Capacitance of the toroidal bowl, v1 PhL 1.25.11

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Overview (1/2 page, written 2.3.11)

This is the first of three versions of this doc: v1,v2,v3.

In **Section 1** I start with the (correct) single-bowl potential and its (perhaps not correct) evaluated form and I make a certain "adjustment" which means a phase $(-1)^{\eta}$ for the two functions of the form $\sqrt{1+\cos(u)}$. I thought this was going to solve my "wrong capacitance" problem encountered at the end of my "charged bowl in toroidals 1_11.doc" doc. I am still using the T1,T2,T3,T4 form (the $\tan^{-1}$ 2-term integral Mehler integral). The adjusted result is stated at the end of this section. The phases are η_2 and η_4.

In **Section 2** I show that this adjusted post-evaluated potential **meets both BC's.**

In **Section 3** I go to the **large r limit** which by now I know means $\xi=0$ and $u = 2\pi$. I find that $\eta_2 = 0$ and $\eta_4 = -1$ for any u_0 except a possible problem when $u_0 = \pi/2$ exactly. I then write out my T1-T4 in this limit with these phases. The T4 term contains $\tan^{-1}[\cot(u_0)]$ and I use $\tan^{-1}[x] + \cot^{-1}[x] = \pi/2$ to simplify this thing and I notice that this identity is only valid for u_0 in 0 to $\pi/2$, but it ought to work within that region. [Much later I will learn more about proper branches of $\tan^{-1}$ and $\cot^{-1}$.] I then go ahead and simplify things in search of the capacitance. This time I get two of the three terms in the capacitance, but the sine term is missing. I added this term in red so I could work backwards to see where it was missing, but I could not solve the mystery. The rest of this section contains scratch old notes that I don't think are worth reviewing. I am going to "start over" soon anyway.

In **Section 4** I successfully obtain the **right capacitance** by taking the limit of the *non-evaluated* single-bowl integral for the potential. I simply set $P_{i\tau-1/2}(\text{ch}\xi) = 1$ and then look up some integrals. So this section is perhaps the only section that is correct in this entire doc, don't lose it!

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1. Adjustment of the evaluated potential.

In our main doc, we first come up with this expression for the bowl potential

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{\text{ch}\xi - \cos u}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(\text{ch}\xi)\ \text{ch}[(\pi-u_0)\tau]\ \text{ch}[(\pi+u_0-u)\tau] / \text{ch}^2(\pi\tau)$$

and the bowl range is u in $(u_0, u_0+2\pi)$. We first break this integral into two terms to get

$$V(\xi,u)= (V_0 /\sqrt{2}\ )\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ \{ \cosh[(2\pi-u)\tau]\ + \cosh[(2u_0-u)\tau]\} / ch^2(\pi\tau)$$

Each of these integrals then yields two terms according to the Boeing lookup, so we end up with four terms. Here are those terms from (***), where I maintain the $\sqrt{1+\cos(2\pi-u)}$ factor in T2 and I do not blindly replace it with $\sqrt{1+\cos(u)}$

$V(\xi,u)= (V_0 /\sqrt{2}\ ) * (T1 + T2 + T3 + T4)$
$T1= 1/\sqrt{2}$
$T2= - \sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{1+\cos(2\pi-u)}\ /\sqrt{ch\xi-\cos(u)}\ ]$
$T3 = \ (1/\sqrt{2}\ )\sqrt{ch\xi-\cos(u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}$
$T4 = -\sqrt{2}\ \pi^{-1}\sqrt{ch\xi-\cos(u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}\ * \tan^{-1}[\sqrt{1+\cos(2u_0-u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}\ ]$

I have since decided that there is an issue with the factors $\sqrt{1+\cos(2\pi-u)}$ and $\sqrt{1+\cos(2u_0-u)}$ which issue results in an alteration of the above formula. The alterations are these

$$\sqrt{1+\cos(x)}\ \rightarrow(-1)^{\eta}\sqrt{1+\cos(x)} \qquad\qquad \eta = floor[(x+\pi)/2\pi]$$

Suppose we start out at $x = \pi/4$, a small value, then $\eta = 0$ and there is no change. When we reach $x = \pi$, however, then there is a sign change. The next change is when we reach $x = 3\pi$. These values $\pi,3\pi,5\pi$ are where my "green bars" live in some Ahlfors doc I have not finished yet. So let's compute η for our two cases of interest

