Capacitance of the Toroidal Bowl, v3 PhL 2.1.11

All the stuff shown below developed in this series of three docs is now somewhere in bowl doc. There are no outstanding problems here.

We shall keep the original document alive, but here I want to take as direct a path as possible.

Overview (3 pages, written 2.2.11) **1**

1. Evaluation of the full integral form into elementary functions. **3**

2. Study of the analytic function f(z) = $\sqrt{1+\cos z}$ and apply to our potential **5**

3. Potential conversion to Cartesian coordinates (too messy) **7**

4. Evaluate the potential "far away" and get C for the charged bowl for $u_0 < \pi/2$ **13**

4a. Evaluate the potential "far away" and get C for the charged bowl for $\pi/2 < u_0 < \pi$ **15**

5. What happens for $u_0 \to 0$ in our post-evaluated form? **17**

6. What happens for $u_0 \to 0$ in the full integral form ? **19**

7. Check boundary conditions using the full integral form. **19**

Overview (3 pages, written 2.2.11)

In **Section 1** I have the benefit of having studied analytic continuation of the ch(bτ) integral, and I realize that you must use the $\cot^{-1}$ instead of the $\tan^{-1}$ form. So, I now use the improved full-b-range Mehler ch(bτ) integral

$$\int_0^\infty d\tau\, P_{i\tau-1/2}(y)\, \mathrm{ch}(b\tau)\, \mathrm{sech}^2(\pi\tau) \;=\; (\sqrt{2}/\pi)\;(1/\sqrt{y-\cos b})\,\cot^{-1}[\sqrt{1+\cos b}/\sqrt{y-\cos b}] \qquad (11.5)$$

to "evaluate" the single-bowl potential integral (bowl label is u_0)

$$V(\xi,u)= V_0\sqrt{2}\;\sqrt{\mathrm{ch}\xi-\cos u}\;\int_0^\infty d\tau\, P_{i\tau-1/2}(\mathrm{ch}\xi)\;\mathrm{ch}[(\pi-u_0)\tau]\,\mathrm{ch}[(\pi+u_0-u)\tau]\,/\,\mathrm{ch}^2(\pi\tau) \qquad (1.1)$$

into elementary functions and I ended up with

$$V(\xi,u)= (V_0/\sqrt{2})\,(\sqrt{2}/\pi)\;\sqrt{\mathrm{ch}\xi-\cos u}\;*$$
$$\{\,(1/\sqrt{\mathrm{ch}\xi-\cos(2\pi-u)})\,\cot^{-1}[\sqrt{1+\cos(2\pi-u)}/\sqrt{\mathrm{ch}\xi-\cos(2\pi-u)}\;]$$
$$+\;(1/\sqrt{\mathrm{ch}\xi-\cos(2u_0-u)})\cot^{-1}[\sqrt{1+\cos(2u_0-u)}/\sqrt{\mathrm{ch}\xi-\cos(2u_0-u)}\,]\;\}$$

u_0 in $(0,\pi)$

u in $(u_0, 2\pi+u_0)$ $\qquad\qquad$ $\mathrm{ch}\xi$ in $(1,\infty)$

which is not in my final useful form yet.

In **Section 2**, as usual, I take into consideration the analytic behavior of two functions in the above formula which have the form $\sqrt{1+\cos(z)}$. These radicals change sign when z passes through odd values of π, so you end up with $\sqrt{1+\cos(z)} = (-1)^{\eta} \left| \sqrt{1+\cos(z)} \right|$ where $\eta = \text{floor}[(z+\pi)/2\pi]$. Then we get

$$V(\xi,u)= (V_0/\pi) \sqrt{ch\xi - cosu} \; *$$
$$\{ (1/\sqrt{ch\xi - cosu}) \quad \cot^{-1}[(-1)^{\eta 1} | \sqrt{1+cosu} | \; / \sqrt{ch\xi - cosu}]$$
$$+ (1/\sqrt{ch\xi - \cos(2u_0 - u)}) \cot^{-1}[(-1)^{\eta 2} | \sqrt{1+\cos(2u_0 - u)} | / \sqrt{ch\xi - \cos(2u_0 - u)}] \; \} \qquad (2.1)$$

where $\eta_1 = \text{floor}[(3\pi - u)/2\pi]$ $\qquad$ u_0 in $(0,\pi)$

$\eta_2 = \text{floor}[(2u_0 - u + \pi)/2\pi]$ $\qquad$ u in $(u_0, u_0 + 2\pi)$

It took a huge amount of work to get the results shown above: (1) I had to find, then derive, then tune the Mehler integral (~ 3 days); (2) I had to find a toroidal coordinates integral for the single-bowl (~ 2 days); (3) I had to understand the $\sqrt{1+\cos(z)}$ stuff (~ 2 days).

In **Section 3** I derive formulas which would allow the potential in (2.1) above to be expressed in Cartesian coordinates. It ain't purdy, here are some of the substitutions you would have to make

$Q \equiv 1/\sqrt{[z^2 + (a-\rho)^2][z^2 + (a+\rho)^2]}$	$\rho^2 = x^2 + y^2$	u_0 = bowl label
$ch\xi - cosu = 2a^2 Q$	$\cos u = (a^2 - \rho^2 - z^2) Q$	
$ch\xi = (a^2 + \rho^2 + z^2) Q$	$\sin u = 2az\, Q$	$a = R \sin u_0$

In **Section 4** I examine 2.1 above in the **"far away" limit r→∞** which is ξ→0 and u→2π. But I find I have to break this limiting into two cases, and this section covers the case $u_0 \le \pi/2$. Here is one intermediate result I get

$$V(\xi,u)= (V_0/\pi) \sqrt{2} \; R \sin(u_0)/ r \; *$$
$$\{ (1/\sqrt{ch\xi - cosu}) \quad \cot^{-1}[\sqrt{2} \; / \sqrt{ch\xi - cosu}]$$
$$+ (1/\sqrt{2} \sin(u_0)) [\; \pi - \cot^{-1}[|\cos(u_0)| / \sin(u_0)] \;] \; \}$$

where I handled the η's using this rule (which just describes the shape of $y = \cot^{-1}x$)

$$\cot^{-1}[(-1)^{\eta} x] = \cot^{-1}(x) \quad \text{of } \eta \text{ is even}$$
$$= \pi - \cot^{-1}(x) \text{ if } \eta \text{ is odd}$$

where $\cot^{-1}(z)$ refers to the principal branch as shown in Schaum. I continue on to obtain

$$V(\xi,u) = (1/r) \{ (V_0/\pi) R (\pi - u_0 + \sin u_0) \}$$

$$C = (R/\pi) (\pi - u_0 + \sin u_0)$$

and this is happily the known correct result (this took me another ~2 days) to get right.

In **Section 4a** I take the same limit for the case $\pi/2 < u_0 < \pi$, and I get the same result.

In **Section 5** I take my general post-evaluated potential shown in (2.1) above, and I just set $u_0 = 0$ in hopes of obtaining the simple potential of a sphere. To my initial surprise, I get $V(\xi,u) = V_0$! I was expecting to get something like this inside the sphere and $V = V_0(R/r)$ outside the sphere, so this was a temporary mystery.

