Paradox 12.4.15 with full sphere limit PhL 12.4.15

Right now bowl doc contains the result of part 1 below. Since V does not contain a, I don't know how to take a limit to maintain fixed R and have a→ 0 with R = a/u_0 holding constant. In my Maple plot, R does get very large and is not fixed. I cannot change the meaning of the u coordinate! So I leave this as it is. // 12.31.15

Here is the paradox.

1. On the one hand:

I start with my solution V which is this

$$V(\xi,u) = (V_0/\pi)\ \{\pi - \cot^{-1}[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - cosu}}] + \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi - cos(2u_0 - u)}}\ \cot^{-1}[\frac{\sqrt{2}\cos(u_0 - u/2)}{\sqrt{ch\xi - cos(2u_0 - u)}}]\ \}\quad (1)$$

I take the limit $u_0 = 0$ and this gives

$\cos(\pi - u/2) = \cos(u/2 - \pi) = -\cos(u/2)$ // for any u

$\cos(2u_0 - u) \to \cos(-u) = \cosu$
$\cos(u_0 - u/2) \to \cos(-u/2) = \cos(u/2)$ // limits as $u_0 \to 0$

$$V(\xi,u) = (V_0/\pi)\ \{\pi - \cot^{-1}[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - cosu}}] + 1*\cot^{-1}[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - cosu}}]\ \} = V_0$$

The two terms are equal and opposite it seems for all values of u in the range 2π, and so the two $\cot^{-1}$ terms cancel and the result is V_0.

2. On the other hand:

I enter exactly the equation (1) above into Maple,

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V := (V0/Pi)*A*(T1+T2);
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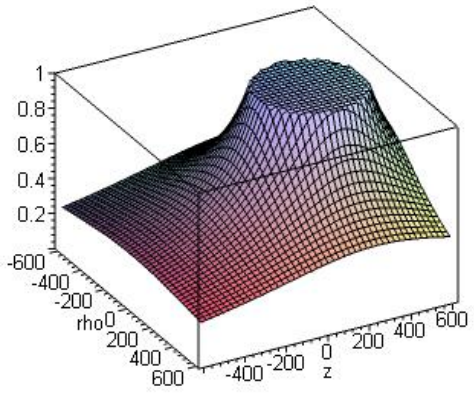
$$V := \frac{V0\sqrt{\cosh(\xi) - \cos(u)}\left(\frac{\pi - \operatorname{arccot}\left(\frac{\sqrt{2}\cos\left(\frac{1}{2}u\right)}{\sqrt{\cosh(\xi) - \cos(u)}}\right)}{\sqrt{\cosh(\xi) - \cos(u)}} + \frac{\operatorname{arccot}\left(\frac{\sqrt{2}\cos\left(-u0 + \frac{1}{2}u\right)}{\sqrt{\cosh(\xi) - \cos(-2\,u0 + u)}}\right)}{\sqrt{\cosh(\xi) - \cos(-2\,u0 + u)}}\right)}{\pi}$$

Maple then does a scan of ρ and z, and at each point it uses these equations to get ξ and u:

$\xi = th^{-1}\,(2a|\rho| / (a^2+\rho^2+z^2)$

u = getu(ρ,z) returns a value of u in range (0,2π)

and then it just plots the above function versus ρ and z, and it shows a dropoff 1/r outside the bowl, for example,



for $u_0 = .01\pi/8$

3. Paradox is this: Why does Maple show a dropoff, but my limit shows no dropoff.

Plan A: I think I know the answer, and Maple helped me out. Notice the *scale* on the above drawing! I had to increase it a lot to "get outside the bowl". If I don't increase the scale, then the entire plot is basically inside the bowl and you get the constant value V_0.

So the confusion is related to the fact that the bowl gets infinitely large in this limit! Basically the bowl fills the entire upper half plane in z,ρ space, and so at all points in that half plane you will have $V = V_0$!

So I have to regulate the limit. I said it was a "thought experiment only" but in fact it is not so.

You have in fact $a = R\,|\sin u_0|$ and if you want to keep R fixed, you have to scale down a as shown. But the parameter a does not appear in the potential!

So here is the problem: How do you take a limit of the above potential in such a way as to keep R fixed?

I could replace $\sin u_0$ by a/R and in the end take the limit $a \to 0$. How would that work? We start with

$$V(\xi,u) = (V_0/\pi)\ \{\ \pi - \cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}}\right]\ +\ \frac{\sqrt{ch\xi-\cos u}}{\sqrt{ch\xi-\cos(2u_0-u)}}\ \cot^{-1}\left[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0-u)}}\right]\ \}\quad (1)$$