For $x = [2\pi-u]$ we have $\eta_2 = floor[(x+\pi)/2\pi] = floor[([2\pi-u] +\pi)/2\pi] = 1 + floor[\ (\pi-u)/2\pi]$
For $x = [2u_0-u]$ we have $\eta_4 = floor[(x+\pi)/2\pi] = \ floor[([2u_0-u]+\pi)/2\pi]$

so that

$$\sqrt{1+\cos(2\pi-u)}\ \rightarrow\ (-1)^{\eta 2}\ \sqrt{1+\cos(2\pi-u)}\ \ = (-1)^{\eta 2}\sqrt{1+\cos(u)}$$
$$\sqrt{1+\cos(2u_0-u)}\rightarrow\ (-1)^{\eta 4}\sqrt{1+\cos(2u_0-u)}$$

where the square roots on the right are taken as being positive (that is the main point!) . We get rid of the 2π in T2 only after the rule is applied. Our potential is now

$V(\xi,u)= (V_0 /\sqrt{2}\ ) * (T1 + T2 + T3 + T4)$
$T1= 1/\sqrt{2}$
$T2= -(-1)^{\eta 2}\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{1+\cos(u)}\ /\sqrt{ch\xi-\cos(u)}\ ]$
$T3 = \ (1/\sqrt{2}\ )\sqrt{ch\xi-\cos(u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}$
$T4 = - (-1)^{\eta 4}\sqrt{2}\ \pi^{-1}\sqrt{ch\xi-\cos(u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}\ * \tan^{-1}[\sqrt{1+\cos(2u_0-u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}\ ]$

$\eta_2 = \ 1 + floor[(\pi-u)/2\pi]$

$\eta_4 = \text{floor}[([2u_0-u]+\pi)/2\pi]$

2. Boundary Condition Check: $u = u_0$ and $u = u_0 + 2\pi$

Suppose $u = u_0$ which is our inside boundary condition. Then

$T1 = 1/\sqrt{2}$

$T2 = -(-1)^{\eta 2}\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{1+\cos(u_0)}\ /\sqrt{ch\xi-\cos(u_0)}\ ]$

$T3 = (1/\sqrt{2})\sqrt{ch\xi-\cos(u_0)}\ /\sqrt{ch\xi-\cos(u_0)}$

$T4 = -(-1)^{\eta 4}\sqrt{2}\ \pi^{-1}\sqrt{ch\xi-\cos(u_0)}\ /\sqrt{ch\xi-\cos(u_0)}\ * \tan^{-1}[\sqrt{1+\cos(u_0)}\ /\sqrt{ch\xi-\cos(u_0)}\ ]$

or

$T1 = 1/\sqrt{2}$

$T2 = -(-1)^{\eta 2}\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{1+\cos(u_0)}\ /\sqrt{ch\xi-\cos(u_0)}\ ]$

$T3 = 1/\sqrt{2}$

$T4 = -(-1)^{\eta 4}\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{1+\cos(u_0)}\ /\sqrt{ch\xi-\cos(u_0)}\ ]$

with a certain η_2 and η_4 which we shall compute below. But first, take note that the four lines above would be exactly the same at the other BC which is $u = u_0 + 2\pi$, except the η's will be different.

First then we look at $u = u_0$: (I use $u_0/\pi = 1/2$ as a "typical value" to get the floor value)

$\eta_2 = 1 + \text{floor}[(\pi-u)/2\pi] = 1 + \text{floor}[(\pi-u_0)/2\pi] = 1+0 = 1$

$\eta_4 = \text{floor}[([2u_0-u]+\pi)/2\pi] = \text{floor}[([u_0]+\pi)/2\pi] = 0$

Second, we look at $u = [u_0 + 2\pi]$:

$\eta_2 = 1 + \text{floor}[(\pi-u)/2\pi] = 1 + \text{floor}[(\pi-[u_0+2\pi])/2\pi] = \text{floor}[(\pi-[u_0])/2\pi] = 0$

