**Capacitance of the Toroidal Bowl, v2** PhL 1.31.11

We shall keep the original document alive, but here I want to take as direct a path as possible.

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**Overview (1.5 pages, written 2.3.11)**

In **Section 1** I "start over" yet again with my single-bowl dτ potential integral. By this time, I have written "doing Mehler integrals" and I use this Mehler integral to evaluate the potential's two terms

$$\int_0^\infty d\tau\, P_{i\tau-1/2}(y)\, \mathrm{ch}(b\tau)\, \mathrm{sech}^2(\pi\tau) \;=\; (\sqrt{2}/\pi)\;(1/\sqrt{y-\cos b})\,\tan^{-1}[\sqrt{y-\cos b}/\sqrt{1+\cos b}\,]$$

This has only a single term because I used $\tan^{-1}(A/B) + \tan^{-1}(B/A) = \pi/2$ (which is valid regardless of the signs of A and B). Just for comparison, here was the integral I used in the 1_11 doc which had the two terms:

$$\int_0^\infty d\tau\, P_{i\tau-1/2}(y)\, \mathrm{ch}[b\tau]\, \mathrm{sech}^2(\pi\tau)$$

$$= 2^{-1/2}/\sqrt{y-\cos b} \;-\sqrt{2}\;\pi^{-1}/\sqrt{y-\cos b}\;*\tan^{-1}[\sqrt{1+\cos b}/\sqrt{y-\cos b}\;]$$

Due to the identity just stated, these two integral forms are exactly the same. I prefer to use the first for two reasons: (1) it has only one term instead of two; (2) it will lead to $\tan^{-1}(\tan(u_0))$ at the end. So I have not corrected any error here. The true error is present in both forms, namely, they are both only valid for $|b| < \pi$, but I apply them outside this range! But of course I don't yet know about this problem!

So I show the usual picture, take the usual starting steps, make the usual simplifications, and I end up with my opening V(ξ,u) = two terms, instead of the former T1,T2,T3,T4 four terms. At this point I study the ranges of the two ch(b) arguments and I find that

$b = (2\pi-u)$ lies in range $(-u_0, 2\pi-u_0)$
$b = (2u_0-u)$ lies in range $(u_0-2\pi, u_0)$

Thus, if I use a value $u_0 = \pi/4$, as u varies both b expressions go "out of range" beyond $|b| < \pi$. This out of range problem occurs no matter what $u_0$ I pick in the range $(0,\pi)$. In particular, if I pick $u = 2\pi$ as I will to study the 1/r limit, for $u_0 < \pi/2$ the second b will be below $-\pi$. But I don't know about this out of range problem yet, so an error is built in from the start. I thought at this time that I was good for $|b| < 2\pi$.

In **Section 2** I do the usual $\sqrt{1+\cos z}$ analytic analysis and I come up with $(-1)^{\eta}$ type phases which are now called $\eta_1$ and $\eta_2$ (one for each of the two terms), and I state the general V(ξ,u) evaluated result at the end of Section 2. As noted above, this thing is wrong near $u \approx 2\pi$ where I am about to use it!

In **Section 3** I then do the large-r limit and carry through to compute the capacitance. This time I get both the $u_0$ and $\sin(u_0)$ terms in C, but I am missing the π constant term! Again I add the missing term in red and trace it back to try to find the error. I am completely stumped, lots of algebra checks fail to produce this missing π term.

I know that in the limit $u_0 \to 0$ the π term is all that is left and we should get the C of a sphere which is just C = (R/π)π. But when I take the $u_0 \to 0$ limit of my evaluated form, I get C = 0! This is consistent with my work just above. Both results have a problem because my general evaluated form is wrong!

Next, I try to take the full-sphere $u_0 \to 0$ limit on the full dτ integral form for V. I find that V(ξ,u) = $V_0$. This is the same mysterious result I got in Section 12 of the 1_11 doc, since I am expecting Q/r. It turns out this is a separate mystery which I will later resolve.

For my last gasp, I check that my general V form satisfies both BC's, and it does.