In **Section 6** I take the same $u_0 = 0$ limit of the full integral formula for the potential (1.1) above, and I get exactly the same conclusion that $V(\xi,u) = V_0$ [duplicating v2 Sec 4] . I realize (actually back in Section 5) that this can be explained as follows. We have

$x = a \cos\varphi \, \mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u)$ $\qquad \rho = a \, \mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u)$
$y = a \sin\varphi \, \mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u)$
$z = a \sin u/(\mathrm{ch}\xi - \cos u)$

If we let $a = R\sin u_0 \rightarrow 0$, then x=y=z=0 for all ξ and u which says the entire toroidal coordinate system has contracted down to that singular point on the bottom of the bowl-now-sphere. If this is correct, then we do in fact expect to get $V = V_0$ for all u,ξ coordinate values, which is what I found above!

In **Section 7** I take the full integral potential form (1.1) above and show that it meets the two boundary conditions of the problem, namely, that $V(\xi,u_0) = V_0$ and $V(\xi,u_0+2\pi) = V_0$. [duplicating v2, very end]

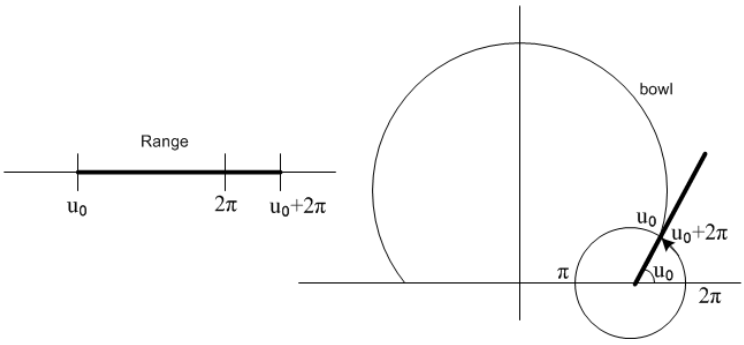
In **Section 8,** while I was unable to get the capacitance to come out right (now repaired above), I was wondering if my large r limit $\xi \rightarrow 0$ and $u \rightarrow 2\pi$ was somehow picking up spurious "near" intersections of the coordinate system H and V circles, but I guess this was not the case.

In **Section 9,** while I was unable to get the capacitance to come out right (now repaired above), I was wondering if my discrepancy was caused by incorrect use of $F(1,1/2; 3/2; z) = [\tanh^{-1}\sqrt{z}] /\sqrt{z}$ when doing the Mehler integral, since this thing seems to have a restriction $|z| < 1$. But that was not it.

---

1. Evaluation of the full integral form into elementary functions.

$$V(\xi,u) = V_0 \sqrt{2} \sqrt{\mathrm{ch}\xi - \cos u} \int_0^\infty d\tau \, P_{i\tau-1/2}(\mathrm{ch}\xi) \, \mathrm{ch}[(\pi - u_0)\tau] \, \mathrm{ch}[(\pi + u_0 - u)\tau] / \mathrm{ch}^2(\pi\tau) \qquad (1.1)$$



where the pictures shows that u lies in the range (u_0, $2\pi+u_0$) and u_0 lies in range $(0,\pi)$.

I now do a ch ch expansion in the integrand

$$\text{ch}[(\pi-u_0)\tau]\ \text{ch}[(\pi+u_0-u)\tau] = (1/2)\ [\ \text{ch(sum)} + \text{ch(diff)}\ ]$$

$$= (1/2)\ \{\ \text{ch}[(2\pi-u)\tau] + \text{ch}[(2u_0-u)\tau]\ \}$$

so we can then write our potential as the sum of two terms,

$$V(\xi,u)= (V_0/\sqrt{2})\ \sqrt{\text{ch}\xi - \text{cos}u}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(\text{ch}\xi)\ \{\ \cosh[(2\pi-u)\tau] + \cosh[(2u_0-u)\tau]\}\ /\ \text{ch}^2(\pi\tau)$$

We now call upon this integral from "doing Mehler transforms...." in math/integrals:

$$\int_0^\infty d\tau\ P_{i\tau-1/2}(y)\ \text{ch}(b\tau)\ \text{sech}^2(\pi\tau) = (\sqrt{2}/\pi)\ (1/\sqrt{y-\cos b}\ )\ \cot^{-1}[\sqrt{1+\cos b}/\sqrt{y-\cos b}\ ] \qquad (11.5)$$

and this integral is known to be valid as stated, with the usual principle branch of $\cot^{-1}(x)$, and with the analytic sign change of $\sqrt{1+\cos b}$ passing through $b = \pi$, for b in the range $(0,2\pi)$. Both sides are symmetric in b, so this is all we care about.

I might note in passing that it took me 2-4 days to get this thing right! Therefore, we can write our potential in this manner

$$V(\xi,u)= (V_0/\sqrt{2})\ (\sqrt{2}/\pi)\ \sqrt{\text{ch}\xi - \text{cos}u}\ *$$
$$\{\ (1/\sqrt{\text{ch}\xi - \cos(2\pi-u)}\ )\ \cot^{-1}[\sqrt{1+\cos(2\pi-u)}\ /\ \sqrt{\text{ch}\xi - \cos(2\pi-u)}\ ]$$
$$+\ (1/\sqrt{\text{ch}\xi-\cos(2u_0-u)}\ )\cot^{-1}[\sqrt{1+\cos(2u_0-u)}\ /\ \sqrt{\text{ch}\xi - \cos(2u_0-u)}\ ]\ \}$$

u_0 in $(0,\pi)$
u in $(u_0, 2\pi+u_0)$ $\qquad$ $\text{ch}\xi$ in $(1,\infty)$

This is a pretty amazing result if true. All in closed form with only elementary functions.

••• I am pretty sure that this result is correct because: (1) each of our two integrals is valid for its range of "parameter b"; (2) because I have worked very hard to obtain the correct integral starting point; (3) and I have worked very hard to derive the actual Mehler integral itself.

Something goes wrong with the large-r limit below causing the capacitance to be missing a term, and I think the error will turn out to be below this line:

We need now to study the insides of the various square roots. In the following square root

$$\sqrt{ch\xi - \cos(2\pi - u)}$$

one never has a singular situation because the inside of the radical never goes negative. This being the case, we can simplify this radical in the obvious manner

$$\sqrt{ch\xi - \cos(2\pi - u)} \quad = \sqrt{ch\xi - \cos u}$$

Doing this in the above and combining constants, we get

$$V(\xi,u) = (V_0/\pi)\sqrt{ch\xi - \cos u} \; *$$
$$\{ (1/\sqrt{ch\xi - \cos u}) \quad \tan^{-1}[\sqrt{ch\xi - \cos u} \; / \sqrt{1+\cos(2\pi - u)}] \qquad \text{// need flip}$$
$$+ \; (1/\sqrt{ch\xi - \cos(2u_0 - u)}) \tan^{-1}[\sqrt{ch\xi - \cos(2u_0 - u)} \; / \sqrt{1+\cos(2u_0 - u)}] \; \}$$

Next, we need some ranges.

