Bug found with σ on the bowl. PhL 1.27.15

see red text below

I was getting wrong results for both the full sphere limit of σ_{in} and for the disk limit as well! I started to doubt my general formula for σ_{in}. But it was just simple mistakes as noted in red below. These limits are now properly done in bowl doc Section 4.3.

(a) The known disk result

The result, apart from constant overall factors, is this (both sides)

$\sigma(\rho) = (1/\pi^2)\,(a-\rho^2)^{-1/2}$ Sneddon page 64 (3.1.7). Also quoted in bowl doc App C.

Look now at the toroidal coordinates,

$$x = a\cos\varphi\,\mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u) \qquad \rho = a\,\mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u) = \sqrt{x^2+y^2}$$
$$y = a\sin\varphi\,\mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u) \qquad h_\xi = h_u = a/(\mathrm{ch}\xi - \cos u)$$
$$z = a\sin u/(\mathrm{ch}\xi - \cos u) \qquad h_\varphi = a\,\mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u) \qquad (5.1)$$

The disk limit is supposed to be $u_0 \to \pi$ in my version of the coordinates, where u_0 is a bowl label.

If $u = \pi$ then $\cos(u) = -1$ and we have

$\rho = a\,\mathrm{sh}\xi/(\mathrm{ch}\xi + 1)$. ok

Then the disk needs

$$a^2 - \rho^2 = a^2[\,1 - \mathrm{sh}^2\xi/(\mathrm{ch}\xi+1)^2] = a^2[\,(\mathrm{ch}\xi+1)^2 - \mathrm{sh}^2\xi\,]/(\mathrm{ch}\xi+1)^2$$

Now,

$[\,(\mathrm{ch}\xi+1)^2 - \mathrm{sh}^2\xi\,] = \mathrm{ch}^2\xi + 2\mathrm{ch}\xi + 1 - \mathrm{sh}^2\xi = 1 + 2\mathrm{ch}\xi + 1 = 2(1+\mathrm{ch}\xi)$ ok

so

$a^2 - \rho^2 = a^2\,2(1+\mathrm{ch}\xi)/(\mathrm{ch}\xi+1)^2 = 2a^2/(\mathrm{ch}\xi+1)$ ok and Maple agrees.

Then the desired limit should be this

$$\sigma(\rho) = (1/\pi^2)\,(a^2-\rho^2)^{-1/2} = (1/\pi^2)\,[2a^2/(\mathrm{ch}\xi+1)]^{-1/2} = (1/\pi^2)\sqrt{\mathrm{ch}\xi+1}\,/(\sqrt{2}\,a)$$

$$= (1/[\pi^2 a\sqrt{2}\,])\sqrt{\mathrm{ch}\xi+1} = \sqrt{\mathrm{ch}\xi+1}\,/(\sqrt{2}\,\pi^2 a) \quad \text{ok}$$

This is for a unit potential, so with potential V_0 we would get

$\sigma(\rho) = V_0\ \sqrt{\mathrm{ch}\xi+1}\,/(\sqrt{2}\,\pi^2 a)$ ok

This is the total charge density on the plane z = 0 for the disk based on Sneddon. Units are correct because dim(V) = q/L and so σ is q/L^2 as shown above.

Meanwhile, **Jackson** green page 93 says this for two sides

$\sigma_{bothsides} = (q/2\pi a)(a^2-\rho^2)^{-1/2}$ and $(q/a) = 2V/\pi$ from page 92, so then

$\sigma_{both} = (1/2\pi)(2V/\pi)(a^2-\rho^2)^{-1/2} = (V/\pi^2)(a^2-\rho^2)^{-1/2}$ // Jackson's claim, both sides

Recall that Sneddon above says this for two sides

$\sigma(\rho) = (V/\pi^2)(a-\rho^2)^{-1/2}$ // Sneddon's claim, both sides

My claim is this both sides, just expressed in toroidal coordinates, derived from the above

$\sigma(\rho) = V_0 \sqrt{ch\xi+1} / (\sqrt{2}\,\pi^2 a)$ // either in T.C, both sides

How about a units check. Since V = q/r in cgs, dim(V) = q/L. Then Jackson's result is q/L^2 which is correct.

My prediction for one side would then be

$\sigma_{in}(\rho) = V_0 \sqrt{ch\xi+1} / (2\sqrt{2}\,\pi^2 a)$ ok

At the center of the disk you have ξ = 0 which gives a finite value at the center of $\sigma_{in}(0) = (1/\pi^2)\sqrt{2}/a$.
At the edge of the disk you have $\xi \to +\infty$ and so σ does diverge as $\sqrt{ch\xi}$

So this is our desired limit at $u_0 \to \pi$.

STOP: I thought we had this rule that the two sides differ by some amount. That amount is this

difference = $V_0/(4\pi R)$ but $R \to \infty$ so the difference is 0

Note added 12/31/5. Here is an interesting question:
(1) the σ's on the two sides of a bowl are supposed to differ by a constant amount
(2) but the σ's on the two sides of a disk (bowl in limit) are the same!
How can this be?