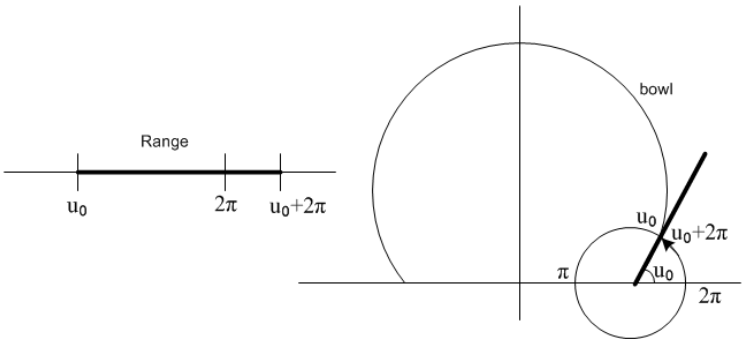
$\eta_4 = \text{floor}[([2u_0-u]+\pi)/2\pi] = \text{floor}[([2u_0-[u_0+2\pi]]+\pi)/2\pi] = -1 + \text{floor}[([2u_0-u_0]+\pi)/2\pi]$

$= -1 + \text{floor}[(u_0+\pi)/2\pi] = -1 + 0 = -1$

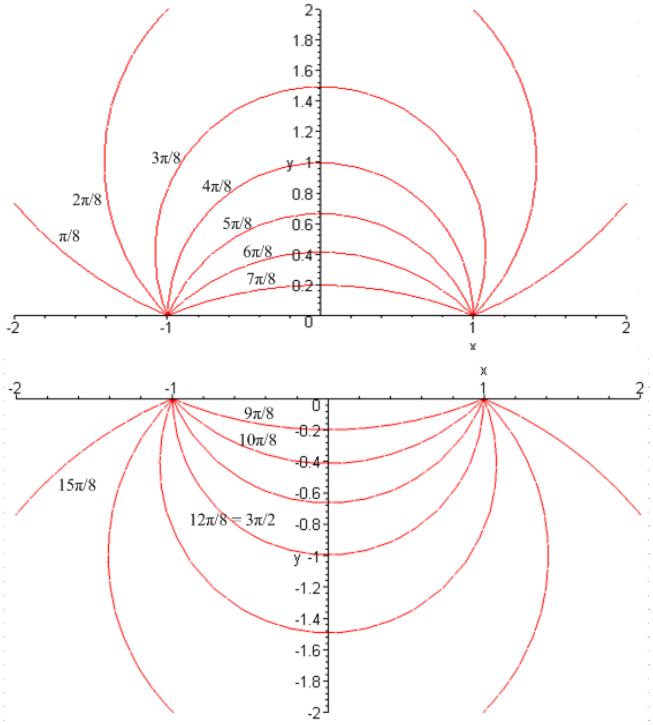
In both cases we get one even and one odd, so in both cases $(-1)^{\eta 2} = -(-1)^{\eta 4}$ and in both cases then the terms T2 and T4 cancel, and $T1+T2 = \sqrt{2}$ and $V(\xi,u) = (V_0/\sqrt{2})\ \sqrt{2} = V_0$ and thus it is that we meet both boundary conditions!

3. Large r (meaning $\xi=0$ and $u = 2\pi$) and Capacitance.

Recall our "coordinate setup" picture for the charged bowl,



The "great sphere" in effect has two halves, an upper and lower. In our parameterization, the upper great sphere has u = [2π+ε] and the lower great sphere has u = [2π-ε]. This follows from our usual pictures:



Let's pause to examine our η factors for these two situations.

$\eta_{2+} = 1 + \text{floor}[(\pi-u)/2\pi] = 1 + \text{floor}[(\pi-[2\pi+\varepsilon])/2\pi] = \text{floor}[(\pi-\varepsilon)])/2\pi] = 0$
$\eta_{2-} = 1 + \text{floor}[(\pi-u)/2\pi] = 1 + \text{floor}[(\pi-[2\pi-\varepsilon])/2\pi] = \text{floor}[(\pi+\varepsilon)])/2\pi] = 0$

$\eta_{4+} = \text{floor}[([2u_0-u]+\pi)/2\pi] = \text{floor}[([2u_0-[2\pi+\varepsilon]]+\pi)/2\pi] = -1 + \text{floor}[(2u_0-\varepsilon+\pi)/2\pi]$
$\eta_{4-} = \text{floor}[([2u_0-u]+\pi)/2\pi] = \text{floor}[([2u_0-[2\pi-\varepsilon]]+\pi)/2\pi] = -1 + \text{floor}[(2u_0+\varepsilon+\pi)/2\pi]$

Unless we happen to have $u_0 = \pi/2$ exactly (the hemisphere) our η_4 will be the same.

