Charged bowl in toroidals PhL 7.12.10

This doc is for historical interest only, pretty much nothing in it is correct! But it is fascinating. 12/31/15.

Overview (written 2.2.11, 1.5 pages)

This was my "first attempt" at doing the charged bowl in toroidals, and it is historically interesting to me for that reason. Lebedev had two problems related to this subject, I thought, but I later realized that 501 just gave the σ results, and problem 503 involves a bowl with a flat lid on it, something I did not realize till much later. I thought therefore that his problem 501 was giving the potential of a charged bowl!

In the first section below, I at least have the right general Smythian form for V, and I learn for the first time why you want a u coordinate range in the form u_0 to $2\pi+u_0$ (in later notation). My Smythian form has the integrand $[A_p ch(u) + B_p sh(u)]$ and later $A_p\ sh(u - a_p)$. I don't think I understood why I was using this particular toroidal atomic form instead of the other form.

The next section is called *My* Solution for the Charged Bowl. Right off the bat I make the big mistake of assuming that $r\to\infty \Leftrightarrow u=2\pi$, though I now know that $r\to\infty \Rightarrow u=2\pi$ only. This seems to imply in the above form that $a_p = 2\pi$ and then my integrand is merely $A_p sh(u-2\pi)$ and I quickly end up with the following wrong solution to the problem: $V(\xi,u,\varphi) = V_0[sh(u-2\pi)/ sh(u_0-2\pi)]$. Since it strongly resembles the charged ellipsoid solution in ellipsoidals, I am optimistic at this point! Although I have a solution here, I still would like to solve for my A_p, and that leads me to the Mehler-Fock. When I apply the Mehler-Fock to the problem of finding A_p, I need a certain Mehler projection integral which I was able to deduce from some horrible GR7 integral, (this is an "easy direction" integral)

$$\int_1^\infty dx\ P_{ip-1/2}(x) / \sqrt{x - \cos u_0} \quad = \sqrt{2}\ ch[p(u_0-\pi)]/ (p\ sh(\pi p))$$

I am then off looking for verification of this thing, and that leads me to downloading the PBM books. In another doc I verify the above equation by a different method. I have then arrived at this point

$$A_p = V_0\sqrt{2}\ ch[p(u_0-\pi)]/ (ch(\pi p)\ sh(u_0-2\pi))$$

$$V(\xi,u,\varphi) = V_0\sqrt{2}\ \left[\sqrt{ch\xi - \cos u}\ \ sh(u - 2\pi)/ sh(u_0-2\pi)\right]\ \int_0^\infty dp\ P_{ip-1/2}(ch\xi)\ ch[p(u_0-\pi)]/(ch(\pi p))\quad (*)$$

Due to my opening mistake, all this stuff is totally wrong, but I was forced to go learn things. I can see now that if you were to look up the above dp integral (I did it much later in 1/11)

$$\int_0^\infty d\tau\ P_{i\tau-1/2}(y)\ \cos(a\tau)\ sech(\pi\tau)\ = (1/\sqrt{2}\ )\ (y+cha)^{-1/2} \qquad \text{// sec 7}$$

you just end up with what I already had , namely, $V(\xi,u,\varphi) = V_0\ [sh(u-2\pi)/ sh(u_0-2\pi)]$. So at least my wrong result is consistent with itself. But at the time I did not know how to do the above integral, and I just assumed it was undoable and part of the answer.

At this point I tried comparing my solution (*) above to Lebedev's 501 and it did not agree. In retrospect, my solution is completely wrong, and Lebedev was doing a completely different problem (!), the bowl with lid, so it is a good thing things did not agree!

Not knowing this, I verified that my result was OK for $r\to\infty$ (wrongly so) and I tried taking that limit of his result ($u = 2\pi$) to see what it did. This sort of led nowhere.

I have tacked on a Note from 1.20.11 where I quote the Deegan two-bowl result, though I didn't really know what it is at that time, the thing with β_1 and β_2. I realize at this time that Prob 503 has a flat lid on it and it is in fact a two-bowl problem, and this is detailed in other docs.

At this point I think my simple result $\text{sh}(u-2\pi)/\text{sh}(u_0-2\pi)$ might still be viable. It seems odd to me that this simple result appears nowhere on the web! I then get the idea of doing $\beta_1 = \beta_2$ in Deegan to get a single-bowl result, and I do get the correct $\text{ch}[(\pi-\beta_0)\tau]\,\text{ch}[(\beta-\beta_0)\tau - \pi\tau] / \text{ch}^2(\pi\tau)$ integrand, but at this time I just ignore it because it does not match my form (*) above. So in fact I had the right single-bowl result in my hands here, but did not realize it yet. And I did not understand either why my simple solution was wrong. At this point, the doc just ends.

I am very confused. I thought Lebedev problem 503 was the charged bowl problem, but now I am not so sure. Adjacent doc "Notes on Kirk on bowl problem + Kelvin review.doc" quotes Lebedev problem 501 as being the charged bowl problem. I show the entire problem there and it sure looks like the charged bowl problem to me. So what is the problem below, 503 ?? Maybe 501 asks just for σ and this one is asking for Φ.

Note added 2.2.11. Lebedev's **Problem 501** asks for σ on the charged bowl surfaces, using toroidals, and he uses the horrible Mehler integral noted below and refers the reader to a Jeans alternate solution. His **Problem 503** I thought was a bowl problem, but it is really the bowl with a flat metal cap! That is why I did not get his answer in this doc. Eventually I did this problem elsewhere and got his answer. Neither of these problems appears in the "solutions" section of this book.