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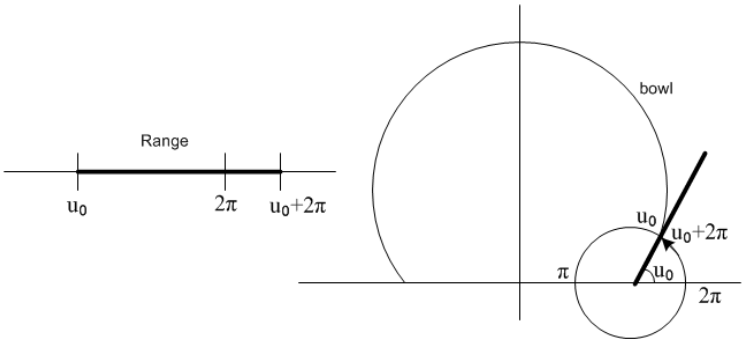
**1. I start with what I claim is the potential for the charged bowl**

$$V(\xi,u) = V_0 \sqrt{2}\ \sqrt{\cosh\xi - \cos u}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(\cosh\xi)\ \cosh[(\pi-u_0)\tau]\ \cosh[(\pi+u_0-u)\tau]\ /\ \cosh^2(\pi\tau)$$




where the pictures shows that u lies in the range $(u_0, 2\pi+u_0)$ and $u_0$ lies in range $(0,\pi)$.

I now do a ch ch expansion in the integrand

$$\cosh[(\pi-u_0)\tau]\ \cosh[(\pi+u_0-u)\tau] = (1/2)\ [\ \text{ch(sum)} + \text{ch(diff)}\ ]$$

$$= (1/2)\ \{\ \cosh[(2\pi-u)\tau] + \cosh[(2u_0-u)\tau]\ \}$$

so we can then write our potential as the sum of two terms,

$$V(\xi,u) = (V_0/\sqrt{2})\ \sqrt{\cosh\xi - \cos u}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(\cosh\xi)\ \{\ \cosh[(2\pi-u)\tau] + \cosh[(2u_0-u)\tau]\}\ /\ \cosh^2(\pi\tau)$$

We now call upon this integral from "doing Mehler transforms...." in math/integrals:

$$\int_0^\infty d\tau\, P_{i\tau-1/2}(y)\, ch(b\tau)\, sech^2(\pi\tau) = (\sqrt{2}/\pi)\ (1/\sqrt{y-cosb})\ \tan^{-1}[\sqrt{y-cosb}/\sqrt{1+cosb}]$$

I might note in passing that it took me 2-3 days to get this thing right! Therefore, we can write our potential in this manner

$$V(\xi,u) = (V_0/\sqrt{2})(\sqrt{2}/\pi)\ \sqrt{ch\xi - cosu}\ *$$
$$\{(1/\sqrt{ch\xi - cos(2\pi-u)})\ \tan^{-1}[\sqrt{ch\xi - cos(2\pi-u)}\ /\sqrt{1+cos(2\pi-u)}\ ]$$
$$+\ (1/\sqrt{ch\xi - cos(2u_0-u)})\ \tan^{-1}[\sqrt{ch\xi - cos(2u_0-u)}/\sqrt{1+cos(2u_0-u)}\ ]\ \}$$

$u_0$ in $(0,\pi)$
u in $(u_0, 2\pi+u_0)$ $\quad$ $ch\xi$ in $(1,\infty)$

This is a pretty amazing result if true. All in closed form with only elementary functions.

We need now to study the insides of the various square roots. In the following square root

$$\sqrt{ch\xi - cos(2\pi-u)}$$

one never has a singular situation because the inside of the radical never goes negative. This being the case, we can simplify this radical in the obvious manner

$$\sqrt{ch\xi - cos(2\pi-u)} = \sqrt{ch\xi - cosu}$$

Doing this in the above and combining constants, we get

$$V(\xi,u) = (V_0/\pi)\sqrt{ch\xi - cosu}\ *$$
$$\{(1/\sqrt{ch\xi - cosu})\ \tan^{-1}[\sqrt{ch\xi - cosu}\ /\sqrt{1+cos(2\pi-u)}\ ]$$
$$+\ (1/\sqrt{ch\xi - cos(2u_0-u)})\ \tan^{-1}[\sqrt{ch\xi - cos(2u_0-u)}/\sqrt{1+cos(2u_0-u)}\ ]\ \}$$

Next, we need some ranges.