It seems to me that the η_4 signs will depend on the value of u_0 which worries me. Let's assume first a small value of u_0 so that we have a large bowl that is more than a hemisphere. Then

$\eta_{4+} = -1 + \text{floor}[(2u_0 - \varepsilon + \pi)/2\pi] = -1 + 0 = -1$
$\eta_{4-} = -1 + \text{floor}[(2u_0 + \varepsilon + \pi)/2\pi] = -1 + 0 = -1$

Luckily, the η factors are the same on either half of the great sphere for modest u_0 :

upper great sphere:	$\eta_{2+} = 0$	$\eta_{4+} = -1$
lower great sphere:	$\eta_{2-} = 0$	$\eta_{4-} = -1$

I think this is now an improved result. First let's copy down our general (adjusted) result:

$T1 = 1/\sqrt{2}$
$T2 = -(-1)^{\eta 2}\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{1+\cos(u)}\ /\sqrt{ch\xi-\cos(u)}\ ]$
$T3 = (1/\sqrt{2}\ )\sqrt{ch\xi-\cos(u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}$
$T4 = -(-1)^{\eta 4}\sqrt{2}\ \pi^{-1}\sqrt{ch\xi-\cos(u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}\ * \tan^{-1}[\sqrt{1+\cos(2u_0-u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}\ ]$

Now we shall make these substitutions

$\sqrt{ch\xi-\cos(u)}$ stays as it is everywhere, since this appears in our large r limit
$\xi = 0$ in other locations $\Rightarrow ch\xi = 1$
$u = 2\pi$ in other locations
$\cos(2u_0-u) = \cos(2u_0-2\pi) = \cos(2u_0)$
$\sqrt{ch\xi-\cos(2u_0-u)} = \sqrt{1-\cos(2u_0-u)} = \sqrt{1-\cos(2u_0)} = [\sqrt{2}\sin(u_0)]$
$\sqrt{1+\cos(2u_0-u)} = \sqrt{1+\cos(2u_0-2\pi)} = \sqrt{1+\cos(2u_0)} = [\sqrt{2}\cos(u_0)]$
$\sqrt{1+\cos(u)} = \sqrt{1+\cos(2\pi)} = \sqrt{2}$
$\eta_2 = 0$
$\eta_4 = -1$

and we then get

$T1 = 1/\sqrt{2}$
$T2 = -(-1)^0\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{2}\ /\sqrt{ch\xi-\cos(u)}\ ]$
$T3 = (1/\sqrt{2}\ )\sqrt{ch\xi-\cos(u)}\ /\ [\sqrt{2}\sin(u_0)]$
$T4 = -(-1)^{-1}\sqrt{2}\ \pi^{-1}\sqrt{ch\xi-\cos(u)}\ /\ [\sqrt{2}\sin(u_0)] * \tan^{-1}[\sqrt{2}\cos(u_0)]/[\sqrt{2}\sin(u_0)]\ ]$

Now use $\tan^{-1}[\cot(u_0)\ ] = \pi/2 - \cot^{-1}[\cot(u_0)\ ] = (\pi/2 - u_0)$ and other simplifications to get

STOP!!! Put on the brakes!! We are using this formula

$\tan^{-1}[x] + \cot^{-1}[x] = \pi/2$

This formula is only valid for x in the range $(0,\infty)$ as Schaum pictures page 18-19 show. If we try applying this to $x = \cot(u_0)$, then $\cot(u_0)$ must be in the range $(0,\infty)$. Looking at page 14, we see that this in turn implies that u_0 must be in the range $(0,\pi/2)$. But we are trying to apply this on $(0,\pi)$. So we have to modify this rule somehow. I am inclined to start over in another doc right now.

Note 12/4/15: As shown in the intro above, I added the missing red terms by working backwards from the known end result. As noted also in the intro, I never solved this mystery, and it is still a mystery today!