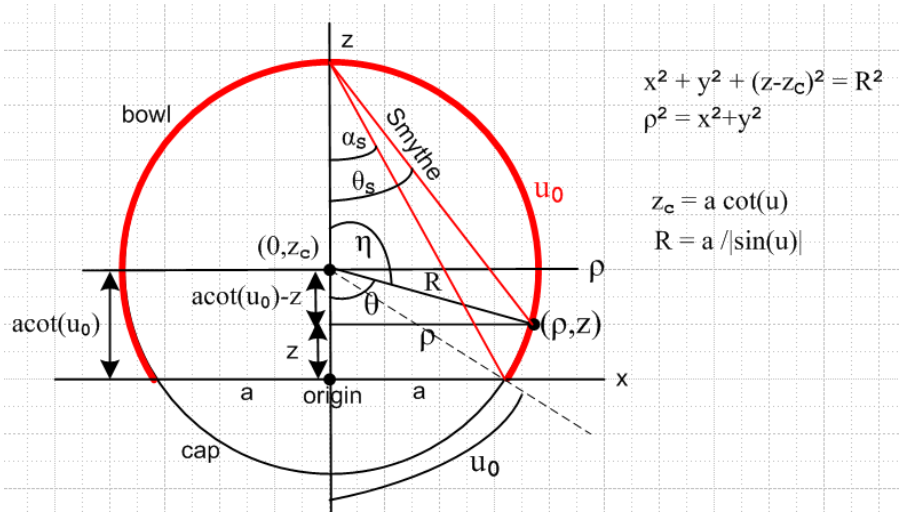
Resolution: The amount they differ by is $V_0/(4\pi R)$ and $R = \infty$ for the disk.

(a1) Limit of my bowl doc results for the full sphere

Here is my original

$$\sigma_{in} = (V_0/4\pi^2R) \{ \sqrt{ch\xi - cosu_0} / (\sqrt{2}\cos(u_0/2)) - \tan^{-1}[\sqrt{ch\xi - cosu_0} / (\sqrt{2}\cos(u_0/2))] \} \qquad (4.7)$$

First, let's make sure the full sphere is OK, which is the limit $u_0 \to 0$



Then $\cos(u_0) = 1$ and $\cos(u_0/2) = 1$ and we get $z_c = a \cot(u_0)$

$$\sigma_{in} = (V_0/4\pi^2R) \{ \sqrt{ch\xi - 1} /\sqrt{2} - \tan^{-1}[\sqrt{ch\xi - 1} /\sqrt{2}] \} \qquad (4.7) \qquad R = a / |\sin(u_0)|$$

STOP. I thought this was supposed to give 0! Both this limit and the disk one below are screwing up, what is going on here? I triple confirmed that the σ_{in} results agrees with Lebedev, Kelvin and Smythe!

I failed to realize here that $\sqrt{ch\xi - 1} \to 0$ in the full sphere limit.

Let's try the Kelvin form of the result in the full sphere limit of $u_0 \to 0$:

$$\sigma_K = (V_0/4\pi^2R) \{ \sqrt{2}\sin(u_0/2)/\sqrt{cosu_0 - cos\theta} - \tan^{-1}[\sqrt{2}\sin(u_0/2)/\sqrt{cosu_0 - cos\theta}] \} \qquad (4.8)$$

$$\sigma_K = (V_0/4\pi^2R) \{ \sqrt{2}\, 0 /\sqrt{1 - cos\theta} - \tan^{-1}[0/\sqrt{cosu_0 - cos\theta}] = 0 + 0 + 0$$

Huh? Look at our **little relation thing**,

$$\sqrt{ch\xi - cosu_0} / (\sqrt{2}\cos(u_0/2)) = \sqrt{2}\sin(u_0/2)/\sqrt{cosu_0 - cos\theta} . \qquad (4.9)$$

Take the limit $u_0 \to 0$ to get

$$\sqrt{ch\xi - 1} / (\sqrt{2}) = \sqrt{2}\, 0/\sqrt{1 - cos\theta} \qquad ??$$

So this relation is failing in this limit!!!!! This is very strange. But look at this equation

$R = a/\sin(u_0)$.

So to maintain radius R, you have also to take $a \to 0$. OK, I will keep that in mind. Now first step in derivation of the above relation is this:

$$\cos\theta = [a \cot u_0 - a \sin u_0/(\cosh\xi - \cos u_0)] / [a/\sin u_0] = \cos u_0 - \sin^2 u_0/(\cosh\xi - \cos u_0)$$

I cancelled the a factors. But suppose I write $a = R\sin(u_0)$. then I have

$$\cos\theta = [a \cot u_0 - a \sin u_0/(\cosh\xi - \cos u_0)] / [a/\sin u_0]$$

$$= [R\sin(u_0) \cot u_0 - R\sin(u_0) \sin u_0/(\cosh\xi - \cos u_0)] / [R\sin(u_0)/\sin u_0]$$

$$= [\cos(u_0) - \sin u_0/(\cosh\xi - \cos u_0)]$$

$$\to 1 \; ??$$

I am staring at my picture. I want R and ρ to stay the same. I just want to stretch the red boundary to get a full sphere. This involves : reducing a, increasing z, reducing u_0. Look

$R = a/\sin(u_0)$ shows how a and $\sin(u_0)$ have to scale down at the same rate to maintain R.