Now replace

$u_0 = \sin^{-1}(a/R)$ $|\sin(u_0)| = a/R$

and then you could write out the above formula in detail. Then

$$\cos(2u_0-u) = \cos(2u_0)\cos(u) + \sin(2u_0)\sin(u)$$

$$= [\,1-2\sin^2(u_0)]\cos(u) + 2\sin(u_0)\cos(u_0)\sin(u)$$

$$= [\,1-2\sin^2(u_0)]\cos(u) + 2\,\mathrm{sign}(u_0)\,|\sin(u_0)|\sqrt{1-\sin^2 u_0}\,\sin(u)$$

$$= [\,1-2a^2/R^2]\cos(u) + 2\,\mathrm{sign}(\sin(u_0))(a/R)\sqrt{1-a^2/R^2}\,\sin(u)$$

and you could then install this in the second factor in two places. Similarly you could write

$$\cos(u_0-u/2) = \cos(u_0)\cos(u/2) + \sin(u_0)\sin(u/2)$$

$$= \sqrt{1-a^2/R^2}\,\cos(u/2) + 2\,\mathrm{sign}(\sin(u_0))(a/R)\,\sin(u/2)$$

So the expression is then very messy, but you could present it this way

$$V(\xi,u;\,u_0) = (V_0/\pi)\,\{\,\pi - \cot^{-1}\!\left[\frac{\sqrt{2}\,\cos(u/2)}{\sqrt{ch\xi-cosu}}\right] + \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi-\cos(2u_0- u)}}\,\cot^{-1}\!\left[\frac{\sqrt{2}\,\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0- u)}}\right]\,\}\quad (1)$$

where $\cos(2u_0-u) = [\,1-2a^2/R^2]\cos(u) + 2\,\mathrm{sign}(\sin(u_0))(a/R)\sqrt{1-a^2/R^2}\,\sin(u)$

$\cos(u_0-u/2) = \sqrt{1-a^2/R^2}\,\cos(u/2) + 2\,\mathrm{sign}(\sin(u_0))(a/R)\,\sin(u/2)$

I see no gain by writing this out as a single formula. But what happens now if we take the limit $u_0 \to 0$?

where $\cos(2u_0-u) = [\,1-2a^2/R^2]\cos(u) + 2\,\mathrm{sign}(\sin(u_0))(a/R)\sqrt{1-a^2/R^2}\,\sin(u)$

$\cos(u_0-u/2) = \sqrt{1-a^2/R^2}\,\cos(u/2) + 2\,\mathrm{sign}(\sin(u_0))(a/R)\,\sin(u/2)$

we know that $|\sin(u_0)| = a/R$ so $a = R|\sin(u_0)|$ so in this limit we have to have $a \to 0$. In this limit, both second terms vanish and we just get

$\cos(0-u) = [\,1]\cos(u)$
$\cos(0-u/2) = 1\,\cos(u/2)$

and everything is just fine. What happens as we take $u_0 \to 2\pi$? Again we have $a \to 0$ and we get the same two little equations as above.

I think we have to "stay away from the limit" a little bit. So lets write

$u_0 = 0 + \varepsilon$ $\qquad \varepsilon > 0$
$a << R$ $\qquad a = R|\sin u_0| = Ru_0 = R\varepsilon \qquad a/R = \varepsilon$

Then we have

$$\cos(2u_0-u) = [\,1-2a^2/R^2]\cos(u) + 2(a/R)\sqrt{1-a^2/R^2}\sin(u)$$

$$= [\,1-2\varepsilon^2]\cos(u) + 2\varepsilon\sqrt{1-\varepsilon^2}\sin(u)$$

$$\approx [\,1-2\varepsilon^2]\cos(u) + 2\varepsilon\,[\,1-(1/2)\varepsilon^2\,]\sin(u)$$

$$\approx [\,1-2\varepsilon^2]\cos(u) - 2\varepsilon\sin(u) \qquad \text{/// dropping the } \varepsilon^3 \text{ small term}$$

$$\approx \cos u - 2\varepsilon \sin u$$

and

$$\cos(u_0-u/2) = \sqrt{1-a^2/R^2}\cos(u/2) + 2\,\text{sign}(\sin(u_0))(a/R)\sin(u/2)$$

$$= \sqrt{1-\varepsilon^2}\cos(u/2) + 2\varepsilon\sin(u/2)$$

$$= \cos(u/2) + 2\varepsilon\sin(u/2)$$

So lets try going just keeping the leading ε terms, so we have

$$\cos(2u_0-u) = \cos u - 2\varepsilon \sin u$$

$$\cos(u_0-u/2) = \cos(u/2) + 2\varepsilon\sin(u/2)$$

and we then want to stick these into

$$V(\xi,u) = (V_0/\pi)\ \{\,\pi - \cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}}\right] + \frac{\sqrt{ch\xi-\cos u}}{\sqrt{ch\xi-\cos(2u_0-u)}}\cot^{-1}\left[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0-u)}}\right]\ \} \quad (1)$$

Well we will need

$$ch\xi - \cos(2u_0-u) = ch\xi - [\cos u - 2\varepsilon \sin u]$$

$$= ch\xi - \cos u + 2\varepsilon \sin u \qquad \sqrt{ch\xi - \cos u + 2\varepsilon \sin u}$$

