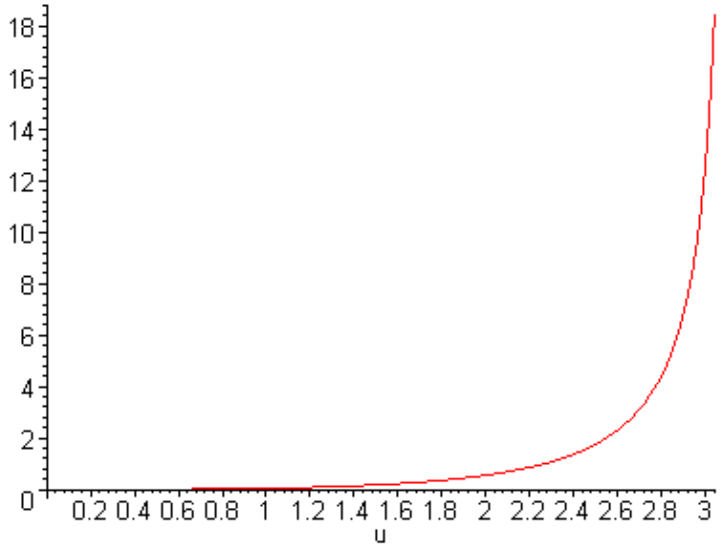
$$= (V_0/4\pi^2 R)\,[\,\tan(u_0/2) - u_0/2]$$

Consider the sequence of upper bowls from large to small so $u_0 \to \pi$ from below.

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f := tan(u/2)-u/2;
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$$f := \tan\left(\frac{1}{2}u\right) - \frac{1}{2}u$$

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plot(f,u=0..Pi-.1);
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This says that σ_{in} (and σ_{out} which differs by a constant) blows up for the flat disk at disk center! This result I am sure is wrong. So we have a new problem.

Time to check my calculation of σ starting with (4.1). I need to show that it replicates the flat disk charge distribution

Question 3: Let's start with the full σ_{in} result and take it to the flat disk limit. Here is my full result

$$\sigma_+ = \sigma_{in} = (V_0/4\pi^2 R)\,[\quad (B/A) - \tan^{-1}(B/A)]$$

$$A = \sqrt{2}\cos(u_0/2) \text{ and } B = \sqrt{ch\xi - cosu_0}$$

The flat disk has $u_0 = \pi$ and then

$$A = \sqrt{2}\cos(\pi/2) = 0 \qquad B = \sqrt{ch\xi + 1}$$

The limit then says

$$\sigma_+ = \sigma_{in} = (V_0/4\pi^2 R)\,[\quad (B/0) - \tan^{-1}(B/0)]$$

$$= (V_0/4\pi^2 R)\,[\quad \infty - \pi/2]$$

So something is WRONG with my σ result! I should have checked this limit when I wrote the doc. Stupid. [again I overlook the fact that $R \to \infty$ in the disk limit]

Check the calculation of σ

I start with

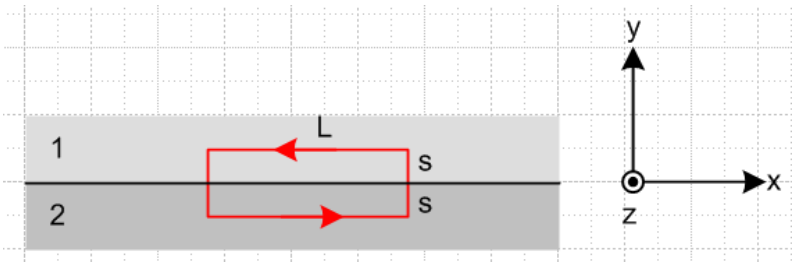
$$V(\xi,u) = V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-u)\tau]/\ ch^2(\pi\tau) \qquad (2.8)$$

$$\sigma_+ = -(1/4\pi)\ (1/h_u)\ \partial_u V(\xi,u)|^{u=u0} \qquad 1/h_u = (ch\xi - cosu)/a$$

Where do I get this second line from? No reference is given. Is this result in any of my web docs? I do have this in SI units,

$$D_{n1} - D_{n2} = n_{free} \qquad \text{or} \qquad [\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}] = n_{free} \qquad (1.1.47)$$

