Finding the large r behavior of the bowl potential PhL 12.5.15

I finally found "the missing term". Will do this again in simpler fashion in v2. Please don't lose it again.

The potential is this

$$V(\xi,u) = (V_0/\pi)\ \{\ \pi - \cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-cosu}}\right]\ +\ \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi-\cos(2u_0- u)}}\ \cot^{-1}\left[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0- u)}}\right]\ \}\quad (1)$$

1. Find region where r is large

Consider

$$r = a\sqrt{sh^2\xi\ + \sin^2u}\ /\ (ch\xi - cosu) \qquad (2)$$

To get large r, we must have small ξ and cosu close to 1. This happens in range near $u = 2\pi$.

Let $u = 2\pi+\varepsilon$ so $cosu = cos\varepsilon\ \approx 1-\varepsilon^2/2$. Also $sinu = sin\varepsilon \approx \varepsilon$

$$numer = a\sqrt{sh^2\xi\ + \sin^2u} = a\sqrt{\xi^2\ + \varepsilon^2}$$

$$denom = ch\xi - cosu = 1+\xi^2/2 - [\ 1-\varepsilon^2/2] = (\xi^2 + \varepsilon^2)/2$$

Thus

$$r = a\sqrt{sh^2\xi\ + \sin^2u}\ /\ (ch\xi - cosu)\ = a\sqrt{\xi^2\ + \varepsilon^2}\ /\ [(\xi^2 + \varepsilon^2)/2]$$

$$= 2a\ /\ \sqrt{\xi^2\ + \varepsilon^2}$$

Now note that

$$z/\rho = sinu/sh\xi = \varepsilon/\xi \equiv 1/f \quad \text{could be any size!!!}$$

$$\xi = f\varepsilon$$

Then $\sqrt{\xi^2\ + \varepsilon^2}\ = \sqrt{f^2\varepsilon^2\ + \varepsilon^2}\ = \sqrt{f^2\ + 1}\ \varepsilon$

so that

$$r = 2a\ /\ \sqrt{\xi^2\ + \varepsilon^2}\ \ = [2a/\sqrt{f^2\ + 1}\ ]\ (1/\varepsilon) \qquad (3)$$

$$1/r\ = \varepsilon\sqrt{f^2\ + 1}\ /\ 2a \qquad (4)$$

Thus, r is large where ε is small.

2. Evaluate the potential

Now go through things as in the previous paradox document:

$$V(\xi,u) = (V_0/\pi)\ \{\ \pi - \underbrace{\cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-cosu}}\right]}_{R1} + \underbrace{\frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi-\cos(2u_0- u)}}\ \cot^{-1}\left[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0- u)}}\right]}_{R2\ \ R3}\ \}\quad (1)$$

$\cos(u/2) = \cos(\pi+\varepsilon/2) = \cos(\varepsilon/2+\pi) = -\cos(\varepsilon/2) \approx -[\ 1 - (\varepsilon/2)^2/2] = \varepsilon^2/8 - 1$

$\cos(u) = \cos(2\pi+\varepsilon) = \cos\varepsilon \approx 1 - \varepsilon^2/2$

$ch\xi - cosu = (1 + \xi^2/2 - 1 + \varepsilon^2/2) = (\xi^2+\varepsilon^2)/2 = (\ f^2\varepsilon^2+\varepsilon^2)/2 = (1/2)(f^2+1)\varepsilon^2$

$\sqrt{ch\xi - cosu} \approx (1/\sqrt{2}\ )\sqrt{f^2+1}\ \varepsilon$

$1/\sqrt{ch\xi - cosu} = 1/[\ (1/\sqrt{2}\ )\sqrt{f^2+1}\ \varepsilon\ ] = \sqrt{2}\ /(\ \sqrt{f^2+1}\ \varepsilon\ )$

$\cos(2u_0-u) = \cos(2u_0- [2\pi+\varepsilon]) = \cos(2u_0- \varepsilon - 2\pi) = \cos(2u_0 - \varepsilon)$

$\approx \cos(2u_0) - \varepsilon\ [\ -\sin(2u_0)\ ] = \cos(2u_0) + \varepsilon\ \sin(2u_0)$