Then the argument of the second cot term is this

$$\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0-u)}} = \frac{\sqrt{2}\,(\cos(u/2) + 2\varepsilon\sin(u/2))}{\sqrt{ch\xi - \cos u + 2\varepsilon \sin u}}$$

Then the potential is given by

$$V(\xi,u) = (V_0/\pi)$$

$$\{\ \pi - \cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - cosu}}\right] + \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi - cosu + 2\varepsilon\ sinu}}\ \cot^{-1}\left[\frac{\sqrt{2}\ (\cos(u/2) + 2\ \varepsilon\ \sin(u/2))}{\sqrt{ch\xi - cosu + 2\varepsilon\ sinu}}\right]\ \}\quad (1)$$

I don't see anything useful happening here! Nice try. Take $\varepsilon \to 0$ and we have the same paradox problem. Nice try, but no cigar.

Plan B. Consider again

$$V(\xi,u;\ u_0) = (V_0/\pi)\ \{\ \pi - \cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - cosu}}\right] + \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi - \cos(2u_0 - u)}}\ \cot^{-1}\left[\frac{\sqrt{2}\cos(u_0 - u/2)}{\sqrt{ch\xi - \cos(2u_0 - u)}}\right]\ \}\quad (1)$$

If we take $u_0 \to 0$ with a fixed, we get $R \to \infty$ and the sphere fills the entire upper half plane! This upper half plane is characterized by u in the range $(0,\pi)$. The lower half plane is $(\pi,2\pi)$. You might think that you should be $V < V_0$ in the lower half plane, but the sphere is so big, that you never move any significant distance from its surface in the lower half plane and V remains at $V = V_0$. So basically you get $V = V_0$ everywhere, and this agrees with what the limit says.

Treated as an abstract function, if you consider $V(\xi,u;\ u_0)$ stated above and take $u_0 \to 0$, you get V_0 and there is nothing you can do about it! You cannot take some kind of limit which keeps the sphere at a constant radius.

I think the only way out is to write the above entirely in terms of z,ρ,R and a. THEN maybe we can do something to get our limits. I might use some results from my Jackson limit work.

I know for example that

$$\sqrt{(\rho-a)^2+z^2} + \sqrt{(\rho+a)^2+z^2} = 2\sqrt{2}\ a\ ch(\xi/2) / \sqrt{ch\xi - cosu}$$

$$r = a\sqrt{sh^2\xi + \sin^2 u}\ /\ (ch\xi - cosu)$$

$$th\xi = 2a\rho/(\rho^2 + z^2 + a^2)$$

$$th^2\xi = [2a\rho/(\rho^2 + z^2 + a^2)]^2 = 1 - 1/ch^2\xi$$

$$1/ch^2\xi = 1 - [2a\rho/(\rho^2 + z^2 + a^2)]^2$$

Lets define

$$g \equiv 2a\rho/(\rho^2 + z^2 + a^2)$$

Then

$$1/ch^2\xi = 1 - g^2$$

$$ch^2\xi = 1/(1-g^2)$$

$$2\,ch^2\xi/2 = 1+ch\xi = 1 + 1/\sqrt{1-g^2}$$

$$ch^2\xi/2 = [\,1 + 1/\sqrt{1-g^2}\,]/2$$

$$\sqrt{2}\,ch\xi/2 = \sqrt{1 + 1/\sqrt{1-g^2}}$$

Then I can use this to get

$$\sqrt{(\rho-a)^2+z^2} + \sqrt{(\rho+a)^2+z^2} = 2\sqrt{2}\,a\,ch(\xi/2) / \sqrt{ch\xi-cosu}$$

$$= 2a\sqrt{1 + 1/\sqrt{1-g^2}} \;/ \sqrt{ch\xi-cosu}$$

and this gives this very messy result

$$1/\sqrt{ch\xi-cosu} = [\sqrt{(\rho-a)^2+z^2} + \sqrt{(\rho+a)^2+z^2}\,]/[\,2a\sqrt{1 + 1/\sqrt{1-g^2}}\,]$$

and the point is that I have written this object entirely in z,ρ coordinates. Now

$$g = 2a\rho/(\rho^2+ z^2 +a^2)$$

$$g^2 = [2a\rho/(\rho^2+ z^2 +a^2)]^2 = 4a^2\rho^2/ (\rho^2+ z^2 +a^2)^2$$

$$1-g^2 = 1 - 4a^2\rho^2/ (\rho^2+ z^2 +a^2)^2 = [(\rho^2+ z^2 +a^2)^2 - 4a^2\rho^2\,] / (\rho^2+ z^2 +a^2)^2$$

$$= [\rho^4+z^4 + a^4 + 2\rho^2z^2 - 2\rho^2a^2 + 2a^2z^2] / (\rho^2+ z^2 +a^2)^2$$

$$\equiv f(\rho,z,a) / (\rho^2+ z^2 +a^2)^2$$

$$\sqrt{1-g^2} = \sqrt{f} / (\rho^2+ z^2 +a^2)$$

$$1/\sqrt{1-g^2} = (\rho^2+ z^2 +a^2)/ \sqrt{f}$$

$$1 + 1/\sqrt{1-g^2} = 1 + (\rho^2+ z^2 +a^2)/ \sqrt{f} = [\sqrt{f} + (\rho^2+ z^2 +a^2)] /\sqrt{f}$$

It is just a real mess. This is not the right Plan. Maple can do it, why can't I do it?