$T1 = 1/\sqrt{2}$ $\quad + (2/\pi)\,(a/r) = \sqrt{2}\,(a/r)(\sqrt{2}/\pi)$

$T2 = -\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{2}/\sqrt{ch\xi - \cos(u)}\ ]$ $\quad = (\sqrt{2}/\pi)\sqrt{ch\xi - cosu}$

$T3 = (1/2)\sqrt{ch\xi - \cos(u)} / [\sin(u_0)]$

$T4 = +\sqrt{ch\xi - \cos(u)} / [\pi\sin(u_0)] * (\pi/2 - u_0)$

As our final step now we want to use our large-r relation which is $\sqrt{ch\xi - cosu} \approx \sqrt{2}\ a/r$ to get

$T1 = 1/\sqrt{2}$

$T2 = -\sqrt{2}\ \pi^{-1} * \tan^{-1}[r/a\ ] = -\sqrt{2}\ \pi^{-1} * \tan^{-1}[+\infty\ ] = -\sqrt{2}\ \pi^{-1} * \pi/2 = -1/\sqrt{2}$

$T3 = (1/2)\,(\sqrt{2}\ a/r) / [\sin(u_0)] = (1/\sqrt{2}\ )\,(a/r) / [\sin(u_0)]$

$T4 = +\sqrt{2}\ a/r / [\pi\sin(u_0)] * (\pi/2 - u_0)$

Now T1 and T2 cancel and our sum of T's is

$\Sigma_T = T3+T4 = (1/\sqrt{2}\ )\,(a/r) / [\sin(u_0)] + \sqrt{2}\ a/r / [\pi\sin(u_0)] * (\pi/2 - u_0) + (2/\pi)\,(a/r)$

$= (a/r)\ \{\ [(1/\sqrt{2}\ )/\sin(u_0) + \sqrt{2}\ (\pi/2 - u_0)/[\pi\ \sin(u_0)] + \sqrt{2}/\pi\ \}$

Now we insert $a = R\sin(u_0)$ where R is the bowl radius

$= (R\sin(u_0)/r)\ \{\ [(1/\sqrt{2}\ )/\sin(u_0) + \sqrt{2}\ (\pi/2 - u_0)/[\pi\ \sin(u_0)] + \sqrt{2}/\pi\ \}$

$= (R/r)\ \{\ [(1/\sqrt{2}\ ) + \sqrt{2}\ (\pi/2 - u_0)/[\pi] + \sqrt{2}\ \sin u_0/\pi\ \}$

Then our large r limit seems to be

$V(\xi,u) = (V_0/\sqrt{2}\ )\Sigma_T = (V_0/\sqrt{2}\ )(R/r)\ \{\ [(1/\sqrt{2}\ ) + \sqrt{2}\ (\pi/2 - u_0)/[\pi] + \sqrt{2}\ \sin u_0/\pi\ \}$

$= (V_0\ )(R/r)\ \{\ [(1/2) + (\pi/2 - u_0)/[\pi] + \sin u_0/\pi\ \}$

$= (V_0\ )(R/r)\ \{\ [(1/2)(\pi/\pi) + (\pi/2 - u_0)/[\pi] + \sin u_0/\pi\}$

$= (V_0\ )(R/\pi r)\ \{\ [(\pi/2) + (\pi/2 - u_0) + \sin u_0\}$

$= (V_0)(R/\pi r)\{\pi - u_0 + \sin u_0\}$

If we think of this as V = Q/r we get

$Q = (V_0)(R/\pi)\{\pi - u_0 + \sin u_0\}$

and if we think $Q = CV_0$ we get

$C = Q/V_0 = (R/\pi)\{\pi - u_0 + \sin u_0\}$

which we compare to the known correct answer from Smythe ($\alpha = u_0$)

$C = (R/\pi)\{\pi - \alpha + \sin\alpha\}$

and the result is correct except **I have somehow lost the sinα term.**

previous:

$T1 = 1/\sqrt{2}$

$T2 = -(-1)^0\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{1+\cos(u_0)}\ /\sqrt{ch\xi-\cos(u_0)}\ ]$

$T3 = 1/\sqrt{2}$

$T4 = -(-1)^{-1}\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{1+\cos(u_0)}\ /\sqrt{ch\xi-\cos(u_0)}\ ]$

This seems to say that T2 and T4 cancel, and the sum of the T's is $\sqrt{2}$ and so $V = V_0$. But how can this be our large-r limit? That limit should be 0 or at least 1/r.