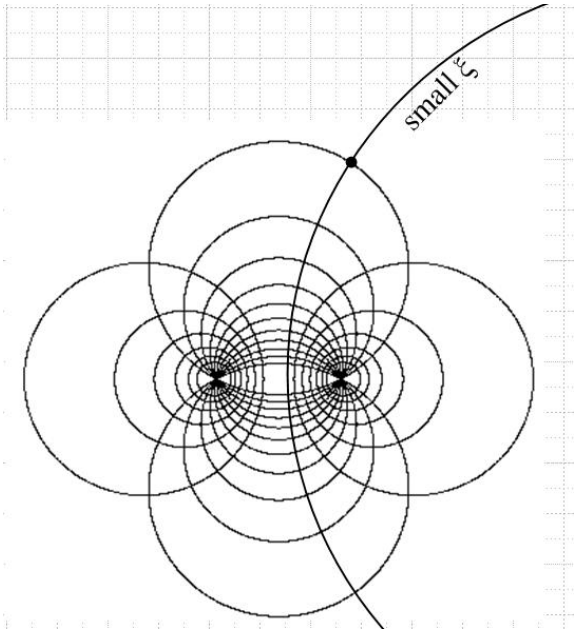
This doc is incomplete because my result disagrees with the Lebedev book result.

I conjecture that the **charged bowl problem** has this form: (m = 0 since azisym)

$$V(\xi,u,\varphi) = \sqrt{\text{ch}\xi - \cos u}\ \int_0^\infty dp\ P_{ip-1/2}(\text{ch}\xi)[\ A_p\text{ch}(u) + B_p\text{sh}(u)]$$

* For the exterior bowl problem, far away would mean small ξ (fat toroid), only P would appear.
? For the interior bowl problem, far away from bowl edge again means large ξ, so again only P.
Far top exterior means small u which suggests only sh(u).
Far bottom exterior means u near 2π, suggests only sh(2π-u).
I could imagine a different way to "range" the two variables.

* Note added 1.20.11. Consider this picture



where bowls have label u and toroids label ξ. Suppose our metal bowl of interest were an inner shallow upper bowl. A point near the top right of the outermost upper bowl with coordinates (u,ξ) would have small ξ because small ξ describes a large toroid . Large ξ would describe points near the foci. Thus, if you want to talk about "going far away" from a charged bowl, you are definitely talking small ξ. Not all points with small ξ are far from the bowl, but all points far from the bowl have small ξ!

Lebedev has the solution of this problem on page 241, where he uses τ for p and **β for u**:
(this is the entire Lebedev problem, I have omitted nothing)

PROB. 506 CURVILINEAR COORDINATES 241

503. Find the potential distribution in the space surrounding a charged conductor shaped like the "spherical zone" shown in Figure 141.

Ans.

$$u(\alpha, \beta) = V\sqrt{2\cosh\alpha - 2\cos\beta}$$
$$\times \int_0^\infty [\sinh(2\pi + \beta_0 - \beta)\tau - \cosh(\pi - \beta_0)\tau$$
$$\times \sinh(\pi - \beta)\tau] \frac{P_{-\frac{1}{2}+i\tau}(\cosh\alpha)}{\sinh(\pi + \beta_0)\tau \cosh\pi\tau} d\tau,$$

where $P_\nu(x)$ is the Legendre function of the first kind, V is the potential of the conductor and $\sin\beta_0 = c/a$.

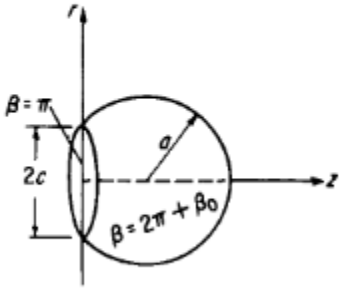


FIGURE 141

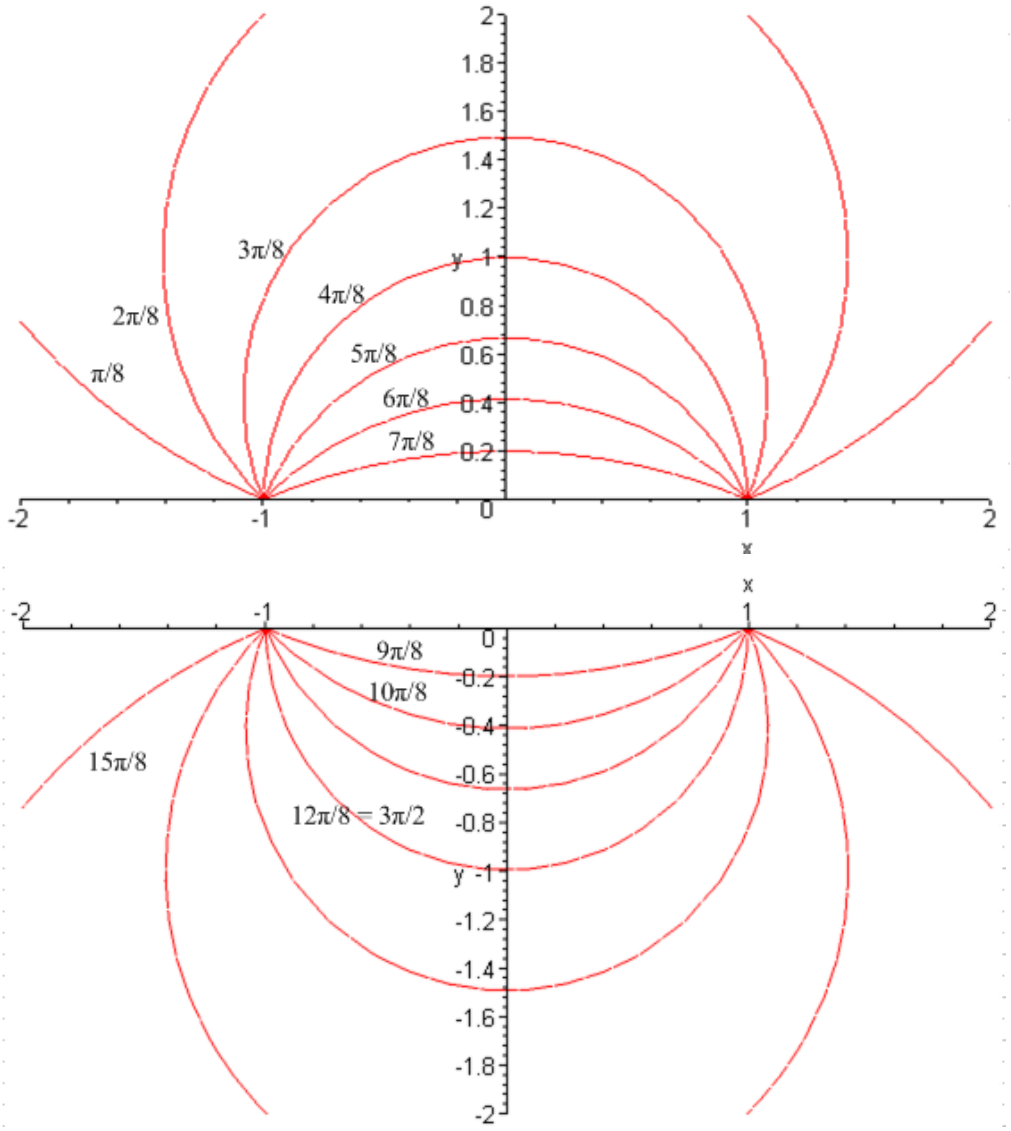
It has my conjectured form with a tricky [$A_p ch(u) + B_p sh(u)$] form. The picture shows β = π on the end slice of the bowl which is correct. The radius of this slice is the toroid radius in the large ξ limit which is called "a" in my notes above, but is called "c" by Lebedev. The bowl radius in my notes above is