Plan C. Recall

$$\xi = \tanh^{-1}[2a\rho/(\rho^2+ z^2 +a^2)] \qquad \rho = \sqrt{x^2+y^2} \qquad (1.3.5)$$

$u = \tan^{-1}[\, 2az/(\rho^2+z^2- a^2)]$

$\varphi = \tan^{-1}(y/x)$ $\qquad\qquad$ $r^2 = \rho^2+ z^2$

Suppose we say that a << ρ,z right from the start. Then we get

$\xi = \tanh^{-1}[2a\rho/r^2] \approx 2a\rho/r^2$ $\qquad$ $a \approx R\, u_0$

$u = \tan^{-1}[\, 2az/r^2] \approx 2az/r^2$ $\qquad\qquad$ assumes r >> R so that $r^2 = z^2+\rho^2$

Then we have simple Cartesian forms for these complicated toroidal parameters.

$\xi = 2a\rho/r^2$

$u = 2az/r^2$ $\qquad$ $a << \rho, |z|$ $\qquad$ $r >> R$

So both ξ and u are going to be small in this Plan C. Now once again

$$V(\xi,u; u_0) = (V_0/\pi)\ \{\ \pi - \cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-cosu}}\right] + \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi-\cos(2u_0- u)}}\ \cot^{-1}\left[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0- u)}}\right]\ \}\quad (1)$$

This is the form Maple uses to make the plot which shows 1/r drop, so I am quite sure it is right.

I know this is a large r regime.

$r = a\sqrt{sh^2\xi + \sin^2u}\ /\ (ch\xi - cosu)$

But when ξ and u are small we have

$ch\xi - cosu = \xi^2/2 + u^2/2$

so

$r = a\sqrt{\xi^2 + u^2}\ /\ [\ (\xi^2 + u^2)/2$

or

$r = 2a/\sqrt{\xi^2 + u^2}$

Now consider since u, u_0 are both small, as is their difference,

$\cos(2u_0-u) = 1 - (2u_0-u)^2/2 + O(u^4)$ $\qquad$ if both u and u_0 are small

$\cos(u_0-u/2) = 1 - (u_0-u/2)^2/2 + O(u^4)$

$ch\xi - \cos(2u_0-u) = [1 + \xi^2/2] - [\ 1 - (2u_0-u)^2/2] = \xi^2/2 + (2u_0-u)^2/2$

$\sqrt{ch\xi - \cos(2u_0-u)} = \sqrt{\xi^2/2 + (2u_0-u)^2/2}$

$$ch\xi - cosu = \xi^2/2 + u^2/2$$

$$\sqrt{ch\xi - cosu} = \sqrt{\xi^2/2 + u^2/2}$$

$$\cos(u/2) = (1 - (u/2)^2)$$

Now go back to the original toroidal potential

$$V(\xi,u;u_0) = (V_0/\pi)\ \{\ \pi - \cot^{-1}[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-cosu}}]\ +\ \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi-\cos(2u_0- u)}}\ \cot^{-1}[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0- u)}}]\ \}\quad (1)$$

r1 r2 r3

Ratio r2 is this

$$r2 = \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi-\cos(2u_0- u)}} = \frac{\sqrt{\xi^2/2 + u^2/2}}{\sqrt{\xi^2/2 + (2u_0-u)^2/2}} = \text{not small}$$

Ratio r3 is this

$$r3 = \frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0- u)}} = \frac{\sqrt{2}\,(1 - (u_0-u/2)^2/2)}{\sqrt{\xi^2/2 + (2u_0-u)^2/2}} = \text{large}$$ (we assume u and u_0 are small)

Ratio r1 is this

$$r1 = \frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-cosu}} = \frac{\sqrt{2}\,(1 - (u/2)^2/2)}{\sqrt{\xi^2/2 + u^2/2}} = \text{large}$$

Here then is our complete potential in this regime

$$V(\xi,u;u_0) = (V_0/\pi)\ \{\ \pi - \cot^{-1}(r1)\ +\ r2\ \cot^{-1}(r3)\ \}$$

All the time I have been thinking

$$\xi = 2a\rho/r^2$$
$$u = 2az/r^2 \qquad a << \rho, |z|$$
$$u_0 = a/R$$

so I guess you could say things are expressed fully in Cartesian coordinates. What do I do with $a = R\sin u_0 = Ru_0$? Well

$$\xi = 2Ru_0\rho/r^2$$
$$u = 2Ru_0z/r^2 \qquad a << \rho, |z|$$

So now our potential has no a showing, but it does have R showing!