$(2\pi-u)_{max} = (2\pi - u_{min}) = 2\pi - u_0$ which lies in the range $(\pi, 2\pi)$
$(2\pi-u)_{min} = (2\pi - u_{max}) = 2\pi - (2\pi + u_0) = -u_0$ which lies in the range $(-\pi, 0)$

=> range of $(2\pi - u)$ is $(-u_0, 2\pi - u_0)$ which lies within $(-\pi, 2\pi)$

$(2u_0-u)_{max} = (2u_0 - u_{min}) = 2u_0 - u_0 = u_0$ which lies in the range $(0, \pi)$
$(2u_0-u)_{min} = (2u_0 - u_{max}) = 2u_0 - (2\pi + u_0) = u_0 - 2\pi$ which lies in the range $(-2\pi, -\pi)$

=> range of $(2u_0 - u)$ is $(u_0 - 2\pi, u_0)$ which lies within $(-2\pi, \pi)$

2. Study of the analytic function $f(z) = \sqrt{1+\cos z}$ and apply to our potential

We must now digress a bit on this subject, since it is crucial in order to more fully elaborate the potential obtained above for the charged bowl. As described elsewhere [where is that?] , we find that

$f(z) \equiv \sqrt{1+\cos z}$ for z in the range $(-\infty, \infty)$

$= (-1)^{\eta} \, | \sqrt{1+\cos(z)} \, |$ where $\eta = \text{floor}[(z+\pi)/2\pi]$

What this says is that, as z moves along the real axis, every time it crosses an odd multiple of π, the square root changes sign, meaning we move from one branch to another of the square root function. For z = π/2 say, η = 0 and we are on the positive branch. So the square root is positive on intervals like (-π,π), (3π,5π), (7π,9π)... and is negative on the complementary intervals. At the boundaries, we have a $\sqrt{0}$ situation, meaning we are at a branch point.

We with to apply this branching rule to our two radicals that begin with 1. Thus

$$\sqrt{1+\cos(2\pi-u)} \;=\; (-1)^{\eta 1}\,|\,\sqrt{1+\cos(2\pi-u)}\,|$$

where

$$\eta_1 = \text{floor}[(z+\pi)/2\pi] \;= \text{floor}[((2\pi-u)+\pi)/2\pi] \;= \text{floor}[(3\pi-u)/2\pi] \qquad //\ u \text{ in } (u_0,\ 2\pi+u_0)$$

Thus we get a sign change when u = π which is the location on the flat end of the bowl.

Once we have the absolute values on, we can simplify to get

$$\sqrt{1+\cos(2\pi-u)} \;=\; (-1)^{\eta 1}\,|\,\sqrt{1+\cos u}\,|$$

The other root is this

$$\sqrt{1+\cos(2u_0-u)} \;= (-1)^{\eta 2}\,|\,\sqrt{1+\cos(2u_0-u)}\,|$$

where

$$\eta_2 = \text{floor}[(z+\pi)/2\pi] \;= \text{floor}[((2u_0-u)+\pi)/2\pi] \;= \text{floor}[(2u_0-u+\pi)/2\pi] \qquad //\ u \text{ in } (u_0,\ 2\pi+u_0)$$

We can then write our potential in this manner, where we extract the phases from the $\tan^{-1}$ functions,

$$V(\xi,u)= (V_0/\pi)\,\sqrt{\text{ch}\xi - \cos u}\;\; *$$
$$\{\ (1/\sqrt{\text{ch}\xi - \cos u}\,)\qquad \cot^{-1}[(-1)^{\eta 1}|\,\sqrt{1+\cos u}\,|\;/\,\sqrt{\text{ch}\xi - \cos u}\,]$$
$$+\ (1/\sqrt{\text{ch}\xi-\cos(2u_0-u)}\,)\ \cot^{-1}[(-1)^{\eta 2}|\,\sqrt{1+\cos(2u_0-u)}\,|/\,\sqrt{\text{ch}\xi - \cos(2u_0-u)}\,]\ \}$$

where $\eta_1 = \text{floor}[(3\pi-u)/2\pi]$

$\eta_2 = \text{floor}[(2u_0-u+\pi)/2\pi]$ (2.1)

NOTE: Looking at our Schaum p 19 principal branch of $\cot^{-1}(x)$, this is NOT true:

$$\cot^{-1}(-x) = -\cot^{-1}(x) \qquad //\ \text{not true!!}$$

Notice that if you "flip it over" then it is true, but that is the whole point here of our branch management!

In fact this is clearly stated in 5.83 on page 18, but I can see it just from the graph just noted

$$\cot^{-1}(-x) = \pi - \cot^{-1}(x)$$

So how are we then going to express:

$$\cot^{-1}[(-1)^{\eta} x] = \cot^{-1}(x) \quad \text{of } \eta \text{ is even}$$
$$= \pi - \cot^{-1}(x) \text{ if } \eta \text{ is odd}$$

Let's just leave it at that, rather than write a complicated expression.

3. Potential conversion to Cartesian coordinates (too messy)

I have written this up in Appendix J of bowl doc.

If we wanted to see our potential in **Cartesian coordinates**, we would use these equations

$$\text{th}\xi = 2a\rho/(a^2+\rho^2+z^2) \qquad \text{// inversions}$$
$$\tan(u) = -2az/(a^2-\rho^2-z^2) \qquad \cot(u) = (\rho^2+z^2-a^2)/(2az)$$
$$a = R|\sin u_0|$$

We know that (Maple for second line, see th.mws)

$$\text{ch}\xi = 1/\sqrt{1-\text{th}^2\xi}$$

$$1 - \text{th}^2\xi = [z^2 + (a-\rho)^2][z^2 + (a+\rho)^2] / (a^2+\rho^2+z^2)^2$$

$$\text{ch}\xi = (a^2+\rho^2+z^2)/\sqrt{[z^2 + (a-\rho)^2][z^2 + (a+\rho)^2]}$$
$$\text{sh}\xi = +2a\rho/\sqrt{[z^2 + (a-\rho)^2][z^2 + (a+\rho)^2]}$$

The reason for the last + sign is that the range for ξ is $(0,\infty)$.

Then in the u department we have

$$\cos u = \pm 1/\sqrt{1+\tan^2 u}$$

$$1 + \tan^2 u = [z^2 + (a-\rho)^2][z^2 + (a+\rho)^2] / (a^2-\rho^2-z^2)^2$$

$$\cos u = \pm (a^2-\rho^2-z^2)/\sqrt{[z^2 + (a-\rho)^2][z^2 + (a+\rho)^2]}$$
$$\sin u = \pm 2az/\sqrt{[z^2 + (a-\rho)^2][z^2 + (a+\rho)^2]}$$

Then we get this combination

$$\text{ch}\xi - \cos u = (a/\rho)\text{sh}\xi = 2a^2/\sqrt{[z^2 + (a-\rho)^2][z^2 + (a+\rho)^2]}$$

What about the signs of cosu and sinu ??? Go back to

$\tan(u) = -2az/(a^2 - \rho^2 - z^2)$

What this really says in my book is

$u = \text{arctan2Pi}(x,y) = \text{arctan2Pi}(a^2 - \rho^2 - z^2, -2az)$ //

This is a very tricky business. Draw a picture. Notice that

arctan2Pi(x,y)	= 1Q	if x,y are positive
	= 4Q	if x,y are negative

So the first issues is that

$\text{arctan2Pi}(x,y) \neq \text{arctan2Pi}(-x,-y)$

I see that I selected a particular way to enter this in Maple: $u = \text{arctan2Pi}(\rho^2+z^2-a^2, 2*a*z)$

In the upper half plane, $2az > 0$ but the first term could be either sign. So I guess u in $(0,\pi)$. You want to have $u = 0$ if you are on the positive x axis just above, far away say. In that case, $2az > 0$ and first argument also positive, so get $u = 0$. So that is how I calibrated this thing. If you were to negate both arguments, then you could get $u = \pi$ for the calibration point. So OK, Maple has it right.