a/|sin(u_o)| so his would be c/|sin(β_o)| and he calls this "a", so he has a = c/|sin(β_o)| . Also, he has used a bowl which is what I call "upper" (positive z side), so $\beta_0 > 0$ and then c = a sin(β_o), just as he states.

So I can regard this problem as "solved by me" if I can just explain the other factors in his result. I don't understand his labeling $\beta = 2\pi + \beta_0$ on the picture. The bowl is defined by $\beta = \beta_0$ in $(0,\pi)$. I will ponder this when I solve this problem. Thank you Lebedev and friends for providing this solution. He has this comment earlier:

> periodic in β and φ, with period 2π. Therefore we can choose $\beta_1 < \beta \leqslant \beta_1 + 2\pi$, $\varphi_1 < \varphi \leqslant \varphi_1 + 2\pi$ instead of $-\pi < \beta \leqslant \pi$, $-\pi < \varphi \leqslant \pi$ (which corresponds to the particular choice $\beta_1 = \varphi_1 = -\pi$), and it is sometimes convenient to do so.

So OK, if the bowl is $\beta = \beta_0$, you could choose this for your β range: $\beta_o < \beta \leq \beta_0 + 2\pi$. Look at the first picture below and imagine that β_0 is one of the bowls shown, perhaps that with $\beta_0 = 4\pi/8$. As you increase β from this value (each β marks a "math bowl" = just a surface in space of bowl shape), you move down in the direction of the shallow math bowls such as $7\pi/8$. This range is the left part of the range $\beta_o < \beta \leq \beta_0 + 2\pi$. When you hit $8\pi/8 = \pi$, you have reached the limit where the math bowl is a disk. As you continue to increase β, you move into the realm of the lower "math bowls" as shown in the second figure. Eventually you reach $\beta = 2\pi$ which is then the limit of a lower iris. Keep going and you have the upper iris, then upper bowls, and eventually when you reach $\beta \leq \beta_0 + 2\pi$, you are back to your starting bowl. So this is just how β is describing one of the coordinates of space with respect to our bowl β_0. Therefore, we have shown that $\beta = \beta_0 + \varepsilon$ describes the inner (lower) surface of our starting upper bowl, and $\beta_o = \beta_0 + 2\pi - \varepsilon$ describes the outer (upper) surface. Later if you want to find charge densities or potentials on these surfaces, you would use these limits! Note that this bowl label β is the one with the traditional range $(0,2\pi)$ and is called u and later η by M&F. It is the ch(u) type label in my form above where I have selected the "second" toroidal atomic form from my general doc on that subject.



Lebedev quotes some integrals along his path:

$$\int_0^{\pi} \frac{\cos n\beta \, d\beta}{\sqrt{2\cosh\alpha_0 - 2\cos\beta}} = Q_{n-\frac{1}{2}}(\cosh\alpha_0).$$

Hint. To calculate the density, use the integral

$$\int_0^{\infty} \frac{\tau \sinh\pi\tau \cosh(\pi-\beta_0)\tau}{\cosh^2\pi\tau} P_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, d\tau = \frac{2}{\pi} \frac{\sin\frac{1}{2}\beta_0}{\sqrt{2\cosh\alpha - 2\cos\beta_0}} \times \left[1 + \frac{2\cos\frac{1}{2}\beta_0}{\sqrt{2\cosh\alpha - 2\cos\beta_0}} \arctan\frac{2\cos\frac{1}{2}\beta_0}{\sqrt{2\cosh\alpha - 2\cos\beta_0}}\right].$$

He also points out that you might want to do problems with regions that are bounded by two bowls, and various lenses (with spherical surfaces!) would be such regions. Pages 233-242 are where Lebedev does all his toroidal stuff. Problems are selected not just from electrostatics by the way!