What can I do next? Consider

$$\cot^{-1}(r1) = \tan^{-1}(1/r_1) \approx 1/r1$$
$$\cot^{-1}(r3) = \tan^{-1}(1/r_3) \approx 1/r3$$

Then the total potential is

$$V(\xi,u;\, u_0) = (V_0/\pi) \quad \{ \pi - 1/r1 \quad + \quad \frac{\sqrt{\xi^2/2 + u^2/2}}{\sqrt{\xi^2/2 + (2u_0-u)^2/2}} \quad 1/r3 \quad \}$$

Now we have

$$\frac{\sqrt{\xi^2/2 + u^2/2}}{\sqrt{\xi^2/2 + (2u_0-u)^2/2}} \quad 1/r3 \quad = \frac{\sqrt{\xi^2/2 + u^2/2}}{\sqrt{\xi^2/2 + (2u_0-u)^2/2}} \Big/ \frac{\sqrt{2}\,(1 - (u_0-u/2)^2/2)}{\sqrt{\xi^2/2 + (2u_0-u)^2/2}}$$

$$= \frac{\sqrt{\xi^2/2 + u^2/2}}{\sqrt{2}\,(1 - (u_0-u/2)^2/2)}$$

and

$$1/r1 = \frac{\xi^2/2 + u^2/2}{\sqrt{2}\,(1 - (u/2)^2/2)} \quad = \text{small}$$

and so

$$V(\xi,u;\, u_0) = (V_0/\pi) \quad \{ \pi - \frac{\xi^2/2 + u^2/2}{\sqrt{2}\,(1 - (u/2)^2/2)} \quad + \quad \frac{\sqrt{\xi^2/2 + u^2/2}}{\sqrt{2}\,(1 - (u_0-u/2)^2/2)} \quad \}$$

Now we do have

$$r \approx 2a/\sqrt{\xi^2 + u^2} \quad = 2Ru_0 / \sqrt{\xi^2 + u^2}$$

$$\sqrt{\xi^2 + u^2} = 2Ru_0/r \qquad \sqrt{\xi^2/2 + u^2/2} = (1/\sqrt{2})\sqrt{\xi^2 + u^2} = (1/\sqrt{2})\, 2Ru_0/r = \sqrt{2}\, Ru_0/r$$

$$\xi^2/2 + u^2/2 = 2R^2u^2_0/r^2$$

So simplifying denominators, we get

$$V(\xi,u;\, u_0) = (V_0/\pi) \quad \{ \pi - (1/\sqrt{2})2R^2u^2_0/r^2 + \quad (1/\sqrt{2})\ \sqrt{2}\, Ru_0/r \ \}$$

$$= (V_0/\pi) \quad \{ \pi - (1/\sqrt{2})(u_0R/r)[2(u_0R/r) \ - \sqrt{2}\ ] \ \}$$

Now $(u_0R/r) << 1$ for large r, and also because u_0 is small, so we have

$= (V_0/\pi)\ \{ \pi - (1/\sqrt{2})(u_0R/r)[- \sqrt{2}]\ \}$

$= (V_0/\pi)\ \{ \pi + (u_0R/r)\ \}$

$= (V_0)\ \{ 1 + (u_0R/\pi r)\ \}$

and for the first time I get something that *at least looks like* a 1/r potential. The result I want is

$V(\xi,u; u_0) = Q/r = (CV_0)/r = (RV_0)/r = V_0\ (R/r)$

In my result if it were correct the first term swamps the second and we get our familiar $V = V_0$.

Summary: I tried to do the case where u,u_0 and ξ are all very small, thinking that would simplify things. But in the end, instead of getting 1/r distant behavior, I get $V = V_0$, conflicting with what I see Maple plotting in front of my eyes.

The far field potential should be K/r for any value of u_0. I should not have to take u_0 small. But of course we have to have $u \geq u_0$ so then u can no longer be small. So in this case, what do you say about an expression for r?

Well, I need to repair my expression for r in the Section 1! This is tricky, you have to choose your r coordinate. I guess I have it right. It is relative to the origin. So I have

$r = a\sqrt{sh^2\xi + \sin^2 u}\ /\ (ch\xi - \cos u)$ $\qquad\qquad$ $r^2 = x^2+y^2+z^2$

and this r is NOT with respect to the center of the bowl!

Now where is the large r limit? Certainly we need $ch\xi - \cos u \rightarrow 0$. Certainly that requires ξ near 0 and u near 0 so I think I have that part correct:

Where is large r? It is in the region of $\xi \approx 0$ and $u \approx 0$ and in that region we have

$r \approx 2a/\sqrt{\xi^2 + u^2} = 2R|\sin u_0| / \sqrt{\xi^2 + u^2}$ $\qquad\qquad$ $r^2 \equiv \rho^2+z^2$

So for fixed u_0 we certainly can associate large r with small ξ and u except we need $u \geq u_0$, so you really cant "get to" small u.

Idea: Maybe H have to stick with the full expression $r = a\sqrt{sh^2\xi + \sin^2 u}\ /\ (ch\xi - \cos u)$ and not make any assumptions about "where this is large". I should play with this a bit.