$ch\xi - \cos(2u_0-u) = 1+\xi^2/2 - \cos(2u_0 - \varepsilon) = \xi^2/2 + [1-\cos(2u_0 - \varepsilon)] = \xi^2/2 + 2\sin^2(u_0 - \varepsilon/2)$

but $\sin(u_0- \varepsilon/2) \approx \sin(u_0) - (\varepsilon/2)\cos(u_0)$

$\sin^2(u_0- \varepsilon/2) \approx \sin^2(u_0) - \varepsilon\ \sin u_0 \cos u_0$

Therefore

$ch\xi - \cos(2u_0-u) \approx \xi^2/2 + 2\sin^2(u_0 - \varepsilon/2) = (f^2/2)\varepsilon^2 + 2\ [\ \sin^2(u_0) - \varepsilon\ \sin u_0 \cos u_0]$

$= 2\sin^2 u_0 - 2\varepsilon\ \sin u_0 \cos u_0 + \text{order}(\varepsilon^2)$ which I will ignore

$\approx 2\sin^2 u_0 - 2\varepsilon\ \sin u_0 \cos u_0$

$= 2\sin u_0\ (\ \sin u_0 - 2\varepsilon \cos u_0) = 2\sin^2 u_0 - 4\varepsilon\ \sin u_0 \cos u_0 = 2\sin^2 u_0[\ 1 - 2\varepsilon \cot u_0]$

$[ch\xi - \cos(2u_0-u)]^{-1} = [\ 2\sin^2 u_0[\ 1 - 2\varepsilon\cot u_0]\ ]^{-1} = (1 + 2\varepsilon\cot u_0)\ /(2\sin^2 u_0)$

$1/\sqrt{ch\xi - \cos(2u_0-u)} = (1 + \varepsilon\cot u_0)/(\sqrt{2}\ |\sin u_0|)$

$$\sqrt{ch\xi - \cos(2u_0-u)} = \sqrt{2}\,|\sin u_0|\sqrt{1 - 2\varepsilon\cot u_0} = \sqrt{2}\,|\sin u_0|\,(1 - \varepsilon\cot u_0)$$

the above agrees with my old v3 doc

$$1/\sqrt{ch\xi - \cos(2u_0-u)} = 1/\left[\sqrt{2}\,|\sin u_0|\,(1 - \varepsilon\cot u_0)\right] = (1+\varepsilon\cot u_0)/(\sqrt{2}\,|\sin u_0|)$$

$$\cos(u_0 - u/2) = \cos(u_0 - \pi - \varepsilon/2) = -\cos(u_0-\varepsilon/2)$$

$$\approx -[\cos(u_0) + (\varepsilon/2)\sin u_0]$$

So here is a **summary** of the above results

$\cos(u/2) = -1 + \varepsilon^2/8$ ok

$\cos(u) = 1 - \varepsilon^2/2$ ok

$ch\xi - \cos u = (1/2)(f^2+1)\varepsilon^2$ ok

$\sqrt{ch\xi - \cos u} = (1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon$ ok

$\cos(2u_0-u) = \cos(2u_0) + \varepsilon\sin(2u_0)$ ok

$1/\sqrt{ch\xi - \cos(2u_0-u)} = (1+\varepsilon\cot u_0)/(\sqrt{2}\,|\sin u_0|)$ ok

$\cos(u_0 - u/2) = -[\cos(u_0) + (\varepsilon/2)\sin u_0]$. ok

in v3 doc I get $|\sqrt{2}\cos(u_0 - u/2)| = |\sqrt{1+\cos(2u_0-u)}| = \sqrt{2}\,|\cos(u_0)|$

so this does not confirm my sign here.