Consider a boundary like this



Suppose 2 = metal and 1 = vacuum. We then have

$$[\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}] = n_{free} \quad \rightarrow \quad \varepsilon_0 E_{1n} = n_{free}$$

Translate this to say

$$\sigma = \varepsilon_0 E_n \qquad \text{SI units}$$

So far so good. Meanwhile,

$$\mathbf{E} = -\ \text{grad}\ V$$

We look this up in tensor doc where we have

$$[\text{grad } f](\mathbf{x}) = (\partial_i f)\ \mathbf{e}^i$$
$$[\text{grad } f](\mathbf{x}) = (\partial^i f)\ \mathbf{e}_i = (1/h_i^2)\ (\partial_i f)\ \mathbf{e}_i$$
$$[\text{grad } f](\mathbf{x}) = h_i(\partial^i f)\ \hat{\mathbf{e}}_i = (1/h_i)\ (\partial_i f)\ \hat{\mathbf{e}} \qquad \text{// M\&S 1.05} \qquad (10.1.17)$$

or

$$[\text{grad } V](\mathbf{x}) = (\partial_i V)\,\mathbf{e}^i$$
$$[\text{grad } V](\mathbf{x}) = (\partial^i V)\,\mathbf{e}_i \quad = (1/h_i^2)\,(\partial_i V)\,\mathbf{e}_i$$
$$[\text{grad } V](\mathbf{x}) = h_i(\partial^i V)\,\hat{\mathbf{e}}_i = (1/h_i)\,(\partial_i V)\,\hat{\mathbf{e}}_i \qquad \text{// M\&S 1.05} \qquad (10.1.17)$$

where italic V means it is a function of the curvilinear coordinates. Assume that $\hat{\mathbf{e}}_i$ is a unit vector in curvilinear coordinates which is normal to our surface of interest, which would be u for our case study. Then

$$[\text{grad } V](\mathbf{x}) = (1/h_u)\,(\partial_u V)\,\hat{\mathbf{e}}_u$$

Then

$$\mathbf{E} = -\text{ grad } V \Rightarrow \quad E_u\hat{\mathbf{e}}_u = -(1/h_u)\,(\partial_u V)\,\hat{\mathbf{e}}_u \quad \Rightarrow \quad E_u = -(1/h_u)\,(\partial_u V)$$

and then finally

$$\sigma = \varepsilon_0 E_u = -\varepsilon_0\,(1/h_u)\,(\partial_u V)$$

and then divide by $4\pi\varepsilon_0$ to get cgs units

$$\sigma = \varepsilon_0 E_u = -(1/4\pi)\,(1/h_u)\,(\partial_u V)$$

and this is what I used.

Question: But there is something unclear to me here. Go back to

$$[\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}] = n_{free}$$

Are these E-field component covariant or contravariant !!! Go back to the derivation (where I have fixed the bad reference which was 1.1.13),

$$\text{div } \mathbf{D} = \rho_{free} \quad \Leftrightarrow \quad \int_V \rho_{free}\, dV = \int_S \mathbf{D} \bullet d\mathbf{S} \qquad \textit{(1.1.32)}$$

Now how do you do this in covariant notation?

$$\mathbf{D} \bullet d\mathbf{S} = D_i dS^i = D^i dS_i$$

My lines doc does not address this question. In the n-piped discussion I do mention area with covariant index stuff, and there fore example I have

$$d\mathbf{A}'^n = J^2(\Pi_{i\neq n} dx'^i)\,\mathbf{e}'^n \qquad \text{since } (\mathbf{e}'^n)_i = \delta^n{}_i \quad \textit{(7.18.1)} \qquad (8.4.c.5)$$

where we are in x'-space. I show in toroidals.doc that

$$J = h_\xi h_u h_\varphi = \frac{a^3 sh\xi}{(ch\xi - cosu)^3}$$

If I then set n = 2 for the u coordinate, we get

$$d\mathbf{A}'^n = J^2(\Pi_{i\neq n} dx'^i)\, \mathbf{e}'^n$$

$$d\mathbf{A}^u = J^2\, d\xi d\varphi\, \mathbf{e}^u$$

$$d\mathbf{S}^u = J^2\, d\xi d\varphi\, \mathbf{e}^u$$

Then we have

$$\mathbf{D} \bullet d\mathbf{S} = (\, D^u\, \mathbf{e}_u) \bullet (\, J^2\, d\xi d\varphi\, \mathbf{e}^u) = D^u\, J^2\, d\xi d\varphi$$

so it is the contravariant component ! Then we really have

$$[\varepsilon_1 E^n{}_1 - \varepsilon_2 E^n{}_2] = n_{free}$$

$$[\varepsilon_1 E^u{}_1 - \varepsilon_2 E^u{}_2] = \sigma$$

$$\varepsilon_o E^u{}_1 = \sigma$$

We now trace through the above discussion.