$(2\pi-u)_{max} = (2\pi-u_{min}) = 2\pi - u_0$ which lies in the range $(\pi,2\pi)$
$(2\pi-u)_{min} = (2\pi-u_{max}) = 2\pi - (2\pi+u_0) = -u_0$ which lies in the range $(-\pi,0)$

=> range of $(2\pi-u)$ is $(-u_0, 2\pi-u_0)$ which lies within $(-\pi, 2\pi)$

$(2u_0-u)_{max} = (2u_0-u_{min}) = 2u_0 - u_0 = u_0$ which lies in the range $(0,\pi)$
$(2u_0-u)_{min} = (2u_0-u_{max}) = 2u_0 - (2\pi+u_0) = u_0-2\pi$ which lies in the range $(-2\pi,-\pi)$

=> range of $(2u_0-u)$ is $(u_0-2\pi, u_0)$ which lies within $(-2\pi, \pi)$

**2. Study of the analytic function $f(z) = \sqrt{1+cosz}$ and apply to our potential**

We must now digress a bit on this subject, since it is crucial in order to more fully elaborate the potential obtained above for the charged bowl. As described elsewhere, we find that

$$f(z) \equiv \sqrt{1+\cos z} \qquad \text{for } z \text{ in the range } (-\infty,\infty)$$

$$= (-1)^{\eta} \left| \sqrt{1+\cos(z)} \right| \qquad \text{where } \eta = \text{floor}[(z+\pi)/2\pi]$$

What this says is that, as z moves along the real axis, every time it crosses an odd multiple of π, the square root changes sign, meaning we move from one branch to another of the square root function. For z = π/2 say, η = 0 and we are on the positive branch. So the square root is positive on intervals like (-π,π), (3π,5π), (7π,9π)... and is negative on the complementary intervals. At the boundaries, we have a $\sqrt{0}$ situation, meaning we are at a branch point.

We with to apply this branching rule to our two radicals that begin with 1. Thus

$$\sqrt{1+\cos(2\pi-u)} = (-1)^{\eta 1} \left| \sqrt{1+\cos(2\pi-u)} \right|$$

where

$$\eta_1 = \text{floor}[(z+\pi)/2\pi] = \text{floor}[((2\pi-u)+\pi)/2\pi] = \text{floor}[(3\pi-u)/2\pi] \qquad // \text{ u in } (u_0,\ 2\pi+u_0)$$

Thus we get a sign change when u = π which is the location on the flat end of the bowl.

Once we have the absolute values on, we can simplify to get

$$\sqrt{1+\cos(2\pi-u)} = (-1)^{\eta 1} \left| \sqrt{1+\cos u} \right|$$

The other root is this

$$\sqrt{1+\cos(2u_0-u)} = (-1)^{\eta 2} \left| \sqrt{1+\cos(2u_0-u)} \right|$$

where

$$\eta_2 = \text{floor}[(z+\pi)/2\pi] = \text{floor}[((2u_0-u)+\pi)/2\pi] = \text{floor}[(2u_0-u+\pi)/2\pi] \qquad // \text{ u in } (u_0,\ 2\pi+u_0)$$

We can then write our potential in this manner, where we extract the phases from the $\tan^{-1}$ functions,

$$V(\xi,u) = (V_0/\pi) \sqrt{ch\xi - \cos u} \ *$$
$$\{ (1/\sqrt{ch\xi - \cos u}) \quad (-1)^{\eta 1} \ \tan^{-1}[\sqrt{ch\xi - \cos u} \quad /| \sqrt{1+\cos u} | ]$$
$$+ \ (1/\sqrt{ch\xi - \cos(2u_0-u)}) \ (-1)^{\eta 2} \ \tan^{-1}[ \sqrt{ch\xi - \cos(2u_0-u)} \ /| \sqrt{1+\cos(2u_0-u)} |] \ \}$$

where $\eta_1 = \text{floor}[(3\pi-u)/2\pi]$

$\eta_2 = \text{floor}[(2u_0-u+\pi)/2\pi]$

**3. Evaluate the potential "far away".**

We are interested in the large r limit and we expect to see V = Q/r in this limit. Far away in our particular toroidal coordinates with the unusual range for u means this:

far away: $\xi \to 0$ $u \to 2\pi$

In this limit we may write

$$|\sqrt{1+\cos u}| = |\sqrt{1+\cos(2\pi)}| = |\sqrt{1+\cos(0)}| = \sqrt{2}$$

and

$$|\sqrt{1+\cos(2u_0-u)}| = |\sqrt{1+\cos(2u_0-2\pi)}| = |\sqrt{1+\cos(2u_0)}| = \sqrt{2}\,|\cos(u_0)|$$

The phases become

$$\eta_1 = \text{floor}[(3\pi-u)/2\pi] = \text{floor}[(3\pi-2\pi)/2\pi] = \text{floor}[(\pi)/2\pi] = 0$$

$$\eta_2 = \text{floor}[(2u_0-u+\pi)/2\pi] = \text{floor}[(2u_0-2\pi+\pi)/2\pi] = \text{floor}[(2u_0-\pi)/2\pi]$$
$= -1$ if $u_0$ lies in $(0,\pi/2)$
$= 0$ if $u_0$ lies in $(\pi/2,\pi)$