I am unsure of how to visualize the capacitor.

Plan A: Let's just try to take $r\to\infty$ in the upper half sphere. Our general potential is this:

$V(\xi,u) = (V_0/\sqrt{2}) * (T1 + T2 + T3 + T4)$

$T1 = 1/\sqrt{2}$

$T2 = -(-1)^{\eta 2}\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{1+\cos(u)}\ /\sqrt{ch\xi-\cos(u)}\ ]$

$T3 = (1/\sqrt{2})\sqrt{ch\xi-\cos(u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}$

$T4 = -(-1)^{\eta 4}\sqrt{2}\ \pi^{-1}\sqrt{ch\xi-\cos(u)}\ /\sqrt{ch\xi-\cos(2u_0-u)} * \tan^{-1}[\sqrt{1+\cos(2u_0-u)}\ /\sqrt{ch\xi-\cos(2u_0-u)}\ ]$

$\eta_2 = \text{floor}(2-u/\pi)$

$\eta_4 = \text{floor}([2u_0-u]/\pi)$

I will try to set $u = 2\pi+\varepsilon$ to evaluate this potential at large upper r. We know that $\xi \rightarrow 0$ as well in our required large r limit.

First, consider the $\tan^{-1}$ in T2. We know $(ch\xi - cosu) \rightarrow 0$ in our limit, so this thing is going to be $\tan^{-1}(\sqrt{2}\,/0) = \tan^{-1}(+\infty) = \pi/2$ where I guess I assume $(ch\xi - cosu) \rightarrow 0+$.

Second, consider the $\tan^{-1}$ in T4. In the limit we have

$$\tan^{-1}[\sqrt{1+\cos(2u_0)}\,/\sqrt{1-\cos(2u_0)}\,]$$

Here is a little theorem concerning a certain $\tan^{-1}$ form (draw triangle picture)

$$\tan^{-1}[\sqrt{1+\cos(x)}\,/\sqrt{1-\cos(x)}\;\;] = \sin^{-1}[\,(\,1+\cos x)/2\,]^{1/2} = \sin^{-1}[\,\cos(x/2)]$$

$$= \pi/2 - \cos^{-1}[\,\cos(x/2)] = \pi/2 - x/2$$

We apply this to the $\tan^{-1}$ above and we get

$$\tan^{-1}[\sqrt{1+\cos(2u_0)}\,/\sqrt{1-\cos(2u_0)}\;\;] = (\pi/2 - u_0)$$

So here is what we now have

$$T1 = 1/\sqrt{2}$$
$$T2 = -(-1)^{\eta 2}\sqrt{2}\;\;\pi^{-1} * (\pi/2)$$
$$T3 = \;\;(1/\sqrt{2}\,)\sqrt{ch\xi-\cos(u)}\,/\sqrt{1-\cos(2u_0)}$$
$$T4 = -\,(-1)^{\eta 4}\sqrt{2}\;\pi^{-1}\sqrt{ch\xi-\cos(u)}\,/\sqrt{1-\cos(2u_0)}\;\;*(\pi/2 - u_0)$$

or

$$T1 = 1/\sqrt{2}$$
$$T2 = -(-1)^{\eta 2}\,(1/\sqrt{2}\,)$$
$$T3 = \;\;\sqrt{2}\,(1/2)\sqrt{ch\xi-\cos(u)}\,/\sqrt{1-\cos(2u_0)}$$
$$T4 = -\,(-1)^{\eta 4}\sqrt{2}\,(1/2 - u_0/\pi)\sqrt{ch\xi-\cos(u)}\,/\,\sqrt{1-\cos(2u_0)}$$

I then insert

upper great sphere: $\eta_{2+} = -1$ $\eta_{4+} = 1$

to get

$$T1 = 1/\sqrt{2}$$
$$T2 = +\,(1/\sqrt{2}\,)$$
$$T3 = \;\;\sqrt{2}\,(1/2)\sqrt{ch\xi-\cos(u)}\,/\sqrt{1-\cos(2u_0)}$$
$$T4 = +\sqrt{2}\,(1/2 - u_0/\pi)\sqrt{ch\xi-\cos(u)}\,/\,\sqrt{1-\cos(2u_0)}$$

This result is immediately problematical. We will have $\sqrt{ch\xi-\cos(u)} \;\approx\; \sqrt{2}\,a/r$ and then our potential will have a residual constant piece that should not be there for large r, coming from T1+T2.