If I take the $u_0 \to 0$ limit of this equation I get

$$\cos\theta = 1 - 0/(\cosh\xi - 1) = 1 \Rightarrow \theta = 0$$

But θ is a free variable, how can it be forced to 0?

(a2) Limit of my bowl doc results for the full sphere: Plan B

I am at least onto the problem. In my figure, if you take u_0 smaller, the red circle wants to get bigger, but I am not allowing it to do so because I am reducing a at the same time. Start again with that little triangle,

$$\cos\theta = (a\cot(u_0)-z)/R = [a \cos u_0/\sin u_0 - a \sin u_0/(\cosh\xi - \cos u_0)] / R$$

$$= [R \cos u_0 - a \sin u_0/(\cosh\xi - \cos u_0)] / R$$

$$= [R \cos u_0 - R \sin^2 u_0/(\cosh\xi - \cos u_0)] / R$$

$$= [\cos u_0 - \sin^2 u_0/(\cosh\xi - \cos u_0)]$$

Notice that θ is a function of both u_0 and ξ. It is not an independent variable. If you want $u_0 \to 0$, you are forcing $\cos\theta \to 1$ regardless of ξ. If I want θ not to move, then I have to adjust ξ along with u_0.

(a3) Limit of my bowl doc results for the full sphere: Plan C

OK, here I just let R grow as we do the limit. It is going into an infinitely large sphere in the limit $u_0 \to 0$, So replace 1/R by a/sin(u_0) right at the start,

$$\sigma_{in} = (V_0/4\pi^2 a) \sin u_0 \{ \sqrt{ch\xi - \cos u_0} / (\sqrt{2} \cos(u_0/2)) - \tan^{-1}[\sqrt{ch\xi - \cos u_0} / (\sqrt{2} \cos(u_0/2))] \}$$

Now do the limit,

$$\sigma_{in} \to (V_0/4\pi^2 a) \; 0 \; \{ \sqrt{ch\xi - 1} / \sqrt{2} - \tan^{-1}[\sqrt{ch\xi - 1} / \sqrt{2}] \} = 0$$

and then we get the right limit. But this is not very interesting because who knows what happens when you make the sphere huge!

In the OTHER limit, you WANT the spheres to shrink down because they then shrink down to become the disk of interest.

(a4) Limit for full sphere in a way that we keep R = constant.

Since R = a/sinu_0, we need to install a = Rsinu_0 somehow so they both go to zero together. Start:

$$\sigma_{in} = (V_0/4\pi^2 R) \{ \sqrt{ch\xi - \cos u_0} / (\sqrt{2} \cos(u_0/2)) - \tan^{-1}[\sqrt{ch\xi - \cos u_0} / (\sqrt{2} \cos(u_0/2))] \} \qquad (4.7)$$

$$= (V_0/4\pi^2 R) \{ A - \tan^{-1} A \}$$

In the limit $u_0 \to 0$ we have

To observe the limit better, let $u_0 \to \varepsilon$, some small positive value. Then

$$\cos(u_0) = \cos(\varepsilon) = 1 - \varepsilon^2/2 \qquad \cos(u_0/2) = \cos(\varepsilon/2) \approx 1 - (\varepsilon/2)^2/2 = 1 - \varepsilon^2/8$$

Meanwhile, in this limit ξ also becomes small. We have

$$\xi = \tanh^{-1}[2a\rho/(\rho^2+z^2+a^2)] = \tanh^{-1}[2 R \sin u_0 \rho/(\rho^2+z^2+R^2 \sin u_0^2)]$$

$$\approx 2 R \sin u_0 \rho/(\rho^2+z^2+R^2 \sin u_0^2) \approx 2 R \varepsilon \rho/(\rho^2+z^2) \to 0$$

Then $ch(\xi) \approx 1 + \xi^2/2$, so

$$ch\xi - \cos u_0 \approx (1 + \xi^2/2) - (1 - \varepsilon^2/2) = (\xi^2+\varepsilon^2)/2$$

$$\sqrt{ch\xi - \cos u_0} / (\sqrt{2} \cos(u_0/2)) \approx \sqrt{\xi^2+\varepsilon^2} / 2 \, (1-\varepsilon^2/8) \approx \sqrt{\xi^2+\varepsilon^2} / 2$$

$$\sigma_{in} = (V_0/4\pi^2 R)\ \{\ \sqrt{\xi^2+\varepsilon^2}\ /\ 2 - \tan^{-1}\ [\sqrt{\xi^2+\varepsilon^2}\ /\ 2$$

$$\sqrt{ch\xi - cosu_0}\ \approx\ \sqrt{1 + \xi^2/2 - 1}\ = \xi/\sqrt{2}\ \text{ and }\qquad \sqrt{ch\xi - cosu_0}\ /\ (\sqrt{2}\ \cos(u_0/2))\ \approx \xi/2$$