Finally we may write

$$\sqrt{ch\xi-\cos(2u_0-u)} = \sqrt{ch\xi-\cos(2u_0-2\pi)} = \sqrt{ch\xi-\cos(2u_0)}$$

But we are taking $\xi \to 0+\varepsilon$ so to speak, so

$= \sqrt{1-\cos(2u_0)} = \sqrt{2}\,\sin(u_0)$ // recall $u_0$ in $(0,\pi)$ so no $|\ |$ needed

In taking our limit, we maintain the factor $(ch\xi-\cos u)$ wherever it appears, for reasons soon to be obvious. So our potential now has this form in our limit

$$V(\xi,u) = (V_0/\pi)\sqrt{ch\xi - \cos u} \;*$$
$$\{ (1/\sqrt{ch\xi - \cos u}) \quad \tan^{-1}[\sqrt{ch\xi - \cos u} \;/\sqrt{2}\;] $$
$$+ (1/[\sqrt{2}\,\sin(u_0)])\,(-1)^{\eta 2}\,\tan^{-1}[\sqrt{2}\,\sin(u_0)/|\,[\sqrt{2}\,|\cos(u_0)|\,]\;\}$$

In our large r limit we have

$$r \approx \sqrt{2}\,a / \sqrt{ch\xi - \cos u} \quad \Rightarrow \quad \sqrt{ch\xi - \cos u} = \sqrt{2}\,a / r$$

So $r \to \infty$ means $\sqrt{ch\xi - \cos u} \to 0$, as is also obvious from our limit $\xi \to 0$ and $u \to 2\pi$ giving. The constant a is the focal distance of the toroidal coordinates which we can write as

$a = R \sin(u_0)$ $R$ = radius of the bowl sphere

So then we have

$$\sqrt{ch\xi - cosu} = \sqrt{2}\, a / r = \sqrt{2}\, R \sin(u_0)/ r$$

So we install this in one location to get

$$V(\xi,u)= (V_0/\pi ) (\sqrt{2}\, R \sin(u_0)/ r) *$$
$$\{ (1/\sqrt{ch\xi - cosu}\,) \quad \tan^{-1}[\sqrt{ch\xi - cosu}\, /\sqrt{2}\, ]$$
$$+ (1/[\sqrt{2}\, \sin(u_0)]) (-1)^{\eta 2}\, \tan^{-1}[\sin(u_0)/| [| \cos(u_0) | ] \}$$

Next, we approximate

$$\tan^{-1}[\sqrt{ch\xi - cosu}\, /\sqrt{2}\, ] \approx \sqrt{ch\xi - cosu}\, /\sqrt{2}$$

which then gives

$$V(\xi,u)= (V_0/\pi ) (\sqrt{2}\, R \sin(u_0)/ r) *$$
$$\{ (1/\sqrt{2}\,) + (1/[\sqrt{2}\, \sin(u_0)]) (-1)^{\eta 2}\, \tan^{-1}[\sin(u_0)/| [| \cos(u_0) | ] \}$$

or

$$V(\xi,u)= (V_0/\pi ) (R \sin(u_0)/ r) *$$
$$\{ 1 + \csc(u_0)\, (-1)^{\eta 2}\, \tan^{-1}[\sin(u_0)/| [| \cos(u_0) | ] \}$$

Certainly for $u_0$ in $(0/\pi/2)$ we get

$$\tan^{-1}[\sin(u_0)/| [| \cos(u_0) | ] = \tan^{-1}[\tan(u_0)] = u_0 - \pi$$ // red shows what I need to happen

and we also know that in this range, $\eta_2 = -1$ so then we have

$$V(\xi,u)= (V_0/\pi ) (R \sin(u_0)/ r) * \{ \pi \csc(u_0) + 1 - \csc(u_0)\, u_0 \}$$

$$= (1/r) (V_0R/\pi ) \{ \pi + \sin(u_0) - u_0 \}$$

Since this must have the form Q/r, we identify

$$Q = (V_0R/\pi ) \{ \pi + \sin(u_0) - u_0 \}$$

$$C = Q/V_0 = (R/\pi ) \{ \pi + \sin(u_0) - u_0 \}$$

The known Smythe result is this

$$C = (R/\pi)\ \{\pi - u_0 + \sin(u_0)\}$$

so we have everything except the constant term, and we have restricted to $u_0$ in $(0,\pi/2)$ which means bowls that are more than a hemisphere.