Note added 12.8.10. Notice that his first integral above arises in my Fourier expansion of $1/\sqrt{a - b\cos(x)}$ and I wrote the integral this way

$$\int_0^{\pi} dx\, \cos(nx)/\sqrt{a - b\cos(x)} \;= \sqrt{2/b}\; Q_{n-1/2}(a/b)$$

so if I set b = 1 and a = cosh(α_0), I duplicate his result.

Unlike the first integral, the second one a Mehler type integral over dτ and this thing might appear in some form in the PBM book which contains such integrals.

Note added 2.2.11. I don't have the above Mehler integral in any table, but I know I could evaluate it if I wanted using methods in math/integral/ "doing Mehler". The form of Lebedev's result is very typical of these integrals.

***My* Solution for the Charged Bowl**

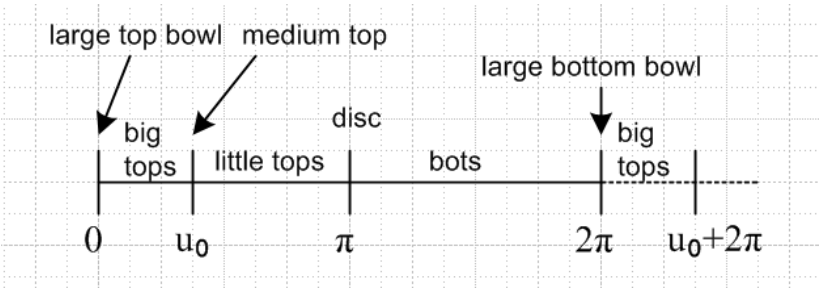
Continuing from above

$$V(\xi,u,\varphi) = \sqrt{ch\xi - cosu}\; \int_0^{\infty} dp\, P_{ip-1/2}(ch\xi)[\, A_p ch(u) + B_p sh(u)]$$

Far top exterior means small u which suggests only sh(u). Ie, far away in the upper red picture.
Far bottom exterior means u near 2π, suggests only sh(2π-u). Ie, far away in lower red picture.
In either case, far away means small ξ. All these facts are noted at the start of this doc.

Normally in toroidal coordinates one takes the u = β to be in the range $0 \le u \le 2\pi$ and this includes ALL the red bowls, upper and lower. The very large upper bowl has u = 0+ε and the red contour lines are (for the most part) "far away" we want V $\rightarrow$ 0 as u $\rightarrow$ 0. BUT, we need the potential to also vanish as we go to the very large lower bowl which is u = 2π.

But how do we get our result to vanish for both u = 0 and u = 2π ? I don't think you can do it! In order to vanish at u = 0, need A_p = 0. But then at u = 2π you have B_p sh(2π) = 0 needed as well, which would force B_p = 0 as well, whole solution collapses to nothing. So maybe you are actually FORCED to change the range of u! Consider this picture, where u_0 is our charged top bowl: ("top" here means my upper Maple red picture in my toroidals doc]



If we start at u_0 instead of 0, as L suggests, then we start with our moderately sized top bowl, go inward through all the smaller top bowls, through the flat disk and then into shallow bottom bowls, all the way through the huge negative bowls, then we reflect around to the large top bowls and end up where we started when we get to $u_0 + 2\pi$. In this scheme, both "huge bowl" regions occur at $u = 2\pi \pm \varepsilon$, so we need only make our formula vanish at that one point (rather than at two points for our starting range). That is to say, $u = 0$ is no longer part of our range, which is still 2π in size. [Here I replicate the discussion above concerning the math bowls for this adjusted range.]

Let's just try a Smythian form here and see what happens with our new range. So here is our form for the atomic form u function showing two unknown coefficients A_p and a_p,

$$f(u) = A_p \; sh(u - a_p)$$

$$f(2\pi) = A_p \; sh(2\pi - a_p) = 0 \qquad \text{// gives one condition on our 4 constants}$$

Note added 1.20.11: Why am I setting this to 0? Because $u = 2\pi$ represents $r = \infty$? But not all points with $u = 2\pi$ are at infinity because $u = 2\pi$ is the lower region iris which has points very close to our charged surface!

$$\Rightarrow \quad a_p = 2\pi$$

$$\Rightarrow \quad f(u) = A_p \; sh(u - 2\pi)$$

and our Smythian form is now

$$V(\xi,u,\varphi) = \sqrt{ch\xi - cosu} \; sh(u - 2\pi) \int_0^\infty dp \; P_{ip-1/2}(ch\xi) \; A_p$$

Our other condition is that V = constant on the bowl, which says (exterior problem ?)

$$V(\xi,u=u_0,\varphi) = \sqrt{ch\xi - cosu_0} \; sh(u_0 - 2\pi) \int_0^\infty dp \; P_{ip-1/2}(ch\xi) \; A_p = V_0 \qquad (***)$$

Something is already wrong! The integral shown here has some value, call it $F(\xi)$. Then we are saying in our last two equations

$$V(\xi,u,\varphi) = \sqrt{ch\xi - cosu} \; sh(u - 2\pi) \; F(\xi)$$

$$V(\xi,u=u_0,\varphi)= \sqrt{ch\xi - cosu_0}\ \ sh(u_0 - 2\pi)\ F(\xi)\ = V_0$$

We could solve the second equation for F(ξ) to get

$$F(\xi) = V_0/\ [\sqrt{ch\xi - cosu_0}\ \ sh(u_0 - 2\pi)]$$

Then our general problem solution is this

$$V(\xi,u,\varphi) \quad = \sqrt{ch\xi - cosu}\ \ sh(u - 2\pi)\ \{\ V_0/\ [\sqrt{ch\xi - cosu_0}\ \ sh(u_0 - 2\pi)]\}$$

$$= V_0\ [\ sh(u - 2\pi)/\ sh(u_0 - 2\pi)]$$

and I KNOW this is not the right solution for a charged bowl! It does vanish at u = 2π. And on the bowl it does have the value V_0. And it is independent of ξ.