So start over with

$u = \text{arctan2Pi}(x,y) = \text{arctan2Pi}(x,y) = \text{arctan2Pi}(\rho^2+z^2-a^2, 2az)$.

Now what does this tell us about signs of cosu and sinu ? π 2π 3π 4π

I can see that

			cosu	sinu
$\rho^2+z^2-a^2 > 0$	$2az > 0$	1Q	+	+
$\rho^2+z^2-a^2 < 0$	$2az > 0$	2Q	-	+
$\rho^2+z^2-a^2 < 0$	$2az < 0$	3Q	-	-
$\rho^2+z^2-a^2 > 0$	$2az < 0$	4Q	+	-

Looking at this table, we see that

$\text{sign}(\text{cosu}) = \text{sign}(\rho^2+z^2-a^2)$

$\text{sign}(\text{sinu}) = \text{sign}(z)$

Now go back to our results below

$\text{cosu} = \pm(a^2-\rho^2-z^2)Q$

$\sin u = \pm 2az\,Q$

The proper version of these equation is this

$$\cos u = -(a^2-\rho^2-z^2)Q = +(\rho^2+z^2-a^2)\,Q$$

So there is nothing arbitrary here! There is no σ_c ! This solves a lot of problems.

Also

$$\sin u = 2az\,Q$$

This then makes sign(sinu) = sign(z). This is consistent with $z/\rho = \sin u/\sh\xi$

So set these signs in below:

In any event, the potential expressed this way is immensely complicated!

$$V(\xi,u)= (V_0/\pi)\sqrt{\ch\xi - \cos u}\ *$$
$$\{ (1/\sqrt{\ch\xi - \cos u}) \quad \cot^{-1}[(-1)^{n1}|\sqrt{1+\cos u}\,|\ /\sqrt{\ch\xi - \cos u}\,]$$
$$+\ (1/\sqrt{\ch\xi-\cos(2u_0-u)})\ \cot^{-1}[(-1)^{n2}|\sqrt{1+\cos(2u_0-u)}\,|/\sqrt{\ch\xi - \cos(2u_0-u)}\,]\ \}$$

where

$Q \equiv 1/\sqrt{[z^2+(a-\rho)^2]\,[z^2+(a+\rho)^2]}$

$\ch\xi - \cos u = 2a^2Q$

$\ch\xi = (a^2+\rho^2+z^2)Q$

$\cos u = -(a^2-\rho^2-z^2)Q$ I think the sign is wrong here!

$\sin u = +2az\,Q$

$a = R|\sin u_0|$

What about those signs? // this is all wrong, see above!

$\rho = a\,\sh\xi/(\ch\xi - \cos u)$

$\ch\xi - \cos u = (a/\rho)\sh\xi$

$\cos u = \ch\xi - (a/\rho)\sh\xi$

Since ρ can be very small, we could have $\cos u < 0$. And if ρ is large, could have $\cos u > 0$

We also know that

$z/\rho = \sin u/\sh\xi \quad \Rightarrow$ sign(z) = sin(sinu)). Let $\sigma = \pm 1$ for upper/lower half plane. Then

$\sigma = +1$ upper half plane $0<u<\pi$ $\sin u > 0$ cosu = either sign

STOP. Make four ranges and recall $z/\rho = \sinu/\sh\xi$:

	σ_s	σ_c	$\sigma_z = \sigma_s$	$= \sigma_z = \sigma_s$
range of u	sinu	cosu	z	cos(u/2)
$0,\pi/2$	+	+	+	+
$\pi/2,\pi$	+	-	+	+
$\pi,3\pi/2$	-	-	-	-
$3\pi/2,2\pi$	-	+	-	-
$2\pi,2\pi+\pi/2$	+	+	+	+
$2\pi+\pi/2,3\pi$	+	-	+	+
$2\pi+\pi,2\pi+3\pi/2$	-	-	-	-
$2\pi+3\pi/2,4\pi$	-	+	-	-

So write in terms of the above table

$$\cosu = \sigma_c(a^2-\rho^2- z^2)Q$$
$$\sinu = \sigma_s(a^2-\rho^2- z^2)Q$$

So parameter u determines all the signs exactly.

Is there a nicer way to present the final results?

$$A = \sqrt{z^2+ (\rho+a)^2} > 0$$
$$B = \sqrt{z^2+ (\rho-a)^2} > 0$$
$$Q = 1/(AB)$$
$$\ch\xi - \cosu = 2a^2Q$$
$$\ch\xi = (a^2+\rho^2+ z^2)Q$$
$$\cosu = \sigma_c(a^2-\rho^2- z^2)Q$$ I think the sign is wrong here!
$$\sinu = \sigma_s\ 2az\ Q$$
$$a = R|\sinu_0|$$

Now consider comparing the **Jackson disk potential** to my disk potential in toroidals.

$$V_{disk}(\xi,u) = (2V_0/\pi)\ \cot^{-1}\left[\frac{-\sqrt{2}\cos(u/2)}{\sqrt{\ch\xi-\cosu}}\right] . \qquad (2.5.4)$$

$$\cos^2(u/2) = (1 + \cosu)/2 = (1/2)(1 + \sigma_c(a^2-\rho^2- z^2)Q)$$

$$\cos(u/2) = \sigma_s \sqrt{(1 + \sigma_c(a^2-\rho^2- z^2)Q)}/\sqrt{2}$$??

Then

$$\frac{-\sqrt{2}\cos(u/2)}{\sqrt{\ch\xi-\cosu}} = \frac{-\sigma_s\sqrt{(1 + \sigma_c(a^2-\rho^2- z^2)Q)}}{\sqrt{2a^2Q}} = -\sigma_s\sqrt{\frac{1 + \sigma_c(a^2-\rho^2- z^2)Q}{2a^2Q}}$$

Pause to do this

Now use

$$A^2 = z^2 + (\rho+a)^2$$
$$B^2 = z^2 + (\rho-a)^2 \qquad\qquad Q = 1/AB$$

$$A^2 + B^2 = 2z^2 + 2\rho^2 + 2a^2 = 2(z^2+\rho^2+a^2)$$

$$-(A^2 + B^2)/2 = -z^2-\rho^2-a^2 = (-z^2-\rho^2+a^2) - 2a^2$$

$$(-z^2-\rho^2+a^2) = 2a^2 - (A^2 + B^2)/2$$

$$2(a^2-z^2-\rho^2) = 4a^2 - (A^2 + B^2)$$

Then resume the above:

$$= -\sigma_s \sqrt{\frac{2AB + 2\sigma_c(a^2-\rho^2- z^2)}{4a^2}} = -\sigma_s \sqrt{\frac{2AB + \sigma_c[4a^2 - (A^2 + B^2)]}{4a^2}}$$