Then

$$\sigma_{in} = (V_0/4\pi^2 R)\ \{\ \xi/2\ -\tan^{-1}[\ \xi/2]\ \ \}$$

(b-pre1) Limit of my bowl doc results for charge density of a disk **no problem here**

Let's now start with,

$$\sigma_{in} = (\ V_0/4\pi^2 a)\ \sin u_0\ \{\ \sqrt{ch\xi - cosu_0}\ /\ (\sqrt{2}\ \cos(u_0/2))\ -\tan^{-1}[\ \sqrt{ch\xi - cosu_0}\ /\ (\sqrt{2}\ \cos(u_0/2))]\ \ \}$$

Now take the limit $u_0 \rightarrow \pi$ but not fully there. So write $u_0 = \pi - x$ where $x > 0$ is very small but finite. Then we get

$$\sin u_0 = \sin(\pi - x) = -\sin(x - \pi) = +\sin(x) \approx x$$

$$\cos u_0 = \cos(\pi - x) = \cos(x - \pi) = -\cos(x) \approx -1$$

$$\cos(u_0/2) = \cos(\pi/2 - x/2) = +\sin(x/2) \approx x/2$$

Then we get

$$\sigma_{in} = (\ V_0/4\pi^2 a)\ x\ \{\sqrt{2}\ \sqrt{ch\xi+1}\ /\ x\ -\tan^{-1}[\ \sqrt{2}\ \sqrt{ch\xi+1}\ /\ x]\ \ \}$$

The $\tan^{-1}[...] = \tan^{-1}[\ +\infty] = \pi/2$, so then

$$\sigma_{in} = (V_0/4\pi^2 a)\ x\ \{\sqrt{2}\ \sqrt{ch\xi+1}\ /\ x\ -\pi/2\ \ \}$$

$$= (V_0/4\pi^2 a)\sqrt{2}\ \sqrt{ch\xi+1}\ +\ (V_0/4\pi^2 a)\ x\ (-\pi/2)$$

$$\approx\ (V_0/4\pi^2 a)\sqrt{2}\ \sqrt{ch\xi+1} \qquad \text{in the limit}$$

$$= (1/2)(V_0/2\pi^2 a)\sqrt{2}\ \sqrt{ch\xi+1}$$

$$= V_0\sqrt{ch\xi+1}\ /\ (2\sqrt{2}\ \pi^2 a) \qquad \textbf{correct}$$

and this agrees with the above !!!!!!!!! (2+ days). $\pi^2 a$

This is very similar to the Sneddon result which is

$\sigma_{in}(\rho) =$

(b) Limit of my bowl doc results for charge density of a disk

Just naively taking the limit $u_0 \to \pi$ from below, say, we get

$\cos(u_0) \to \cos(\pi) = -1$ $\qquad\qquad$ $\sqrt{ch\xi - cosu_0} \to \sqrt{ch\xi + 1}$

$\cos(u_0/2) \to \cos(\pi/2) = 0$ $\qquad\qquad$ $\sqrt{ch\xi - cosu_0} \to \sqrt{ch\xi - cosu_0}$

This $\cos(u_0/2) \to 0$ causes both terms to be singular so I need to be just a bit careful. Let

$u_0 = \pi - x$ $\qquad\qquad$ where later we will take $x \to 0+$.

Then

$\cos(u_0/2) = \cos(\pi/2 - x/2) = \sin(x/2) \approx x/2$

$\sqrt{2}\cos(u_0/2)) = x/\sqrt{2}$

$1/[\sqrt{2}\cos(u_0/2))] = \sqrt{2}/x$

$\sqrt{ch\xi - cosu_0} / (\sqrt{2}\cos(u_0/2))] \to \sqrt{2}\sqrt{ch\xi + 1} / x$.

Then we have

$$\sigma_{in} = (V_0/4\pi^2 R)\{\sqrt{ch\xi+1}\sqrt{2}/x - \tan^{-1}[\sqrt{ch\xi+1}\sqrt{2}/x]$$

But I failed to say here that 1/R = x/a which makes all the difference!!

As $x \to 0+$, the $\tan^{-1}$ argument [] goes to $+1/\infty$ and we can take

$\lim_{x \to 0+}\{\tan^{-1}[\sqrt{ch\xi+1}\sqrt{2}/x]\} = \pm\pi/2$ depending on where you want to be on the branch.

But the first term blows up, and here is what my result gives,

$\sigma_{in} = (V_0/4\pi^2 R)\{\sqrt{ch\xi+1}\sqrt{2}/x) \to \infty$ $\quad x = \pi - u_0$.

whereas the result is supposed to be

$$\sigma_{in} = (1/\pi^2)\sqrt{ch\xi+1}\,/(\sqrt{2}\,a)$$

The conclusion is that my bowl charge density result is WRONG. [no, it is not wrong]

(c) Plan A: Suppose I am off by one or two factors of h_u

Recall that

$$h_u = a/(ch\xi - cosu)$$

In my limit $u \to \pi$ this becomes

$$h_u \to a/(ch\xi + 1).$$

This number is finite, so no number of such factors will repair my problem!

(d) Plan B: I thought Lebedev confirmed the charge result! And so did Kelvin!