Note added 1.20.11. Recall in ellipsoidals how the potential of a charged ellipsoid is a function only of the ellipsoid label. Here we are saying that the potential of a charged bowl is a function only of the bowl label. Maybe that is exactly right! Maybe the problem is completely solved right here!! I don't know A_p, but I think I know V(ξ,u,φ) which is what I really want! You could then find A_p from (***) which involves inverting a Mehler transform and this is described below.

How on earth are we going to arrange for this to be true, so integral is independent of ξ ? Let's think of this in the following way [where we replace A_p by f(p; u_o)]

$$V(\xi,u=u_0,\varphi)= \sqrt{ch\xi - cosu_0}\ \ \int_0^\infty dp\ P_{ip-1/2}(ch\xi)\ f(p;\ u_o)\ \ = V_0 \qquad (*)\ \text{expansion}$$

where $f(p;\ u_o)\ \equiv\ A_p\ sh(u_0 - 2\pi)$

We of course need the Mehler-Fock transform at this point. I have not yet developed this from the SL problem, so let's steal the result from somewhere. Lebedev calls it the MF theorem, but I want another source. I have it above from a certain book. Bateman (wrong) and AS and WW do NOT have it. Nor in GR7. Only a book that talks toroidals would have this thing I suspect, hence nothing in MF. Nothing in CH. But it is in Dan Z's book on ODE's! I copied his transforms into my transforms.doc, and here is the MF:

This following is from Dan Z ODE, page 351, and I think it is wrong! He quote Chapter 7 of Sneddon's book The Use of Integral Transforms. Dan has a 3rd Ed errata list and the end of his book, and nothing appears there.

Mehler–Fock transform of order m, (see Sneddon [22, Chapter 7])

$$v(\xi) = \int_0^\infty \sinh(x) P^m_{i\xi-1/2}(\cosh x)\, u(x)\, dx,$$
$$u(x) = \int_0^\infty \xi \tanh(\pi\xi) P^m_{i\xi-1/2}(\cosh x)\, v(\xi)\, d\xi.$$

so this form has the cosh stuff preloaded for us. I will translate to my current variables: [Note: this transform is correctly stated, see "doing Mehler transform integrals.doc"]

$G(p) = \int_0^\infty d\xi\ sh(\xi)\ P_{ip-1/2}(ch\xi)\ g(\xi)$ // projection

$g(\xi) = \int_0^\infty dp\ p\ th(\pi p)\ P_{ip-1/2}(ch\xi)\ G(p)$ // expansion

Rewrite the above (*) as:

$V_o/\sqrt{ch\xi - cosu_0} = \int_0^\infty dp\ P_{ip-1/2}(ch\xi)\ f(p; u_o)$ (**)

So here then is the plan. Let

$f(p; u_o) = p\ th(\pi p)\ G(p)$

$g(\xi) = V_o/\sqrt{ch\xi - cosu_0}$

Then (**) is our standard expansion, and the projection coefficient is then

$G(p) = \int_0^\infty d\xi\ sh(\xi)\ P_{ip-1/2}(ch\xi)\ g(\xi)$

or

$f(p; u_o)/[\ p\ th(\pi p)] = V_0 \int_0^\infty d\xi\ sh(\xi)\ P_{ip-1/2}(ch\xi) / \sqrt{ch\xi - cosu_0}$

Let $x = ch\xi$ so $dx = sh\xi d\xi$ and this is

$G(p) = f(p; u_o)/[\ p\ th(\pi p)] = V_0 \int_1^\infty dx\ P_{ip-1/2}(x) / \sqrt{x - cosu_0}$

So if we could do this integral we would have G(p), hence $f(p; u_o)$, hence A_p.

Aside: It was at this point that I found the difference between Russian and English Bateman ET II, see Bateman notes on that volume.

How about this GR7 offering, p 774 with $\lambda = 1$, $\mu=0$ and $\rho = 1/2$

3. $$\int_1^\infty (x-1)^{\lambda-1}\left(x^2-1\right)^{-\mu/2}(x+z)^{-\rho}\,P_\nu^\mu(x)\,dx$$
$$= -\frac{\sin(\nu\pi)\,\Gamma(\lambda-\mu-\rho)\,\Gamma(\rho-\lambda+\mu-\nu)\,\Gamma(\rho-\lambda+\mu+\nu+1)}{2^{\rho-\lambda+\mu}\pi\,\Gamma(1+\rho-\lambda)}$$
$$\times {}_3F_2\left(\rho,\rho-\lambda+\mu-\nu,\rho-\lambda+\mu+\nu+1;1+\rho-\lambda,1+\rho-\lambda+\mu;\frac{1+z}{2}\right)$$
$$+\frac{\Gamma(\lambda-\mu)\,\Gamma(\rho-\lambda+\mu)}{\Gamma(\rho)\,\Gamma(1-\mu)}(z+1)^{\lambda-\rho-\mu}$$
$$\times {}_3F_2\left(\lambda-\mu,-\nu,\nu+1;1+\lambda-\mu-\rho,1-\mu;\frac{1+z}{2}\right)$$
$$[\mathrm{Re}(\lambda-\mu)>0,\quad \mathrm{Re}(\rho-\lambda+\mu-\nu)>0,\quad \mathrm{Re}(\rho-\lambda+\mu+\nu+1)>0,\quad |\arg(z+1)|<\pi]$$

ET II 322(10)

It is a bit heavy duty, but I would set λ = 1, μ = 0. I did a *very serious* Maple entry job (bowl in toroidals.mws) and careful processing, and I ended up with this nice result:

$$\frac{\sqrt{2}\,\cosh(p\,u0-\pi\,p)}{\sinh(\pi\,p)\,p}$$

$$\int_1^\infty dx\; P_{ip-1/2}(x) / \sqrt{x - \cos u_0} \quad = \sqrt{2}\;\mathrm{ch}[p(u_0-\pi)]/\,(p\;\mathrm{sh}(\pi p))$$

I am assuming u_0 is in $(0,\pi)$ and this means $u_0/2$ is in $(0,\pi/2)$ and that means $\sin(u_0/2)$ and $\cos(u_0/2)$ are positive, something I had to tell Maple. I bet I can find this somewhere now that I know the answer.