STOP. We have a problem right here because we will not get the right answer if $\sigma_c = +1$ as I show below.

$$= -\sigma_s(1/2a)\sqrt{\sigma_c 2AB + \sigma_c 4a^2 - \sigma_c(A^2+ B^2)}$$

Now examine the argument of the radical.

$$\sigma_c 2AB + \sigma_c 4a^2 - \sigma_c(A^2+ B^2) = -\sigma_c [\, A^2+B^2-2\sigma_c AB - 4a^2 \,]$$

$$= -\sigma_c[\, (A-\sigma_c B)^2 - 4a^2]$$

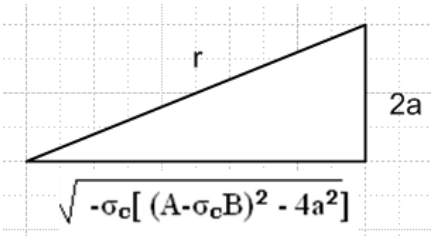
So we then have

$$\frac{-\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-cosu}} = -\sigma_s(1/a)\sqrt{-\sigma_c[\, (A-\sigma_c B)^2 - 4a^2]}$$

$$A = \sqrt{z^2+ (\rho+a)^2} > 0$$
$$B = \sqrt{z^2+ (\rho-a)^2} > 0$$

I presume the inside of the radical is always plus. I could maple verify if need be.

Now draw the famous triangle with edges . Make vertical be 2a and horizontal be $\sqrt{-\sigma_c[\, (A-\sigma_c B)^2 - 4a^2]}$ which is the abs value. Then,



Then we get

$$r^2 = -\sigma_c[(A-\sigma_c B)^2 - 4a^2] + 4a^2$$

and we will NOT get cancellation of the $4a^2$ for arbitrary σ_c, so something is wrong.

It seems to require that $\sigma_c = -1$. Then we get

$$r^2 = +[(A+B)^2 - 4a^2] + 4a^2 = (A+B)^2 \qquad r = A+B$$

Then we find that

$$\sin\theta = 2a/r = 2a/(A+B)$$

$$\theta = \sin^{-1}(\ 2a/(A+B)\ ) \qquad \text{and this is what Jackson gives.}$$

The problem solution is then

$$V_{disk}(\xi,u) = (2V_0/\pi)\ \cot^{-1}\left[\frac{-\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - cosu}}\right]$$

and the ratio is, if $\sigma_c = +1$,

$$\theta = \frac{-\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - cosu}} = -\sigma_s(1/a)\sqrt{-\sigma_c[(A-\sigma_c B)^2 - 4a^2]} = -\sigma_s(1/a)\sqrt{[(A+B)^2 - 4a^2]}$$

This angle θ can presumably be anywhere. In fact I know that

$$sign(\theta) = -sign(\cos(u/2)) = -\sigma_s$$

So when I draw my picture with a positive θ, I am assuming $\sigma_s = -1$, so I am happy with the σ_s situation.

I am not happy with the σ_c situation.

Bug Track Down. I have a problem for example when u = 3π/2,2π range, say u = 15π/8 looking at my figure (1.2.2) on bowl doc. At this u, I claim that σ_c = + and so

$$\text{cosu} = +(a^2-\rho^2- z^2)Q \qquad Q = 1/(AB) > 0$$

But it seems that at u = 15π/8 I could have ρ and z very large, so that cosu is negative. So even though I have said that $\sigma_c = +1$, I have cosu < 0, so my interpretation of σ_c is wrong.

4. Evaluate the potential "far away" and get C for the charged bowl for $u_0 < \pi/2$

We are interested in the large r limit and we expect to see V = Q/r in this limit. Far away in our particular toroidal coordinates with the unusual range for u means this:

far away: $\xi \to 0$ $u \to 2\pi$

In this limit we may write

$$|\sqrt{1+\text{cosu}}| = |\sqrt{1+\cos(2\pi)}| = |\sqrt{1+\cos(0)}| = \sqrt{2}$$

and

$$|\sqrt{1+\cos(2u_0-u)}| = |\sqrt{1+\cos(2u_0-2\pi)}| = |\sqrt{1+\cos(2u_0)}| = \sqrt{2}\ |\cos(u_0)|$$

The phases become

$$\eta_1 = \text{floor}[(3\pi-u)/2\pi] = \text{floor}[(3\pi-2\pi)/2\pi] = \text{floor}[(\pi)/2\pi] = 0$$

$$\eta_2 = \text{floor}[(2u_0-u+\pi)/2\pi] = \text{floor}[(2u_0-2\pi+\pi)/2\pi] = \text{floor}[(2u_0-\pi)/2\pi]$$
$= -1$ if u_0 lies in $(0,\pi/2)$
$= 0$ if u_0 lies in $(\pi/2,\pi)$

Finally we may write

$$\sqrt{\text{ch}\xi-\cos(2u_0-u)} = \sqrt{\text{ch}\xi-\cos(2u_0-2\pi)} = \sqrt{\text{ch}\xi-\cos(2u_0)}$$

But we are taking $\xi \to 0+\varepsilon$ so to speak, so

$$= \sqrt{1-\cos(2u_0)} = \sqrt{2}\ \sin(u_0) \qquad \text{// recall } u_0 \text{ in } (0,\pi) \text{ so no } |\ | \text{ needed}$$

In taking our limit, we maintain the factor (chξ-cosu) wherever it appears, for reasons soon to be obvious. So our potential now has this form in our limit

$$V(\xi,u) = (V_0/\pi)\sqrt{\text{ch}\xi - \text{cosu}}\ *$$
$$\{ (1/\sqrt{\text{ch}\xi - \text{cosu}}) \qquad \cot^{-1}[(-1)^{\eta_1}|\sqrt{1+\text{cosu}}|\ /\sqrt{\text{ch}\xi - \text{cosu}}]$$

$$+\ (1/\sqrt{ch\xi-\cos(2u_0-u)}\ )\ \cot^{-1}[(-1)^{\eta 2}|\ \sqrt{1+\cos(2u_0-u)}\ |/\ \sqrt{ch\xi-\cos(2u_0-u)}\ ]\ \}$$

$$V(\xi,u)= (V_0/\pi)\ \sqrt{ch\xi-\cos u}\ *$$
$$\{\ (1/\sqrt{ch\xi-\cos u}\ )\qquad \cot^{-1}[(-1)^{\eta 1}\sqrt{2}\ \ /\sqrt{ch\xi-\cos u}\ ]$$
$$+\ (1/\sqrt{2}\ \sin(u_0))\ \cot^{-1}[(-1)^{\eta 2}\sqrt{2}\ \ |\cos(u_0)|/\sqrt{2}\ \sin(u_0)]\ \}$$

In our large r limit we have

$$r \approx \sqrt{2}\ a\ /\ \sqrt{ch\xi-\cos u}\qquad \Rightarrow\qquad \sqrt{ch\xi-\cos u}\ = \sqrt{2}\ a\ /\ r$$ // where from? toroidals.doc!