I quote both these results in (4.8) and (4.9).

$$\sigma_L = (V_0/4\pi^2R)\,\{\sqrt{ch\xi-cos\alpha}\,/\,(\sqrt{2}\cos(\alpha/2)) - \tan^{-1}[\sqrt{ch\xi-cos\alpha}\,/\,(\sqrt{2}\cos(\alpha/2))]\,\}$$ // Lebedev (4.8)

$$\sigma_K = (V_0/4\pi^2R)\,\{\sqrt{1-cos\alpha}\,/\,\sqrt{cos\alpha - cos\theta} - \tan^{-1}[\sqrt{1-cos\alpha}\,/\sqrt{cos\alpha - cos\theta}\,]\,\}$$ // Kelvin (4.9)

$$\sigma_{in} = (V_0/4\pi^2R)\,\{\sqrt{ch\xi-cosu_0}\,/\,(\sqrt{2}\cos(u_0/2)) - \tan^{-1}[\sqrt{ch\xi-cosu_0}\,/\,(\sqrt{2}\cos(u_0/2))]\,\}$$ // me (4.7)

These other guys are not going to have it wrong. Lebedev has this

501. Find the surface density of free charge on a thin charged conductor shaped like a spherical bowl of radius a (see Figure 139). Calculate the capacitance of the bowl (see J2, p. 250).

Ans. The charged density is

$$\sigma_i = \frac{V}{4\pi^2 a}\left[\frac{\sqrt{2\cosh\alpha - 2\cos\beta_0}}{2\cos\frac{1}{2}\beta_0} - \arctan\frac{\sqrt{2\cosh\alpha - 2\cos\beta_0}}{2\cos\frac{1}{2}\beta_0}\right]$$

on the inner surface of the bowl, and

$$\sigma_o = \sigma_i + \frac{V}{4\pi a}$$

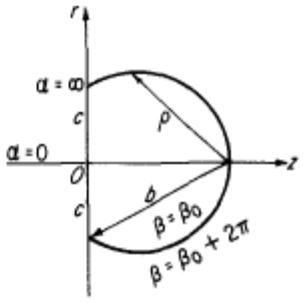


FIGURE 139

on the outer surface, where V is the potential of the bowl and

$$\sin\beta_0 = \frac{c}{a}.$$

Using the formula

$$\frac{4a^2 - b^2}{b^2 - \rho^2} = \frac{\cosh\alpha - \cos\beta_0}{2\cos^2\frac{1}{2}\beta_0},$$

where the distances b and ρ are shown in Figure 139, we find that

$$\sigma_i = \frac{V}{4\pi^2 a}\left[\sqrt{\frac{4a^2 - b^2}{b^2 - \rho^2}} - \arctan\sqrt{\frac{4a^2 - b^2}{b^2 - \rho^2}}\right].$$

The capacitance of the bowl is

$$C = \frac{b}{2\pi a}\sqrt{4a^2 - b^2} + \frac{2a}{\pi}\arctan\frac{b}{\sqrt{4a^2 - b^2}}.$$

Hint. To calculate the density, use the integral

$$\int_0^\infty \frac{\tau\sinh\pi\tau\cosh(\pi - \beta_0)\tau}{\cosh^2\pi\tau} P_{-\frac{1}{2}+i\tau}(\cosh\alpha)\,d\tau = \frac{2}{\pi}\frac{\sin\frac{1}{2}\beta_0}{2\cosh\alpha - 2\cos\beta_0} \times \left[1 + \frac{2\cos\frac{1}{2}\beta_0}{\sqrt{2\cosh\alpha - 2\cos\beta_0}}\arctan\frac{2\cos\frac{1}{2}\beta_0}{\sqrt{2\cosh\alpha - 2\cos\beta_0}}\right].$$

Notice that he says

$\sin\beta_0$ = lip circle radius / bowl radius = a/R for me

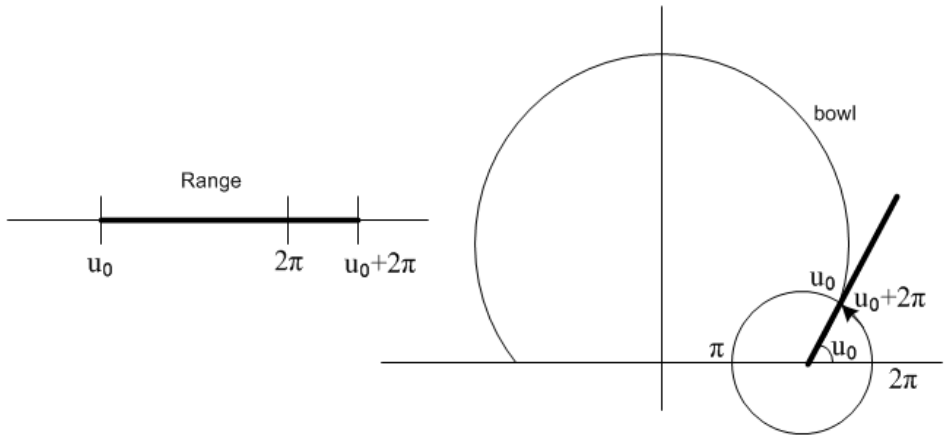
IN my bipolar doc (2.6) I have

sinu = a/R

so that is why I think u = β_0.