$r = a\sqrt{sh^2\xi + \sin^2 u}\ /\ (ch\xi - \cos u)$ $\qquad\qquad$ // correct from bowl doc

$= a\ sh\xi \sqrt{1 + \sin^2 u/sh^2\xi}\ /\ (ch\xi - \cos u)$

$= a\,sh\xi \sqrt{1 + z^2/\rho^2} / (ch\xi - cosu)$

$$r = a\sqrt{1 + z^2/\rho^2}\,\frac{sh\xi}{ch\xi - cosu}$$

How do you make r be large? You MUST have the denominator small here, so you need these ingredients:

$\xi \approx$ small $u \approx 2\pi$ which always lies in the range u_0, $u_0+2\pi$

So maybe my u = 0 was the bad turn I took in all the above! I need to fix this wherever I say it in bowl doc. Keeping u as is, we end up with

$$r = a\sqrt{1 + z^2/\rho^2}\,\frac{\xi}{1+\xi^2/2 - cosu}$$

Now let

$u = 2\pi + \varepsilon$ ε could have either sign

$\cos(2\pi+\varepsilon) = \cos(\varepsilon) = 1-\varepsilon^2/2$

Then we have

$$r = a\sqrt{1 + z^2/\rho^2}\,\frac{\xi}{1+\xi^2/2 - [1-\varepsilon^2/2]} = a\sqrt{1 + z^2/\rho^2}\,\frac{\xi}{\xi^2/2+\varepsilon^2/2} = 2a\sqrt{1 + z^2/\rho^2}\,\frac{\xi}{\xi^2+\varepsilon^2}$$

again,

$$r = 2a\sqrt{1 + z^2/\rho^2}\,\frac{\xi}{\xi^2+\varepsilon^2}$$

So "large r" is a certain region of (u,ξ) space where

$\xi >> \xi^2 + \varepsilon^2$

$1 >> \xi + \varepsilon^2/\xi$

$\xi + \varepsilon^2/\xi << 1$

For this last to be true, since both terms are positive, they must both be small. So we need

$\xi << 1$ AND $\varepsilon^2 << \xi$

or

$\varepsilon^2 << \xi << 1$

Implications? $\xi << 1$ $\xi^2 << 1$

$$\varepsilon << 1 \qquad \varepsilon^2 << 1$$
$$\xi + \varepsilon << 1 \qquad \xi^2 + \varepsilon^2 << 1 \qquad \sqrt{\xi^2+\varepsilon^2} << 1$$

So this is something new in this document.

$$\xi << 1 \qquad \text{AND} \qquad (u-2\pi)^2 << \xi$$

This certainly is a very strange region of ξ,u space!

$$(u-2\pi)^2 << \xi << 1$$

Certainly Maple "finds" this region during its plot.

Fact: Here is the region of (u,ξ) space where one finds large r:

$$(u-2\pi)^2 << \xi << 1 \qquad \varepsilon^2 << \xi << 1 \qquad u = 2\pi + \varepsilon$$

So if you want to examine the potential for large r, you *have to go* to this region!

Question: do we know that $\varepsilon << \xi$? Well we know that $\varepsilon^2 << \xi$ so $\varepsilon << \sqrt{\xi}$.

Example: Suppose $\varepsilon = 10^{-4}$ and $\xi = 10^{-5}$. Then $\varepsilon^2 = 10^{-8} << 10^{-5}$ but $\varepsilon = 10^{-4}$ is NOT $<< 10^{-5}$. And also $\varepsilon^2 = 10^{-8}$ is NOT $<< \xi^2 = 10^{-10}$.

Fact: We do NOT know that $\varepsilon << \xi$ so we do NOT know that $\varepsilon^2 << \xi^2$.

So I guess I could try to study up on the various terms in this select region

$$V(\xi,u) = (V_0/\pi) \{ \pi - \cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - \cos u}}\right] + \frac{\sqrt{ch\xi - \cos u}}{\sqrt{ch\xi - \cos(2u_0 - u)}} \cot^{-1}\left[\frac{\sqrt{2}\cos(u_0 - u/2)}{\sqrt{ch\xi - \cos(2u_0 - u)}}\right] \} \quad (1)$$
$$\qquad\qquad R1 \qquad\qquad R2 \qquad\qquad R3$$

$\cos(u/2) = \cos(\pi+\varepsilon/2) = \cos(\varepsilon/2+\pi) = -\cos(\varepsilon/2) \approx -[1 - (\varepsilon/2)^2/2] = \varepsilon^2/8 - 1$

$\cos(u) = \cos(2\pi+\varepsilon) = \cos\varepsilon \approx 1 - \varepsilon^2/2$

$ch\xi - \cos u = (1 + \xi^2/2 - 1 + \varepsilon^2/2) = (\xi^2+\varepsilon^2)/2$ cannot replace with $\xi^2/2$!!!