So $r\to\infty$ means $\sqrt{ch\xi-\cos u}\to 0$, as is also obvious from our limit $\xi\to 0$ and $u\to 2\pi$ giving. The constant a is the focal distance of the toroidal coordinates which we can write as

$a = R\sin(u_0)$ R = radius of the bowl sphere

So then we have

$$\sqrt{ch\xi-\cos u}\ = \sqrt{2}\ a\ /\ r = \sqrt{2}\ R\sin(u_0)/\ r$$

So we install this in one location to get

$$V(\xi,u)= (V_0/\pi)\ \sqrt{2}\ R\sin(u_0)/\ r\ *$$
$$\{\ (1/\sqrt{ch\xi-\cos u}\ )\qquad \cot^{-1}[(-1)^{\eta 1}\sqrt{2}\ \ /\sqrt{ch\xi-\cos u}\ ]$$
$$+\ (1/\sqrt{2}\sin(u_0))\ \cot^{-1}[(-1)^{\eta 2}\ |\cos(u_0)|/\ \sin(u_0)]\ \}$$

Now is the time to deal with the η's. From above we found that

$\eta_1 = 0$
$\eta_2 = -1$ if u_0 lies in $(0,\pi/2)$

So let's now specialize to this range of u_0 (just for the moment). Then we use our rule

$$\cot^{-1}[\ (-1)^{\eta}\ x] = \cot^{-1}(x)\quad \text{of } \eta \text{ is even}$$
$$= \pi - \cot^{-1}(x) \text{ if } \eta \text{ is odd}$$

and we then have

$$V(\xi,u)= (V_0/\pi)\ \sqrt{2}\ R\sin(u_0)/\ r\ *$$
$$\{\ (1/\sqrt{ch\xi-\cos u}\ )\qquad \cot^{-1}[\sqrt{2}\ \ /\sqrt{ch\xi-\cos u}\ ]$$
$$+\ (1/\sqrt{2}\sin(u_0))\ [\ \pi\ -\ \cot^{-1}[|\cos(u_0)|/\ \sin(u_0)]\ \ ]\ \ \}$$

Now $\cot^{-1}(x)$ for large positive x can be written $\tan^{-1}(1/x)\ \approx\ 1/x$ and then

Next, we approximate

$$\cot^{-1}[\sqrt{2}\ /\sqrt{ch\xi - cosu}\ ] = \tan^{-1}[\sqrt{ch\xi - cosu}\ /\sqrt{2}\ ] \approx \sqrt{ch\xi - cosu}\ /\sqrt{2}$$

which then gives

$$V(\xi,u) = (V_0/\pi)\sqrt{2}\ R \sin(u_0)/ r *$$
$$\{ (1/\sqrt{ch\xi - cosu}\ )\sqrt{ch\xi - cosu}\ /\sqrt{2}$$
$$+ (1/\sqrt{2}\sin(u_0))[\ \pi - \cot^{-1}[|\cos(u_0)|/\sin(u_0)]\ ]\ \}$$

$$V(\xi,u) = (V_0/\pi)\sqrt{2}\ R \sin(u_0)/ r *$$
$$\{ 1/\sqrt{2}$$
$$+ (1/\sqrt{2}\sin(u_0))[\ \pi - \cot^{-1}[|\cot(u_0)|\ ]\ ]\ \}$$

Now for u_0 in our range of interest u_0 in $(0,\pi/2)$, we find that $|\cot(u_0)| = \cot(u_0)$ and thus we have

$$V(\xi,u) = (V_0/\pi)\sqrt{2}\ R \sin(u_0)/ r *$$
$$\{ 1/\sqrt{2}$$
$$+ (1/\sqrt{2}\sin(u_0))[\ \pi - u_0\ ]\ \} \qquad (3.1)$$

and finally our magic π term has appeared! (it only took about 3 days). Now we finish up like so:

$$V(\xi,u) = (V_0/\pi)\ R \sin(u_0)/ r *$$
$$\{ 1$$
$$+ (1/\sin(u_0))[\ \pi - u_0\ ]\ \}$$

$$= (V_0/\pi)\ R \sin(u_0)/ r * \{ 1 + (1/\sin(u_0))(\pi - u_0)\ \}$$

$$= (V_0/\pi)\ R / r * \{ \sin(u_0) + (\pi - u_0)\ \}$$

$$= (1/r)\ \{ (V_0/\pi)\ R\ (\pi - u_0 + \sin u_0)\ \}$$

Which tells us that

$$Q = (V_0/\pi)\ R\ (\pi - u_0 + \sin u_0)$$

and then

$$C = Q/V_0 = (R/\pi)\ (\pi - u_0 + \sin u_0)$$ // 12.5.15 I got right result, just back-tracing it.

and this agrees with the known Smythe result.

4a. Evaluate the potential "far away" and get C for the charged bowl for $\pi/2 < u_0 < \pi$

I will copy, paste then edit the tail end of the preceding section, which we join "in progress" :

$$V(\xi,u)= (V_0/\pi)\sqrt{2}\ R\sin(u_0)/r\ *$$
$$\{ (1/\sqrt{ch\xi - cosu}) \quad \cot^{-1}[(-1)^{\eta 1}\sqrt{2}\ / \sqrt{ch\xi - cosu}]$$
$$+ (1/\sqrt{2}\sin(u_0))\cot^{-1}[(-1)^{\eta 2} |\cos(u_0)| / \sin(u_0)] \}$$

Now is the time to deal with the η's. From above we found that

$$\eta_1 = 0$$
$$\eta_2 = 0 \text{ if } u_0 \text{ lies in } (\pi/2,\pi)$$

So let's now specialize to this range of u_0 (for this section). Then we don't need our rule

$$\cot^{-1}[(-1)^{\eta} x] = \cot^{-1}(x) \text{ of } \eta \text{ is even}$$
$$= \pi - \cot^{-1}(x) \text{ if } \eta \text{ is odd}$$

and we just have

$$V(\xi,u)= (V_0/\pi)\sqrt{2}\ R\sin(u_0)/r\ *$$
$$\{ (1/\sqrt{ch\xi - cosu})\cot^{-1}[\sqrt{2}\ / \sqrt{ch\xi - cosu}]$$
$$+ (1/\sqrt{2}\sin(u_0))\cot^{-1}[|\cos(u_0)| / \sin(u_0)] \}$$

Now $\cot^{-1}(x)$ for large positive x can be written $\tan^{-1}(1/x) \approx 1/x$ and then

Next, we approximate

$$\cot^{-1}[\sqrt{2}\ / \sqrt{ch\xi - cosu}] = \tan^{-1}[\sqrt{ch\xi - cosu}\ /\sqrt{2}\ ] \approx \sqrt{ch\xi - cosu}\ /\sqrt{2}$$

which then gives (just as in the previous section)

$$V(\xi,u)= (V_0/\pi)\sqrt{2}\ R\sin(u_0)/r\ *$$
$$\{ 1/\sqrt{2}$$
$$+ (1/\sqrt{2}\sin(u_0))\cot^{-1}[|\cos(u_0)| / \sin(u_0)] \}$$

$$V(\xi,u)= (V_0/\pi)\sqrt{2}\ R\sin(u_0)/r\ *$$
$$\{ 1/\sqrt{2}$$
$$+ (1/\sqrt{2}\sin(u_0))\cot^{-1}[|\cot(u_0)|] \}$$