So I am totally stumped (as usual). How can all three of us agree on the σ result, but the disk limit fails?? Is it somehow related to the range of u? That might be it.

What range did I use? It is in my picture and it is same as Lebedev.



Now what does this mean for taking the disk limit? Imagine we have almost reached the disk limit, so u_0 has reached $\pi-\varepsilon$. Then what is the range for u?

$$\pi-\varepsilon \le u \le 3\pi-\varepsilon$$

Then the lower surface of the disk is at $u = \pi-\varepsilon$, as I used above. So $u \to \pi$ is then the correct limit, and I continue to be getting the wrong result!

****

Question 1: I say that σ is infinite at the bowl edge. How does that appear in my result? I never checked on that.

$$\sigma_+ = \sigma_{in} = (V_0/4\pi^2R)\,[\quad (B/A) - \tan^{-1}(B/A)]$$

$$A = \sqrt{2}\cos(u_0/2) \text{ and } B = \sqrt{ch\xi - cosu_0}$$

At the bowl edge, we are at the focal point so we have $\xi \to +\infty$. Then $B \to \infty$ and

$$\sigma_{in}(\text{near edge}) = (V_0/4\pi^2R)\,[\quad (B/A) - \pi/2\,] \approx (V_0/4\pi^2R)\,(B/A)$$

Now $B/A = (ch\xi)^{1/2}/ [\sqrt{2}\cos(u_0/2)]$ so the answer close to the edge is

$$\sigma_{in}(\text{near edge}) = (V_0/4\pi^2R)\ (ch\xi)^{1/2}/\ [\sqrt{2}\cos(u_0/2)]$$

which does indeed blow up as claimed as $\xi \to \infty$.

Question 2. How about at the bowl center? There $\xi \to 0$ and we get $B = \sqrt{ch\xi - \cos u_0} = \sqrt{1-\cos u_0}$. Then

$$B/A = \sqrt{1-\cos u_0} / [\sqrt{2}\cos(u_0/2)]$$

Now $\sqrt{1-\cos u_0} = \sqrt{2}\ \sin(u_0/2)$

$$B/A = \sqrt{2}\ \sin(u_0/2)/ [\sqrt{2}\cos(u_0/2)] = \tan(u_0/2)$$

and then we have

$$\sigma_{in}(\text{at center}) = (V_0/4\pi^2R)[\ \tan(u_0/2) - \tan^{-1}(\tan(u_0/2))]$$

$$= (V_0/4\pi^2R)[\ \tan(u_0/2) - u_0/2]$$

Consider the sequence of upper bowls from large to small so $u_0 \to \pi$ from below.

```
f := tan(u/2)-u/2;
```