$\sqrt{ch\xi - \cos u} \approx \sqrt{\xi^2+\varepsilon^2}/\sqrt{2}$ in my region

$\cos(2u_0-u) = \cos(2u_0 - [2\pi+\varepsilon]) = \cos(2u_0 - \varepsilon - 2\pi) = \cos(2u_0 - \varepsilon)$

$ch\xi - \cos(2u_0-u) = 1+\xi^2/2 - \cos(2u_0 - \varepsilon) = \xi^2/2 + [1-\cos(2u_0 - \varepsilon)] = \xi^2/2 + 2\sin^2(u_0 - \varepsilon/2)$

$$\sqrt{ch\xi - \cos(2u_0-u)} = \sqrt{\xi^2/2 + 2\sin^2(u_0 - \varepsilon/2)}$$

$\cos(u_0- u/2) = \cos(u_0 - \pi - \varepsilon/2) = - \cos(u_0-\varepsilon/2)$ // rechecked all on 12/5/15 made fixes

Now look at the ratios,

$$R1 = \frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-cosu}} = \frac{\sqrt{2}\,(-1+\varepsilon^2/8)}{\sqrt{\xi^2+\varepsilon^2}\,/\sqrt{2}} \approx -2/\sqrt{\xi^2+\varepsilon^2} \quad = \text{large} \quad \text{in my region}$$

$$R2 = \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi-\cos(2u_0- u)}} = \frac{\sqrt{\xi^2+\varepsilon^2}\,/\sqrt{2}}{\sqrt{\xi^2/2 + 2\sin^2(u_0 - \varepsilon/2)}} \approx \frac{\sqrt{\xi^2+\varepsilon^2}\,/\sqrt{2}}{\sqrt{2\sin^2(u_0 - \varepsilon/2)}} \quad = \text{small}$$

$$= (\sqrt{\xi^2+\varepsilon^2}\,/2) \,/\, |\sin(u_0- \varepsilon/2)| \approx (\sqrt{\xi^2+\varepsilon^2}\,/2)/\,|\sin(u_0)|$$

$$R3 = \frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0- u)}} = \frac{-\sqrt{2}\cos(u_0-\varepsilon/2)}{\sqrt{2\sin^2(u_0)}} \approx -\frac{\cos(u_0)}{|\sin(u_0)|} = -\,\text{sign}(u_0)\cot(u_0) = \text{moderate size say}$$

Note: I have made various approximations in getting final results for R1,R2,R3.

So here then is my potential in the special region // repairs done!!!

$$V(\xi,u) = (V_0/\pi)\ \{\pi - \cot^{-1}[R1] + R2 * \cot^{-1}[R3]\ \}$$

$$= (V_0/\pi)\ \{\pi - \cot^{-1}[-2/\sqrt{\xi^2+\varepsilon^2}\ ] + (\sqrt{\xi^2+\varepsilon^2}\,/2)/\,|\sin(u_0)|\,\cot^{-1}[-\,\text{sign}(u_0)\cot(u_0)]\ \}$$

Now I happily replace $\pi - \cot^{-1}(-2/\sqrt{\xi^2+\varepsilon^2}) = \cot^{-1}(2/\sqrt{\xi^2+\varepsilon^2})$ so we then have

$$V(\xi,u) = (V_0/\pi)\ \{\cot^{-1}(2/\sqrt{\xi^2+\varepsilon^2}) + (\sqrt{\xi^2+\varepsilon^2}\,/2) \,/\, |\sin(u_0)|\,\cot^{-1}[-\,\text{sign}(u_0)\cot(u_0)]\ \}$$

Now **assume** for the moment that u_0 lies in 0 to π so $\text{sign}(u_0) = 1$, then we have:

$$V(\xi,u) = (V_0/\pi)\ \{\cot^{-1}(2/\sqrt{\xi^2+\varepsilon^2}) + (\sqrt{\xi^2+\varepsilon^2}\,/2)/\,|\sin(u_0)|\,\cot^{-1}[-\cot(u_0)]\ \}$$

and one of my "problems" is solved, the first free-standing term is now gone! Now

$$\cot^{-1}(2/\sqrt{\xi^2+\varepsilon^2}) \approx \cot^{-1}(+\infty) = 0$$

so we are left with

$$V(\xi,u) = (V_0/\pi)\ (\sqrt{\xi^2+\varepsilon^2}\,/2)/\,|\sin(u_0)|\,\cot^{-1}[-\cot(u_0)]$$ // in my region of interest

Now $\cot^{-1}(-\cot(u_0)) = \pi - \cot^{-1}[\cot(u_0)] = \pi - u_0$

So I then have

$V(\xi,u) = (V_0/\pi)\,[\,(\sqrt{\xi^2+\varepsilon^2}\,/2)/\,|\sin(u_0)|\,]\,(\pi-u_0)$ // in my region of interest

Now recall that

$r = 2a\sqrt{1+z^2/\rho^2}\,\dfrac{\xi}{\xi^2+\varepsilon^2}$ // correctly quoted from above