Now for u_0 in our new range of interest u_0 in $(\pi/2,\pi)$, we see that $\cot(u_0)$ ranges $(0,-\infty)$. The abs value changes this then to $(0,+\infty)$ and the $\cot^{-1}(..)$ then gives us a result in the range $(\pi/2,0)$. Clearly this is not u_0 which is in range $(\pi/2,\pi)$. It must be that $\cot^{-1}[|\cot(u_0)|] = \pi - u_0$. I could prove this formally, but let's assume it is right and move on. Our result is then

$$V(\xi,u)= (V_0/\pi)\sqrt{2}\ R\sin(u_0)/r\ *$$
$$\{\ 1/\sqrt{2}$$
$$+\quad (1/\sqrt{2}\sin(u_0))(\pi-u_0)\quad \}$$

But this result is the same as (3.1) in the previous section, so we are done, C comes out the same!

5. What happens for $u_0 \rightarrow 0$ in our post-evaluated form?

Well, we start with our general post-evaluated form (2.1)

$$V(\xi,u)= (V_0/\pi)\sqrt{ch\xi - cosu}\ *$$
$$\{ (1/\sqrt{ch\xi - cosu})\quad \cot^{-1}[(-1)^{\eta 1}|\sqrt{1+cosu}|\ /\sqrt{ch\xi - cosu}]$$
$$+\ (1/\sqrt{ch\xi-cos(2u_0-u)})\cot^{-1}[(-1)^{\eta 2}|\sqrt{1+cos(2u_0-u)}|/\sqrt{ch\xi - cos(2u_0-u)}]\ \}$$

where $\eta_1 = floor[(3\pi-u)/2\pi]$
$\eta_2 = floor[(2u_0-u+\pi)/2\pi]$

What happens when u_0 is very small? Let's try just setting $u_0 = 0$ everywhere:

$$V(\xi,u)= (V_0/\pi)\sqrt{ch\xi - cosu}\ *$$
$$\{ (1/\sqrt{ch\xi - cosu})\quad \cot^{-1}[(-1)^{\eta 1}|\sqrt{1+cosu}|\ /\sqrt{ch\xi - cosu}]$$
$$+\ (1/\sqrt{ch\xi-cos(u)})\cot^{-1}[(-1)^{\eta 2}|\sqrt{1+cos(u)}|/\sqrt{ch\xi - cos(u)}]\ \}$$

where $\eta_1 = floor[(3\pi-u)/2\pi]$
$\eta_2 = floor[(\pi-u)/2\pi]$

Our range for u is now u in $(0,2\pi)$, But let's first try u in $(0,\pi)$, so that

$\eta_1 = floor[(3\pi-u)/2\pi] = 1$
$\eta_2 = floor[(\pi-u)/2\pi] = 0$

We then use our rule

$$\cot^{-1}[(-1)^{\eta}\ x] = \cot^{-1}(x) \quad \text{if } \eta \text{ is even}$$
$$= \pi - \cot^{-1}(x) \text{ if } \eta \text{ is odd}$$

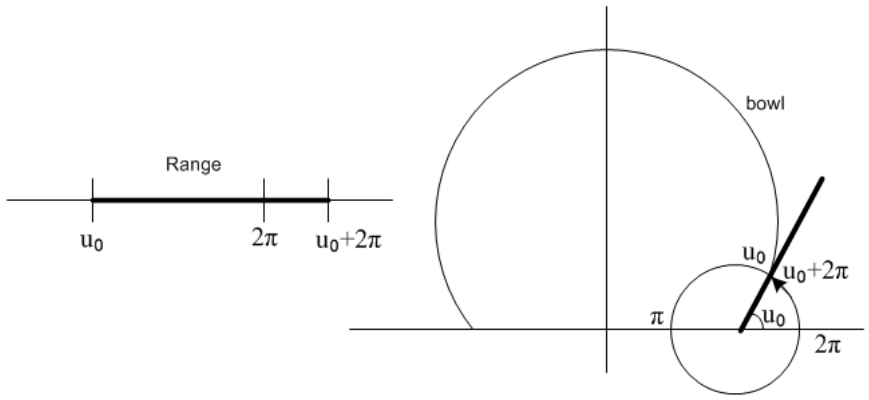
to get

$$V(\xi,u)= (V_0/\pi)\sqrt{ch\xi - cosu}\ *$$
$$\{ (1/\sqrt{ch\xi - cosu})\quad [\ \pi - \cot^{-1}[|\sqrt{1+cosu}|\ /\sqrt{ch\xi - cosu}]\quad]$$
$$+\ (1/\sqrt{ch\xi-cos(u)})\cot^{-1}[|\sqrt{1+cos(u)}|/\sqrt{ch\xi - cos(u)}]\ \}$$

The $\cot^{-1}$ terms cancel, and we are left with just

$V(\xi,u) = V_0$

Now consider our basic picture:



For u in the range u_0, $\pi = (0,\pi)$, all the u contours lie inside the sphere! So yes, we expect the potential to be V_0 everywhere inside our full sphere!

Now let's try u in $(\pi,2\pi)$ which are going to be exterior contours of u. This time we get

$\eta_1 = \text{floor}[(3\pi-u)/2\pi] = 0$
$\eta_2 = \text{floor}[(\pi-u)/2\pi] = -1$

so the sense are now reversed. But this then gives exactly the same answer that $V = V_0$, and I am not sure how to explain this. One thing I can say: there are no u contours at all in the upper half space if $u_0 = 0$. But I think they exist in the lower half space and have labels in the range $(\pi,2\pi)$.

Perhaps it is this: a bowl with $u_0 = 0$ fills the entire upper half space for some ε setting of 2a.

But *suppose* for some reason I have the signs wrong and both η are really 0. Then we would get

$$V(\xi,u) = (V_0/\pi)\sqrt{ch\xi - cosu} \; *$$
$$\{ (1/\sqrt{ch\xi - cosu}) \; \cot^{-1}[\,|\sqrt{1+cosu}\,| \; / \sqrt{ch\xi - cosu}\,]$$
$$+ \; (1/\sqrt{ch\xi - cos(u)})\cot^{-1}[\,|\sqrt{1+cos(u)}\,|/\sqrt{ch\xi - cos(u)}\,] \; \}$$

$$= (V_0/\pi)\, 2\cot^{-1}[\,|\sqrt{1+cosu}\,| \; / \sqrt{ch\xi - cosu}\,]$$

We would maybe like to see this be $V = V_0(R/r)$ which would be the potential outside a sphere. But here is what we know

$r^2 = a^2 (sh^2\xi + \sin^2 u)/(ch\xi - cosu)^2$

$a = R\,|\sin(u_0)| = 0$ // a bit of a problem!