$$f := \tan\left(\frac{1}{2}u\right) - \frac{1}{2}u$$

```
plot(f,u=0..Pi-.1);
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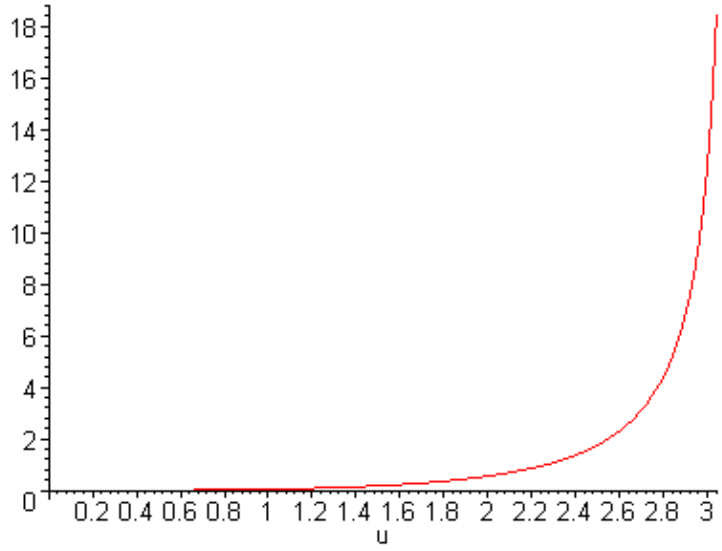


This says that σ_{in} (and σ_{out} which differs by a constant) blows up for the flat disk at disk center!

This result I am sure is wrong. So we have a new problem.

Time to check my calculation of σ starting with (4.1). I need to show that it replicates the flat disk charge distribution

Question 3: Let's start with the full σ_{in} result and take it to the flat disk limit. Here is my full result

$$\sigma_+ = \sigma_{in} = (V_0/4\pi^2R) [\quad (B/A) - \tan^{-1}(B/A)]$$

$$A = \sqrt{2}\cos(u_0/2) \text{ and } B = \sqrt{ch\xi - cosu_0}$$

The flat disk has $u_0 = \pi$ and then

$$A = \sqrt{2}\cos(\pi/2) = 0 \qquad B = \sqrt{ch\xi + 1}$$

The limit then says

$$\sigma_+ = \sigma_{in} = (V_0/4\pi^2R) [\quad (B/0) - \tan^{-1}(B/0)]$$

$$= (V_0/4\pi^2R) [\quad \infty - \pi/2]$$

So something is WRONG with my σ result! I should have checked this limit when I wrote the doc. Stupid.

Check the calculation of σ

I start with

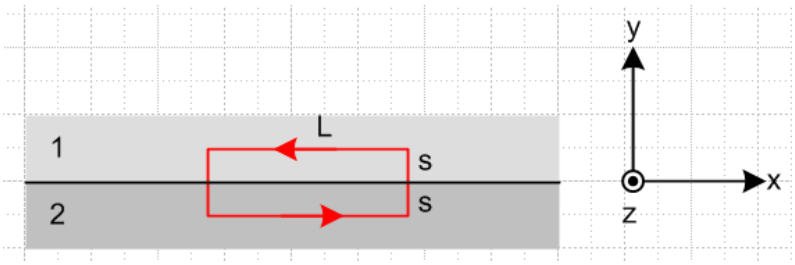
$$V(\xi,u) = V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi - u_0)\tau]\ ch[(\pi + u_0 - u)\tau] / ch^2(\pi\tau) \qquad (2.8)$$

$$\sigma_+ = -(1/4\pi)\ (1/h_u)\ \partial_u V(\xi,u)|^{u=u0} \qquad 1/h_u = (ch\xi - cosu)/a$$

Where do I get this second line from? No reference is given. Is this result in any of my web docs? I do have this in SI units,

$$D_{n1} - D_{n2} = n_{free} \qquad \text{or} \qquad [\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}] = n_{free} \qquad (1.1.47)$$

Consider a boundary like this



Suppose 2 = metal and 1 = vacuum. We then have

$$[\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}] = n_{free} \quad \rightarrow \quad \varepsilon_0 E_{1n} = n_{free}$$

Translate this to say

$$\sigma = \varepsilon_0 E_n \qquad \text{SI units}$$

So far so good. Meanwhile,

$$\mathbf{E} = -\text{ grad } V$$

We look this up in tensor doc where we have

$$[\text{grad } f](\mathbf{x}) = (\partial_i f)\, \mathbf{e}^i$$
$$[\text{grad } f](\mathbf{x}) = (\partial^i f)\, \mathbf{e}_i = (1/h_i^2)\, (\partial_i f)\, \mathbf{e}_i$$
$$[\text{grad } f](\mathbf{x}) = h_i (\partial^i f)\, \hat{\mathbf{e}}_i = (1/h_i)\, (\partial_i f)\, \hat{\mathbf{e}} \qquad \text{// M\&S 1.05} \qquad (10.1.17)$$

or

$$[\text{grad } V](\mathbf{x}) = (\partial_i V)\, \mathbf{e}^i$$
$$[\text{grad } V](\mathbf{x}) = (\partial^i V)\, \mathbf{e}_i = (1/h_i^2)\, (\partial_i V)\, \mathbf{e}_i$$
$$[\text{grad } V](\mathbf{x}) = h_i (\partial^i V)\, \hat{\mathbf{e}}_i = (1/h_i)\, (\partial_i V)\, \hat{\mathbf{e}}_i \qquad \text{// M\&S 1.05} \qquad (10.1.17)$$

where italic V means it is a function of the curvilinear coordinates. Assume that $\hat{\mathbf{e}}_i$ is a unit vector in curvilinear coordinates which is normal to our surface of interest, which would be u for our case study. Then

$$[\text{grad } V](\mathbf{x}) = (1/h_u)\, (\partial_u V)\, \hat{\mathbf{e}}_u$$

Then

$$\mathbf{E} = -\text{ grad } V \Rightarrow \quad E_u \hat{\mathbf{e}}_u = -(1/h_u)\, (\partial_u V)\, \hat{\mathbf{e}}_u \quad \Rightarrow \quad E_u = -(1/h_u)\, (\partial_u V)$$

and then finally

$$\sigma = \varepsilon_0 E_u = -\varepsilon_0 \, (1/h_u) \, (\partial_u V)$$

and then divide by $4\pi\varepsilon_0$ to get cgs units

$$\sigma = \varepsilon_0 E_u = -(1/4\pi) \, (1/h_u) \, (\partial_u V)$$

and this is what I used.

Question: But there is something unclear to me here. Go back to

$$[\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}] = n_{free}$$

Are these E-field component covariant or contravariant !!! Go back to the derivation (where I have fixed the bad reference which was 1.1.13),