Let's back up and examine a previous paragraph:

" First, consider the $\tan^{-1}$ in T2. We know $(ch\xi - cosu) \rightarrow 0$ in our limit, so this thing is going to be $\tan^{-1}(\sqrt{2}/0) = \tan^{-1}(+\infty) = \pi/2$ where I guess I assume $(ch\xi - cosu) \rightarrow 0+$."

Had we assumed $(ch\xi - cosu) \rightarrow 0-$, we would not have this problem, so everything hinges on this seeming quirk. But I think of $ch\xi > 1$ and the 0+ really is the correct thing here.

STOP.

4. Show that V→0 as r→∞, and compute the Capacitance directly from the integral.

Here is our presumed potential

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-u)\tau] / ch^2(\pi\tau)$$

If we go to our large r limit right away, we set

$$r \approx \sqrt{2}\ a / \sqrt{ch\xi - cosu} \qquad => \qquad \sqrt{ch\xi - cosu} = \sqrt{2}\ a / r$$

and we would then have

$$V(\xi,u) = V_0 \sqrt{2}\ (\sqrt{2}\ a / r)\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-u)\tau] / ch^2(\pi\tau)$$

$$= (\ 2aV_0 / r)\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-u)\tau] / ch^2(\pi\tau) \equiv (\ 2aV_0 / r)\ I$$

so we expect the integral to somehow be a constant in this limit. The limit is $\xi \rightarrow 0$ and $u \rightarrow 2\pi$ so if we are lucky, we can just do it directly, ignoring order interchanges,

$$I = \int_0^\infty d\tau\ P_{i\tau-1/2}(1)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-2\pi)\tau] / ch^2(\pi\tau)$$

$$= \int_0^\infty d\tau\ ch[(\pi-u_0)\tau]\ ch[(u_0-\pi)\tau] / ch^2(\pi\tau)$$

$$= \int_0^\infty d\tau\ ch^2[(\pi-u_0)\tau] / ch^2(\pi\tau)$$

$$= \int_0^\infty dt\ ch^2[(\pi-A)t] / ch^2(\pi t) \qquad A \equiv u_0 \qquad t = \tau$$

Let $x = \pi t$ and $dx = \pi dt$

$= \int_0^\infty dx/\pi \ * ch^2[x - Ax/\pi] / ch^2x$ Let $b \equiv 1 - A/\pi$

$= (1/\pi) \int_0^\infty dx\ ch^2[bx] / ch^2x$ $b < 1$

Maple still won't do it. Now expand

$2ch^2[bx] = ch(2bx)+1$

$I = (1/2\pi) \int_0^\infty dx\ (ch(2bx)+1)/ ch^2x = (1/2\pi)(I_1 + I_2)$

Let's try GR7 p372

3.512

1. $$\int_0^\infty \frac{\cosh 2\beta x}{\cosh^{2\nu} ax}\,dx = \frac{4^{\nu-1}}{a} \mathrm{B}\left(\nu + \frac{\beta}{a}, \nu - \frac{\beta}{a}\right) \qquad [\mathrm{Re}\,(\nu \pm \beta) > 0, \quad a > 0, \quad \beta > 0]$$

LI(27)(17)a, EH I 11(26)

Want $\nu = 1$ and $\beta = b$. Then need $Re(1 \pm b) > 0$ which is OK, $a = 1 > 0$, $b > 0$ yes!