Aside: Can I get **verification** of the above integral somewhere? Not as such in GR or ET II or HT I. GR7 p 980 has the same as HT I really. This led to me downloading the **PBM** integral books, and in **conical function data.doc** I do verify the above integral from a completely different starting point!

So let's finish off our problem now. We then have

$$f(p;\,u_o)/[\;p\;\mathrm{th}(\pi p)] \;= V_0\sqrt{2}\;\mathrm{ch}[p(u_0-\pi)]/\,(p\;\mathrm{sh}(\pi p))$$

$$f(p;\,u_o) \;= p\;\mathrm{th}(\pi p)\;V_0\sqrt{2}\;\mathrm{ch}[p(u_0-\pi)]/\,(p\;\mathrm{sh}(\pi p)) \;= V_0\sqrt{2}\;\mathrm{ch}[p(u_0-\pi)]/\,(\mathrm{ch}(\pi p))$$

but we had earlier that

$$f(p;\,u_o) = \;A_p\mathrm{sh}(u_0 - 2\pi)$$

so we find that

$$A_p = V_0\sqrt{2}\;\mathrm{ch}[p(u_0-\pi)]/\,(\mathrm{ch}(\pi p)\;\mathrm{sh}(u_0-2\pi))$$

and there is our coefficient! Our solution is then

$$V(\xi,u,\varphi) = \sqrt{ch\xi - cosu} \int_0^\infty dp\, P_{ip-1/2}(ch\xi)\, A_p\, sh(u - 2\pi)$$

$$= \sqrt{ch\xi - cosu} \int_0^\infty dp\, P_{ip-1/2}(ch\xi)\, sh(u - 2\pi) \{ V_0\sqrt{2}\, ch[p(u_0-\pi)]/ (ch(\pi p)\, sh(u_0-2\pi))\}$$

$$= V_0\sqrt{2} [\sqrt{ch\xi - cosu}\ sh(u - 2\pi)/ sh(u_0-2\pi)] \int_0^\infty dp\, P_{ip-1/2}(ch\xi)\, ch[p(u_0-\pi)]/(ch(\pi p))$$

Maybe I can **check** what happens when $u = u_0$:

$$V(\xi,u_0,\varphi) = V_0\sqrt{2} [\sqrt{ch\xi - cosu_0} \int_0^\infty dp\, P_{ip-1/2}(ch\xi)\, ch[p(u_0-\pi)]/(ch(\pi p))$$

but then I need this integral which I don't know without doing some work.

Compare my result to his result

Here is Lebedev's quoted result,

$$u(\alpha, \beta) = V\sqrt{2\cosh\alpha - 2\cos\beta}$$
$$\times \int_0^\infty [\sinh(2\pi + \beta_0 - \beta)\tau - \cosh(\pi - \beta_0)\tau$$
$$\times \sinh(\pi - \beta)\tau] \frac{P_{-\frac{1}{2}+i\tau}(\cosh\alpha)}{\sinh(\pi + \beta_0)\tau \cosh\pi\tau} d\tau,$$

where $P_\nu(x)$ is the Legendre function of the first kind, V is the potential of the conductor and $\sin\beta_0 = c/a$.

which I now translate

$$u(\xi,u) = V_0\sqrt{2}\sqrt{ch\xi - cosu} * \int_0^\infty dp\ P_{ip-1/2}(ch\xi) *$$

$$\{ sh[(2\pi+u_0-u)p] - ch[(\pi-u_0)p]\, sh[(\pi-u)p] \}/ (sh[(\pi+u_0)p]\, ch(\pi p))$$

Our results would manifestly agree if I could show that

$$sh(u - 2\pi)/ sh(u_0-2\pi) * ch[p(u_0-\pi)]/(ch(\pi p)) \qquad \text{// mine}$$

$$= \{ sh[(2\pi+u_0-u)p] - ch[(\pi-u_0)p]\, sh[(\pi-u)p] \}/ (sh[(\pi+u_0)p]\, ch(\pi p)) \qquad \text{// his}$$

or

$$sh(u - 2\pi)/ sh(u_0-2\pi) * ch[p(u_0-\pi)] = \{ sh[(2\pi+u_0-u)p] - ch[(\pi-u_0)p]\, sh[(\pi-u)p] \}/ (sh[(\pi+u_0)p]) \quad (*)$$

or

$$\underset{h1}{\sh(u - 2\pi)\, \ch[p(u_0-\pi)]\, \sh[(\pi+u_0)p]} = \underset{h2}{\sh(u_0-2\pi)\, \{\, \sh[(2\pi+u_0-u)p] - \ch[(\pi-u_0)p]\, \sh[(\pi-u)p]\, \}}$$

If these two expressions are to be the same, they must be the same for any special case I can concoct. For example, if u = 2π, the LHS = 0. Is the RHS zero as well ? The only way that could happen is if the bracket vanishes. That would say

$$\{\, \sh[(2\pi+u_0-2\pi)p] - \ch[(\pi-u_0)p]\, \sh[(\pi-2\pi)p]\, \} \quad =?= 0$$

$$\{\, \sh[(u_0)p] - \ch[(\pi-u_0)p]\, \sh[(-\pi)p]\, \} \quad =?= 0$$

But now let $u_0 = 0$ and this becomes

$$\{\, \ch[(\pi)p]\, \sh[(\pi)p]\, \} \quad =?= 0$$

and this does not fly. So either one of us has made a mistake, or the integrals are the same even though the integrands are different.