$$[\text{grad } V](\mathbf{x}) = (\partial_i V)\, \mathbf{e}^i$$
$$[\text{grad } V](\mathbf{x}) = (\partial^i V)\, \mathbf{e}_i = (1/h_i{}^2)\, (\partial_i V)\, \mathbf{e}_i$$
$$[\text{grad } V](\mathbf{x}) = h_i (\partial^i V)\, \hat{\mathbf{e}}_i = (1/h_i)\, (\partial_i V)\, \hat{\mathbf{e}}_i \qquad \text{// M\&S 1.05} \qquad (10.1.17)$$

$$\mathbf{E} = -\text{ grad } V \Rightarrow \quad E_u \hat{\mathbf{e}}_u = -\,(1/h_u)\, (\partial_u V)\, \hat{\mathbf{e}}_u \quad \Rightarrow \quad E_u = -\,(1/h_u)\, (\partial_u V)$$

$$E^u = -\,(1/h_u)\, (\partial^u V)$$

Then

$$\sigma = \varepsilon_o E^u{}_1 = -\,\varepsilon_0\, (1/h_u)(\partial^u V) \qquad \text{// with an upper index!}$$

So my document should say not this

$$\sigma_+ = -\,(1/4\pi)\, (1/h_u)\, \partial_u V(\xi,u)|^{u=u0} \qquad 1/h_u = (ch\xi - cosu)/a$$
$$\sigma_- = +(1/4\pi)\, (1/h_u)\, \partial_u V(\xi,u)|^{u=u0+2\pi} \qquad \text{// cgs units selected here} \qquad (4.1)$$

but this

$\sigma_+ = -(1/4\pi)\,(1/h_u)\;\partial^u V(\xi,u)|^{u=u0}$ $\qquad 1/h_u = (ch\xi - cosu)/a$

$\sigma_- = +(1/4\pi)\,(1/h_u)\;\partial^u V(\xi,u)|^{u=u0+2\pi}$ $\qquad$ // cgs units selected here $\qquad$ (4.1)

So this is the first thing I have found to be done wrong. Now we have

$$\partial_i = \Sigma_j\, g_{ij}\partial^j \;=\; g_{ii}\partial^i$$

For i = 2 = u this says

$$\partial_u = g_{uu}\partial^i = h_u^2\,\partial^u$$

$$\partial^u = h_u^{-2}\partial_u$$

Then I would seem to get

$\sigma_+ = -(1/4\pi)\,(1/h_u)^3\;\partial_u V(\xi,u)|^{u=u0}$ $\qquad 1/h_u = (ch\xi - cosu)/a$

$\sigma_- = +(1/4\pi)\,(1/h_u)^3\;\partial_u V(\xi,u)|^{u=u0+2\pi}$ $\qquad$ // cgs units selected here $\qquad$ (4.1)

Basically this adds an extra factor of $(1/h_u)^2$ to my charge densities. Note that

$$1/h_u = (ch\xi - cosu)/a$$

$$(1/h_u)^2 = (ch\xi - cosu)^2/a^2 = B^4/a^2$$

Then my new charge density results (if everything ELSE was done correctly) would be

$$\sigma_+ = \sigma_{in} = (\,V_0/4\pi^2R)\,[\;(B/A) - \tan^{-1}(B/A)]\;(B^4/a^2)$$

$$A = \sqrt{2}\,\cos(u_0/2) \;\text{ and }\; B = \sqrt{ch\xi - cosu_0} \qquad B^2 = ch\xi - cosu_0$$

How does this change the limit $u_0 \neq \pi$? The problem remains unrepaired! $A \to 0$ and $\sigma \to \infty$ for any ξ.

I have some major problems here. I will respond to the guy first, and then tackle these problems. This might result in a major rewrite of bowl doc!