$I_1 = B(1+b,1-b) = \Gamma(1+b)\Gamma(1-b)/\Gamma(2) = \Gamma(1+b)\Gamma(1-b) = \pi b/\sin(\pi b)$

Meanwhile,

$I_2 = \int_0^\infty dx/ ch^2x = 1$ says GR7 p 372

So we then have

$I = (1/2\pi)[\pi b/\sin(\pi b) + 1]$

$V = (2aV_0 /r) I = (2aV_0 /r) (1/2\pi)[\pi b/\sin(\pi b) + 1] = (aV_0 /r) (1/\pi)[\pi b/\sin(\pi b) + 1]$

$= (aV_0 /r) [b/\sin(\pi b) + 1/\pi] = (aV_0 /r) [b + \sin\pi b/\pi]/\sin(\pi b)$

as our large r limit. If we think of this as $V = Q/r$ we then have

$Q = (aV_0) [b + \sin\pi b/\pi]/\sin(\pi b) = (aV_0) [b/\sin(\pi b) + 1/\pi]$

Then capacitance is

$C = Q/V_0 = (a) [b/\sin(\pi b) + 1/\pi]$

We know that $R = a/\sin(u_0) \Rightarrow$ $a = R\sin(u_0)$

Also we had

$b = 1-u_0/\pi \qquad \pi b = \pi-u_0 \qquad \sin(\pi b) = \sin(\pi-u_0) = -\sin(-u_0) = \sin(u_0)$

$C = (a)\{ b/\sin(\pi b) + 1/\pi \}$

$= R\sin(u_0)\{ (1-u_0/\pi)/[\sin(u_0)] + 1/\pi \}$

$= R\{ (1-u_0/\pi) + \sin(u_0)/\pi \}$

$= R\{1- u_0/\pi + \sin(u_0)/\pi \}$

$= R\{\pi/\pi- u_0/\pi + \sin(u_0)/\pi \}$

$= (R/\pi)\{ \pi - u_0 + \sin(u_0) \}$ // hurray !!!!

Now I know from Smythe that the capacitance of a bowl is this (A is bowl radius)

$C = (R/\pi)\{\pi - \alpha + \sin\alpha \}$

and we agree exactly.

So if we want to think of our charged bowl as forming a capacitor with the "great sphere", we have to think of the great sphere as two great hemispheres, upper and lower. The upper great hemisphere has u = 0 while the lower great hemisphere has u = 2π. Here are the various equations relating r to the toroidal coordinates first in general, and then when r is large.

$r^2 = a^2 (\mathrm{sh}^2\xi + \sin^2 u)/(\mathrm{ch}\xi - \cos u)^2$

$r^2 = a^2 (sh^2\xi + \sin^2 u)/(ch\xi - cosu)^2 \approx a^2 (\xi^2+u^2) / [(1/2)(\xi^2+u^2)]^2 = (2a)^2/ (\xi^2+u^2)$

$(ch\xi - cosu) \approx (1/2)(\xi^2+u^2)$ => $\sqrt{\xi^2+u^2} \approx \sqrt{2}\sqrt{ch\xi - cosu}$

$r \approx 2a/\sqrt{\xi^2+u^2}$ $r \approx \sqrt{2}\, a / \sqrt{ch\xi - cosu}$ in the large r limit

My ansatz for the moment is that we have to think of our bowl "being two capacitors" , one with the great upper, and the other with the great lower, and that we then add the capacitances.

Upper Great Hemisphere Capacitor.

Our general potential is this

$V(\xi,u)= (V_0 /\sqrt{2}) * (T1 + T2 + T3 + T4)$
$T1= 1/\sqrt{2}$
$T2= -(-1)^{\eta 2}\sqrt{2}\ \pi^{-1} * \tan^{-1}[\sqrt{1+cos(u)} /\sqrt{ch\xi-cos(u)}\ ]$
$T3 = (1/\sqrt{2})\sqrt{ch\xi-cos(u)} /\sqrt{ch\xi-cos(2u_0-u)}$
$T4 = - (-1)^{\eta 4}\sqrt{2}\ \pi^{-1}\sqrt{ch\xi-cos(u)} /\sqrt{ch\xi-cos(2u_0-u)} * \tan^{-1}[\sqrt{1+cos(2u_0-u)} /\sqrt{ch\xi-cos(2u_0-u)}\ ]$

$\eta_2 = floor(2-u/\pi)$
$\eta_4 = floor([2u_0-u]/\pi)$

For the upper capacitor we have $u \approx 0$ on the great hemisphere so $cos(u) \rightarrow 1$ etc.