$$\text{div } \mathbf{D} = \rho_{free} \quad \Leftrightarrow \quad \int_V \rho_{free} \, dV = \int_S \mathbf{D} \bullet d\mathbf{S} \qquad (1.1.32)$$

Now how do you do this in covariant notation?

$$\mathbf{D} \bullet d\mathbf{S} = D_i dS^i = D^i dS_i$$

My lines doc does not address this question. In the n-piped discussion I do mention area with covariant index stuff, and there fore example I have

$$d\mathbf{A}'^n = J^2 (\Pi_{i \neq n} dx'^i) \, \mathbf{e}'^n \qquad \text{since } (\mathbf{e}'^n)_i = \delta^n{}_i \quad (7.18.1) \qquad (8.4.c.5)$$

where we are in x'-space. I show in toroidals.doc that

$$J = h_\xi h_u h_\varphi = \frac{a^3 \text{sh}\xi}{(\text{ch}\xi - \text{cos}u)^3}$$

If I then set n = 2 for the u coordinate, we get

$$d\mathbf{A}'^n = J^2 (\Pi_{i \neq n} dx'^i) \, \mathbf{e}'^n$$

$$d\mathbf{A}^u = J^2 \, d\xi d\varphi \, \mathbf{e}^u$$

$$d\mathbf{S}^u = J^2 \, d\xi d\varphi \, \mathbf{e}^u$$

Then we have

$$\mathbf{D} \bullet d\mathbf{S} = (D^u \mathbf{e}_u) \bullet (J^2 \, d\xi d\varphi \, \mathbf{e}^u) = D^u \, J^2 \, d\xi d\varphi$$

so it is the contravariant component ! Then we really have

$$[\varepsilon_1 E^n{}_1 - \varepsilon_2 E^n{}_2] = n_{free}$$

$$[\varepsilon_1 E^u{}_1 - \varepsilon_2 E^u{}_2] = \sigma$$

$$\varepsilon_o E^u{}_1 = \sigma$$

We now trace through the above discussion.

$$[\text{grad } V](\mathbf{x}) = (\partial_i V)\, \mathbf{e}^i$$
$$[\text{grad } V](\mathbf{x}) = (\partial^i V)\, \mathbf{e}_i = (1/h_i{}^2)\,(\partial_i V)\, \mathbf{e}_i$$
$$[\text{grad } V](\mathbf{x}) = h_i(\partial^i V)\, \hat{\mathbf{e}}_i = (1/h_i)\,(\partial_i V)\, \hat{\mathbf{e}}_i \qquad \text{// M\&S 1.05} \qquad (10.1.17)$$

$$\mathbf{E} = -\text{grad } V \Rightarrow \quad E_u \hat{\mathbf{e}}_u = -(1/h_u)\,(\partial_u V)\, \hat{\mathbf{e}}_u \quad \Rightarrow \quad E_u = -(1/h_u)\,(\partial_u V)$$

$$E^u = -(1/h_u)\,(\partial^u V)$$

Then

$$\sigma = \varepsilon_o E^u{}_1 = -\varepsilon_0\,(1/h_u)(\partial^u V) \qquad \text{// with an upper index!}$$

So my document should say not this

$$\sigma_+ = -(1/4\pi)\,(1/h_u)\,\partial_u V(\xi,u)|^{u=u0} \qquad 1/h_u = (ch\xi - cosu)/a$$
$$\sigma_- = +(1/4\pi)\,(1/h_u)\,\partial_u V(\xi,u)|^{u=u0+2\pi} \qquad \text{// cgs units selected here} \qquad (4.1)$$

but this

$$\sigma_+ = -(1/4\pi)\,(1/h_u)\,\partial^u V(\xi,u)|^{u=u0} \qquad 1/h_u = (ch\xi - cosu)/a$$
$$\sigma_- = +(1/4\pi)\,(1/h_u)\,\partial^u V(\xi,u)|^{u=u0+2\pi} \qquad \text{// cgs units selected here} \qquad (4.1)$$

So this is the first thing I have found to be done wrong. Now we have

$$\partial_i = \Sigma_j\, g_{ij}\partial^j = g_{ii}\partial^i$$

For i = 2 = u this says

$$\partial_u = g_{uu}\partial^i = h_u{}^2\,\partial^u$$

$$\partial^u = h_u{}^{-2}\partial_u$$

Then I would seem to get

$$\sigma_+ = -(1/4\pi)\,(1/h_u)^3\,\partial_u V(\xi,u)|^{u=u0} \qquad 1/h_u = (ch\xi - cosu)/a$$

$\sigma_- = +(1/4\pi)\,(1/h_u)^3\,\partial_u V(\xi,u)|^{u=u0+2\pi}$ // cgs units selected here (4.1)

Basically this adds an extra factor of $(1/h_u)^2$ to my charge densities. Note that

$1/h_u = (ch\xi - cosu)/a$

$(1/h_u)^2 = (ch\xi - cosu)^2/a^2 = B^4/a^2$

Then my new charge density results (if everything ELSE was done correctly) would be

$\sigma_+ = \sigma_{in} = (V_0/4\pi^2R)\,[\,(B/A) - \tan^{-1}(B/A)]\,(B^4/a^2)$

$A = \sqrt{2}\cos(u_0/2)$ and $B = \sqrt{ch\xi - cosu_0}$ $B^2 = ch\xi - cosu_0$

How does this change the limit $u_0 \neq \pi$? The problem remains unrepaired! $A \rightarrow 0$ and $\sigma \rightarrow \infty$ for any ξ.

I have some major problems here. I will respond to the guy first, and then tackle these problems. This might result in a major rewrite of bowl doc!