Check some Limits of My Result

First, here is my result from above,

$$V(\xi,u,\varphi) = V_0\sqrt{2}\ \left[\ \sqrt{\ch\xi - \cos u}\ \ \sh(u - 2\pi)/\ \sh(u_0-2\pi)\ \right]\ \int_0^\infty\ dp\ P_{ip-1/2}(\ch\xi)\ \ch[p(u_0-\pi)]/(\ch(\pi p))$$

This thing is supposed to vanish when we take $u \to 2\pi$ since that is "far away". My result manifestly vanishes in this limit.

How does *his* result vanish in this limit? His result becomes

$$u(\xi,u) = V_0\sqrt{2}\ \sqrt{\ch\xi - 1}\ \ *\ \int_0^\infty\ dp\ \ P_{ip-1/2}(\ch\xi)\ \ *$$

$$\{\, \sh[(u_0)p] + \ch[(\pi-u_0)p]\, \sh[\pi p]\, \}/\ (\ \sh[(\pi+u_0)p]\ \ch(\pi p)\ )$$

It is not obvious that this vanishes. Suppose we ALSO take the limit $u_0 = 0$ which is the disk as I did it above. Then my result of course still vanishes, and his becomes

$$u(\xi,u) = V_0\sqrt{2}\ \sqrt{\ch\xi - 1}\ \ *\ \int_0^\infty\ dp\ \ P_{ip-1/2}(\ch\xi)\ \ *$$

$$\{\, \ch[\pi p]\, \sh[\pi p]\, \}/\ (\ \sh[\pi p]\ \ch(\pi p)\ )$$

$$= V_0\sqrt{2}\ \sqrt{\ch\xi - 1}\ \ *\ \int_0^\infty\ dp\ \ P_{ip-1/2}(\ch\xi)$$

PBM gives this integral

$$1.\quad \int_0^{\infty} P^{\mu}_{ix-1/2}(c)\,dx = \sqrt{\frac{\pi}{2}}\,\frac{(c-1)^{-(\mu+1)/2}}{\Gamma(1/2-\mu)}\,(c+1)^{\mu/2}$$

which I can translate for $\mu = 0$ to say

$$\int_0^{\infty} dp\ P_{ip-1/2}(ch\xi) = \sqrt{\pi/2}\ (ch\xi-1)^{-1/2}/\Gamma(1/2) = (1/\sqrt{2})\ 1/\sqrt{ch\xi - 1}$$

and then we get

$$u(\xi,u) = V_0\sqrt{2}\ \sqrt{ch\xi - 1}\ *\ (1/\sqrt{2})\ 1/\sqrt{ch\xi - 1} = V_0$$

So his result is going to V_0 when $u = 2\pi$ and $u_0 = 0$.

Notes Added 1.20.11. While doing a web search, I found an obscure paper from which I quote:

5.2. Comparison with the Exact Solution. Deegan et al.[14] reported an analytical solution for the Laplace equation, which was derived by Lebedev.[9] Lebedev considered a charged surface formed by the union of two spherical domains, and derived the electrostatic potential distribution in the space exterior to this surface by using toroidal coordinates. In toroidal coordinates, the potential distribution is given by

$$u = V\sqrt{2\cosh\alpha - 2\cos\beta}\int_0^{\infty} \frac{\cosh[(\pi-\beta_1)\tau]\sinh[(\beta-\beta_2)\tau] + \cosh[(\pi-\beta_2)\tau]\sinh[(2\pi+\beta_1-\beta)\tau]}{\cosh(\pi\tau)\sinh[(2\pi+\beta_1-\beta_2)\tau]} P_{-(1/2)+i\tau}(\cosh\alpha)\,d\tau \quad (22)$$

where α and β are the toroidal coordinates, V is the potential on the surface, and β_1 and β_2 are two angles of $\pi - \theta$ and $\pi + \theta$, $P_{-(1/2)+i\tau}(\cosh\alpha)$ is the Legendre function of the first kind and is expressed by

$$P_{-(1/2)+i\tau}(\cosh\alpha) = \frac{2}{\pi}\coth(\pi\tau)\int_{\alpha}^{\infty} \frac{\sin(\tau\, t)}{\sqrt{2\cosh t - 2\cosh\alpha}}\,dt \quad (23)$$

(9) Lebedev, N. N. *Special Functions and Their Application*; Prentice-Hall: Englewood Cliffs, New Jersey, 1965.

(14) Deegan, R. D.; Bakajin, O.; Dupont, T. F.; et al. *Phys. Rev. E* **2000**, *62*, 756.