I know that the $\pi$ term above is what causes the "full sphere" to have capacitance C = R. So let's look at $u_0 \rightarrow 0$ and see if we find this constant capacitance.

$$V(\xi,u)= (V_0/\sqrt{2})(\sqrt{2}/\pi)\ \sqrt{ch\xi - cosu}\ *$$
$$\{ (1/\sqrt{ch\xi - cos(2\pi-u)})\ \tan^{-1}[\sqrt{ch\xi - cos(2\pi-u)}\ /\sqrt{1+cos(2\pi-u)}\ ]$$
$$+\ (1/\sqrt{ch\xi-cos(2u_0-u)})\ \tan^{-1}[\sqrt{ch\xi - cos(2u_0-u)}/\sqrt{1+cos(2u_0-u)}\ ]\ \}$$

$$V(\xi,u)= (V_0/\sqrt{2})(\sqrt{2}/\pi)\ \sqrt{ch\xi - cosu}\ *$$
$$\{ (1/\sqrt{ch\xi - cos(u)})\ \tan^{-1}[\sqrt{ch\xi - cos(u)}\ /\sqrt{1+cos(u)}\ ]$$
$$+\ (1/\sqrt{ch\xi-cos(u)})\ \tan^{-1}[\sqrt{ch\xi - cos(u)}/\sqrt{1+cos(u)}\ ]\ \}$$

$$V(\xi,u)= (V_0/\sqrt{2})(\sqrt{2}/\pi)\ \sqrt{ch\xi - cosu}\ *$$
$$\{ (1/\sqrt{ch\xi - cos(u)})\ \tan^{-1}[\sqrt{ch\xi - cos(u)}\ /\sqrt{2}\ ]$$
$$+\ (1/\sqrt{ch\xi-cos(u)})\ \tan^{-1}[\sqrt{ch\xi - cos(u)}/\sqrt{2}\ ]\ \}$$

$$V(\xi,u)= (V_0/\sqrt{2})(\sqrt{2}/\pi)\ \sqrt{ch\xi - cosu}\ *$$
$$\{ (1/\sqrt{ch\xi - cos(u)})\ [\sqrt{ch\xi - cos(u)}\ /\sqrt{2}\ ]$$
$$+\ (1/\sqrt{ch\xi-cos(u)})\ [\sqrt{ch\xi - cos(u)}/\sqrt{2}\ ]\ \}$$

$$V(\xi,u)= (V_0/\sqrt{2})(\sqrt{2}/\pi)\ \sqrt{ch\xi - cosu}\ *$$
$$\{ (1/\sqrt{2}\ ]$$
$$+\ (1/\sqrt{2}\ ]\ \}$$

$$V(\xi,u)= (V_0/\sqrt{2})(\sqrt{2}/\pi)\ \sqrt{ch\xi - cosu}\ *\ 2/\sqrt{2}$$

$$V(\xi,u)= (\sqrt{2}\ V_0/\pi)\ \sqrt{ch\xi - cosu}$$

$$V(\xi,u)= (\sqrt{2}\ V_0/\pi)\sqrt{2}\ R\sin(u_0)/r \rightarrow 0/r \quad \Rightarrow \text{ no capacitance}$$

What happens if you do this directly to the full integral, ie, set $u_0 \rightarrow 0$ here:

$$V(\xi,u)= V_0\sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-u)\tau]\ /\ ch^2(\pi\tau)$$

$$V(\xi,u)= V_0\sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi)\tau]\ ch[(\pi-u)\tau]\ /\ ch^2(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi)\tau]\ ch[(\pi-2\pi)\tau] / ch^2(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch^2[(\pi)\tau] / ch^2(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ (1/\sqrt{2}\ )\ (1/\sqrt{ch\xi - 1}\ )$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - 1}\ (1/\sqrt{2}\ )\ (1/\sqrt{ch\xi - 1}\ )\ = V_0$$

This says the potential is constant $V_0$ at r=∞, so no wonder we have not capacitance.

Now the capacitance seems to be infinite!

Check BC's. First $u = u_0$ :

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-u)\tau] / ch^2(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi)\tau] / ch^2(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau] / ch(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ (1/\sqrt{2}\ )\ (ch\xi+cos(\pi-u_0))^{-1/2}$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu_0}\ (1/\sqrt{2}\ )\ (ch\xi-cos(u_0))^{-1/2}$$

$$V(\xi,u)= V_0$$

Second check $u = 2\pi + u_0$ :

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-u)\tau] / ch^2(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-[2\pi+u_0]\ )\tau] / ch^2(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi)\tau] / ch^2(\pi\tau)\ = \text{same}$$