$(\xi^2+\varepsilon^2) = 2a\xi\sqrt{1+z^2/\rho^2}$

STOP. Redo r this way $\sin(u) = \sin(2\pi+\varepsilon)$ in $\sin(\varepsilon) = \varepsilon$

$r = a\sqrt{sh^2\xi + \sin^2 u}\,/\,(ch\xi - \cos u)$ // correct from bowl doc

$= a\sqrt{\xi^2+\varepsilon^2}\,/\,[\,(\xi^2+\varepsilon^2)/2\,] = 2a/\sqrt{\xi^2+\varepsilon^2}$ // as I state in bowl doc

So here is where we stand right now

$V(\xi,u) = (V_0/\pi)\,[\,(\sqrt{\xi^2+\varepsilon^2}\,/2)/\,|\sin(u_0)|\,]\,(\pi-u_0)$ // in my region of interest

$r = 2a/\sqrt{\xi^2+\varepsilon^2}$ $0 < u_0 < \pi$

So now replace

$\sqrt{\xi^2+\varepsilon^2} = 2a/r$

to get

$V(\xi,u) = (V_0/\pi)\,[\,(a/r)/\,|\sin(u_0)|\,]\,(\pi-u_0)$ // in my region of interest

But $a = R|\sin u_0|$ so write again as

$V(\xi,u) = (V_0/\pi)\,[\,(R|\sin u_0|/r)/\,|\sin(u_0)|\,]\,(\pi-u_0)$ // in my region of interest

$= (V_0R/\pi)(\pi-u_0)\,(1/r)$

Side comment:

$z/\rho = \sin u/sh\xi = \varepsilon/\xi$ and we do NOT know that this is small, problem fixed!

Now what? What is the capacitance of our bowl? I have that at the end of section 5

$$C = (R/\pi)\ \{ (\pi - u_0) + \sin(u_0) \}$$

Then

$$\pi C/R = (\pi - u_0) + \sin(u_0)$$

So my result is not quite right, but this is the closest I have ever gotten. The result I expect is this

$$Q = CV_0 \qquad V = Q/r = V_0(C/r) = V_0\ \{ (R/\pi) [\ (\pi - u_0) + \sin(u_0)]\}/r$$

$$= (V_0R/\pi) [\ (\pi - u_0) + \sin(u_0)] / r \qquad \text{// What I expected}$$

$$= (V_0R/\pi)[(\pi - u_0)] / r \qquad \text{// What I got}$$

This is looking very promising, I seem to have lost a term somewhere!!!! Otherwise very good!

Back up to this point and try undoing certain approximations:

$$R1 = \frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - cosu}} = \frac{\sqrt{2}\,(-1 + \varepsilon^2/8)}{\sqrt{\xi^2+\varepsilon^2}\,/\sqrt{2}} \approx -(1-\varepsilon^2/8)\,2/\sqrt{\xi^2+\varepsilon^2}$$

$$= -2/\sqrt{\xi^2+\varepsilon^2} + (1/4)\,\varepsilon^2/\sqrt{\xi^2+\varepsilon^2} \qquad \text{recall} \quad \varepsilon^2 << \xi << 1$$

So here is a candidate "extra term" which may be "finite" . Can I claim that

$$\varepsilon^2 << \sqrt{\xi^2+\varepsilon^2} \qquad ?$$

$$\varepsilon^4 << \xi^2 + \varepsilon^2 \qquad ?$$

Not obvious to me, so let's see what this particular extra term does

$$\pi - \cot^{-1}[\ -(1-\varepsilon^2/8)\,2/\sqrt{\xi^2+\varepsilon^2}\ ]$$

Well, this extra term really has no effect because we still get argument = -∞. So this is not it. Move to R2 which I think is the mostly likely factor which will have that extra term.

$$R2 = \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi - cos(2u_0 - u)}} = \frac{\sqrt{\xi^2+\varepsilon^2}\,/\sqrt{2}}{\sqrt{\xi^2/2 + 2\sin^2(u_0 - \varepsilon/2)}}$$

Now write

$$\xi^2/2 + 2\sin^2(u_0 - \varepsilon/2) = 2\sin^2(u_0 - \varepsilon/2) + \xi^2/2$$

$$= 2\sin^2(u_0 - \varepsilon/2)\ [\ 1 + (\xi^2/2) / [\ 2\sin^2(u_0 - \varepsilon/2)]\ ]$$

$= 2\sin^2(u_0 - \varepsilon/2)\,[\,1 + (1/4)\,\xi^2 / \sin^2(u_0 - \varepsilon/2)]$

It would be nice if I had some common order thing, like saying $\varepsilon = \varepsilon$ and $\xi = f\,\varepsilon$ and then do order ε. Maybe do that on the final presentation. Now consider

$$\sin(u_0 - \varepsilon/2) \approx \sin(u_0) - (\varepsilon/2)\cos(u_0)$$

$$2\sin^2(u_0 - \varepsilon/2) \approx 2\,[\,]^2$$