I think it is actually ref (14) that gets this formula. I am now trying Marriott on line PROLA and I get at least this far

- **Contact line deposits in an evaporating drop**
  Robert D. Deegan, Olgica Bakajin, Todd F. Dupont, Greg Huber, Sidney R. Nagel, and Thomas A. Witten
  pp. 756-765 [**View** PDF (383 kB)]

But I can't get in of course, too bad. But then I found it on line and I have it, a preprint. But the preprint has a different result, and does not contain any figures, ouch! I will have to go there in person. Found while browsing:

$$\int_1^\infty P_{-1/2+i\tau_1}(x)\,P_{-1/2+i\tau_2}(x)\,dx = \frac{\delta(\tau_1-\tau_2)}{\tau_1\tanh(\pi\tau_1)}. \qquad \text{(A3)}$$

This is exactly the book I have! For comparison, here is Leb's problem 503 solution

$$u(\alpha,\beta) = V\sqrt{2\cosh\alpha - 2\cos\beta}$$
$$\times\int_0^\infty [\sinh(2\pi+\beta_0-\beta)\tau - \cosh(\pi-\beta_0)\tau$$
$$\times \sinh(\pi-\beta)\tau]\,\frac{P_{-\frac{1}{2}+i\tau}(\cosh\alpha)}{\sinh(\pi+\beta_0)\tau\cosh\pi\tau}\,d\tau,$$

I don't see (22) appearing in the book, but the above makes me think that this Problem 503 is really not about a spherical bowl, but a spherical bowl with a flat metal lid on it! This must be what Lebedev means by a spherical zone!!! I suspect that (22) above is a generalization of Lebedev's result which applies to the surface defined by two bowls of β_1 and β_2. If in (22) I were to set $\beta_2 = \pi$ to apply the flat lid, then the two results agree exactly with $\beta_1 = \beta_0$. Mystery is now explained!!!

The paper in which (22) appears concerns itself with the physics of the evaporation of a drop of liquid which exists on some surface, like a drop of water on the top of your car. It seems that such a "sessile drop" assumes the shape of a spherical cap (certainly not obvious to me), and somehow the evaporation theory is analogous to potential theory (perhaps evaporation is diffusion) and the electrostatic potential of the analogous problem tells you something about evaporation.

(1) So the reason that my spherical bowl result does not agree with Lebedev's result is that we have done different problems! So now I am motivated to find a web verification of my simple answer for the bowl which is this:

$$V(\xi,u,\varphi) = V_0 [\, sh(u - 2\pi)/ sh(u_0 - 2\pi)]$$

Again, this is strikingly similar to the result for the charged ellipsoid. I think I could derive the above afresh in a similar manner, perhaps apart from the R-separation factor issue.

What about taking a limit of (22) as $\beta_1 = \beta_2$. This would describe the potential outside a bowl I think.

Here is the big factor (22) written out by me where I set $\beta_1 = \beta_2 = \beta_0$:

$$\{ ch[(\pi-\beta_0)\tau]\, sh[(\beta-\beta_0)\tau] + ch[(\pi-\beta_0)\tau]sh[(2\pi+\beta_0-\beta)\tau] \}/ \{ ch(\pi\tau)sh(2\pi\tau) \}$$

which I now try to simplify

$$= ch[(\pi-\beta_0)\tau]\{ sh[(\beta-\beta_0)\tau]+ sh[(2\pi+\beta_0-\beta)\tau]/ \{ ch(\pi\tau)sh(2\pi\tau) \}$$

I can try the Schaum sh + sh formula which says

$$shx + shy = 2\, sh(x+y)/2 * ch(x-y)/2$$

Set $x = (\beta-\beta_0)\tau$ and $y = (2\pi+\beta_0-\beta)\tau$ so

$$(x+y)/2 = [(\beta-\beta_0)\tau + (2\pi+\beta_0-\beta)\tau]/2 = \pi\tau$$
$$(x- y)/2 = [(\beta-\beta_0)\tau - (2\pi+\beta_0-\beta)\tau]/2 = (\beta-\beta_0)\tau - \pi\tau$$

$$\Rightarrow sh[(\beta-\beta_0)\tau]+ sh[(2\pi+\beta_0-\beta)\tau] = 2\, sh(\pi\tau)\, ch[(\beta-\beta_0)\tau - \pi\tau]$$

Then our big factor has become

$$= ch[(\pi-\beta_0)\tau]\{ sh[(\beta-\beta_0)\tau]+ sh[(2\pi+\beta_0-\beta)\tau]/ \{ ch(\pi\tau)sh(2\pi\tau) \}$$

$$= 2\, ch[(\pi-\beta_0)\tau]\{ sh(\pi\tau)\, ch[(\beta-\beta_0)\tau - \pi\tau] / \{ ch(\pi\tau)sh(2\pi\tau) \}$$

Now write $sh(2\pi\tau) = 2\, sh(\pi\tau)ch(\pi\tau)$ and we have

$$= ch[(\pi-\beta_0)\tau]\, ch[(\beta-\beta_0)\tau - \pi\tau] / ch^2(\pi\tau)$$

which is not very pleasant. I was hoping for better. I was hoping for a result of this form

$$V(\xi,\beta,\varphi) = V_0\sqrt{2}\ [\, \sqrt{ch\xi - cos\beta}\ sh(\beta - 2\pi)/ sh(\beta_0-2\pi)\,]\ \int_0^\infty d\tau\, P_{i\tau-1/2}(ch\xi)\, ch[\tau(\beta_0-\pi)]/(ch(\pi\tau))$$

and in order to get this, I have to show that

$$ch[(\pi-\beta_0)\tau]\, ch[(\beta-\beta_0)\tau - \pi\tau] / ch^2(\pi\tau) = [sh(\beta - 2\pi)/ sh(\beta_0-2\pi)]\, ch[\tau(\beta_0-\pi)]/(ch(\pi\tau))$$

or

$$ch[(\beta-\beta_0)\tau - \pi\tau] / ch(\pi\tau) = [sh(\beta - 2\pi)/ sh(\beta_0-2\pi)]$$

or

$$sh(\beta_0-2\pi)\, ch[(\beta-\beta_0)\tau - \pi\tau] = ch(\pi\tau)sh(\beta - 2\pi)$$

This is clearly not true since if $\beta = 2\pi$ the RHS = 0 but LHS is not 0. So the limit of the general formula does not give my result, too bad.