In fact, the whole toroidal coordinate system has a problem:

$x = a\cos\varphi\ \mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u)$ $\qquad \rho = a\ \mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u)$

$y = a\sin\varphi\ \mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u)$

$z = a\sin u/(\mathrm{ch}\xi - \cos u)$

If we let $a \to 0$, then $x=y=z=0$ for all ξ and u which says the entire toroidal coordinate system has contracted down to that singular point on the bottom of the bowl-now-sphere. If this is correct, then we do in fact expect to get $V = V_0$ for all u,ξ coordinate values, which is what I found above!

So this explains "what happens" if your opening gambit is to set $u_o = 0$! Somehow if we take the limit slowly, we do get the correct capacitance of the sphere.

6. What happens for $u_0 \to 0$ in the full integral form ?

What happens if you do this directly to the full integral, ie, set $u_0 \to 0$ here: (sphere limit)

$$V(\xi,u) = V_0\sqrt{2}\ \sqrt{\mathrm{ch}\xi - \cos u}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(\mathrm{ch}\xi)\ \mathrm{ch}[(\pi - u_0)\tau]\ \mathrm{ch}[(\pi + u_0 - u)\tau] / \mathrm{ch}^2(\pi\tau)$$

$$V(\xi,u) = V_0\sqrt{2}\ \sqrt{\mathrm{ch}\xi - \cos u}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(\mathrm{ch}\xi)\ \mathrm{ch}[(\pi)\tau]\ \mathrm{ch}[(\pi - u)\tau] / \mathrm{ch}^2(\pi\tau)$$

$$V(\xi,u) = V_0\sqrt{2}\ \sqrt{\mathrm{ch}\xi - \cos u}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(\mathrm{ch}\xi)\ \mathrm{ch}[(\pi)\tau]\ \mathrm{ch}[(\pi - 2\pi)\tau] / \mathrm{ch}^2(\pi\tau)$$

$$V(\xi,u) = V_0\sqrt{2}\ \sqrt{\mathrm{ch}\xi - \cos u}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(\mathrm{ch}\xi)\ \mathrm{ch}^2[(\pi)\tau] / \mathrm{ch}^2(\pi\tau)$$

$$V(\xi,u) = V_0\sqrt{2}\ \sqrt{\mathrm{ch}\xi - \cos u}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(\mathrm{ch}\xi)$$

$$V(\xi,u) = V_0\sqrt{2}\ \sqrt{\mathrm{ch}\xi - \cos u}\ (1/\sqrt{2})\ (1/\sqrt{\mathrm{ch}\xi - 1})$$

$$V(\xi,u) = V_0\sqrt{2}\ \sqrt{\mathrm{ch}\xi - 1}\ (1/\sqrt{2})\ (1/\sqrt{\mathrm{ch}\xi - 1}) = V_0$$

This is the exact same conclusion reached in the previous section!

7. Check boundary conditions using the full integral form.

Check BC's. First $u = u_0$:

$$V(\xi,u) = V_0\sqrt{2}\ \sqrt{\mathrm{ch}\xi - \cos u}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(\mathrm{ch}\xi)\ \mathrm{ch}[(\pi - u_0)\tau]\ \mathrm{ch}[(\pi + u_0 - u)\tau] / \mathrm{ch}^2(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi)\tau] / ch^2(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau] / ch(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ (1/\sqrt{2}\ )\ (ch\xi+\cos(\pi-u_0))^{-1/2}$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu_0}\ (1/\sqrt{2}\ )\ (ch\xi-\cos(u_0))^{-1/2}$$

$$V(\xi,u)= V_0$$

Second check $u = 2\pi + u_0$:

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-u)\tau] / ch^2(\pi\tau)$$

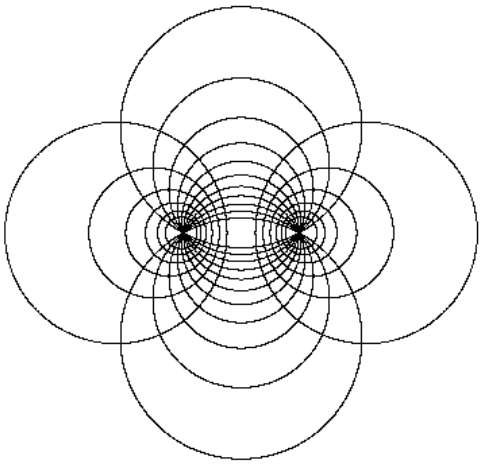
$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi+u_0-[2\pi+u_0]\ )\tau] / ch^2(\pi\tau)$$

$$V(\xi,u)= V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ ch[(\pi-u_0)\tau]\ ch[(\pi)\tau] / ch^2(\pi\tau)\ = \text{same}$$

So at least something is right. Of course meeting the BC's was how I obtained the full integral form in the first place.

8. Possible Idea A

Consider the toroidal coordinates picture



We know "far away" is the combination of small ξ and $u = 2\pi$. This means fat torus and large sphere, or shall we say, this means large H circle and large V circle. Officially the V circles are just bowls and not complete spheres, but we could be picking up the spurious intersections between the foci. Thus we could have $u = 2\pi$ and small ξ and accidentally end up "close in" rather than "far away".

9. Possible Idea B

In our Mehler integral evaluation we got to this point in the long derivation:

Maple tells us that

$$F(1,1/2;\ 3/2;\ z) = [\ \tanh^{-1}\sqrt{z}\ ]\ /\sqrt{z} \qquad (10.26)$$

Note added: for real z, both sides of 10.26 require that $|z| < 1$.

Applying this to our case, we have

$$z = -(y-\alpha)/(1+\alpha).$$
$$\sqrt{z} = i\sqrt{y-\alpha}\ /\sqrt{1+\alpha} \qquad (10.27)$$

where we make a certain choice in the branch of $\sqrt{z}$, but we make the same choice in both $\sqrt{z}$ factors, so the choice does not matter. Then we have

$$[\tanh^{-1}\sqrt{z}\ ]/\sqrt{z} \ = \ \tan^{-1}[\ \sqrt{y-\alpha}\ /\sqrt{1+\alpha}\ ]\ \ (\ \sqrt{1+\alpha}\ /\ \sqrt{y-\alpha}\ ) \qquad (10.28)$$

and our integral of interest, adding back the leading $(1/\pi)\ ch(a/2)$ factor becomes

$$I = \ (2/\pi)\ ch(a/2)\ (1+\alpha)^{-1}\ \tan^{-1}[\ \sqrt{y-\alpha}\ /\sqrt{1+\alpha}\ ]\ \ (\ \sqrt{1+\alpha}\ /\ \sqrt{y-\alpha}\ ) \qquad (10.29)$$

We now set α back to cha, and use $ch(a/2)/\ \sqrt{1+cha}\ = (1/\sqrt{2}\ )$ to state the final result

$$I = \ (\sqrt{2}\ /\pi\ )\ \ (1/\sqrt{y-cha}\ )\ \tan^{-1}[\ \sqrt{y-cha}\ /\sqrt{1+cha}\ ] \qquad y>cha \qquad (10.30a)$$

$$= (1/\pi)\ \ (1/\sqrt{(1/2)y-(1/2)cha}\ )\ \tan^{-1}[\ \sqrt{y-cha}\ /\sqrt{1+cha}\ ] \qquad y>cha \qquad (10.30b)$$

Now what about that $|z| < 1$ business? For $y >> \alpha = cha$, this condition won't be met.