Now write again

$$V(\xi,u) = (V_0/\pi)\left\{\pi - \underbrace{\cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}}\right]}_{R1} + \underbrace{\frac{\sqrt{ch\xi - \cos u}}{\sqrt{ch\xi-\cos(2u_0- u)}}}_{R2}\ \underbrace{\cot^{-1}\left[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0- u)}}\right]}_{R3}\right\} \quad (1)$$

$$R1 = \frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}} = \frac{\sqrt{2}\,(-1+\varepsilon^2/8)}{(1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon} = \frac{2(-1+\varepsilon^2/8)}{\sqrt{f^2+1}\,\varepsilon} = -2/[\sqrt{f^2+1}\,\varepsilon] + (1/4\sqrt{f^2+1})\,\varepsilon$$ ok

$$R2 = \frac{\sqrt{ch\xi - \cos u}}{\sqrt{ch\xi-\cos(2u_0- u)}} = (1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon * (1+\varepsilon\cot u_0)/(\sqrt{2}\,|\sin u_0|)$$

$$\approx (1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon\,/\,(\sqrt{2}\,|\sin u_0|) + \text{order}(\varepsilon^2)$$ ok

$$R3 = \frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0-u)}} = -\sqrt{2}\ [\cos(u_0) + (\varepsilon/2)\sin u_0] * (1+\varepsilon\cot u_0)/(\sqrt{2}\,|\sin u_0|)$$

$$= -\ [\cos(u_0) + (\varepsilon/2)\sin u_0]\ [(1+\varepsilon\cot u_0]/|\sin u_0|$$

$$= -\ \mathrm{sign}(\sin u_0)\ [\cot(u_0) + (\varepsilon/2)]\ [1+\varepsilon\cot u_0]$$

$$= -\ \mathrm{sign}(\sin u_0)\ \{\cot u_0 + \varepsilon\ [\ 1/2 + \cot^2 u_0]\} + \mathrm{order}(\varepsilon^2) \qquad \text{ok}$$

To summarize:

$$R1 = -\ 2/[\sqrt{f^2+1}\ \varepsilon] + (1/4\sqrt{f^2+1}\ )\ \varepsilon \qquad \text{ok}$$

$$R2 = (1/\sqrt{2}\ )\ \sqrt{f^2+1}\ \varepsilon\ /\ (\sqrt{2}\ |\sin u_0|) \qquad \text{ok}$$

$$R3 = -\ \mathrm{sign}(\sin u_0)\ \{\cot u_0 + \varepsilon\ [\ 1/2 + \cot^2 u_0]\} \qquad \text{ok}$$

The potential is then

$$V(\xi,u) = (V_0/\pi)\ \{\ \pi -\ \cot^{-1}[R1]\ +\ R2\ \cot^{-1}[R3]\ \}$$

So let's now look at the individual terms. First

$$\pi -\ \cot^{-1}[R1] = \cot^{-1}[-R1]\ = \cot^{-1}\ [\ 2/[\sqrt{f^2+1}\ \varepsilon] - (1/4\sqrt{f^2+1}\ )\ \varepsilon]$$

As $\varepsilon\to 0$, the first term becomes $+\infty$ and the second term does not matter at all, So

$$\pi -\ \cot^{-1}[R1] = \cot^{-1}(+\infty)\ = 0 \qquad \text{that term is completely gone!!!\ (wrong!!!!)}$$

STOP!!! Consider this alternative evaluation of the first term!

$$\pi -\ \cot^{-1}[R1] = \cot^{-1}[-R1]\ = \cot^{-1}\ [\ 2/[\sqrt{f^2+1}\ \varepsilon] - (1/4\sqrt{f^2+1}\ )\ \varepsilon]$$

$$\approx\ \cot^{-1}2/[\sqrt{f^2+1}\ \varepsilon]\ \approx \tan^{-1}\ (\sqrt{f^2+1}\ \varepsilon\ /2)\ \approx\ \sqrt{f^2+1}\ \varepsilon\ /2$$

and there suddenly we have a new linear ε term that was not there before! So I will add this "new term" in red below.

Next, consider

$$\cot^{-1}[R3] = \cot^{-1}[-\ \mathrm{sign}(\sin u_0)\ \{\cot u_0 + \varepsilon\ [\ 1/2 + \cot^2 u_0]\}\ ]$$

Let's now assume that $0 \le u_0 < \pi$. This is the normal range to describe any "upper bowl" over the full range from full sphere to disk! Then $\mathrm{sign}(\sin u_0) = +1$ and we have

$$\cot^{-1}[R3] = \cot^{-1}[- \{ \text{cotu}_0 + \varepsilon [1/2 + \cot^2 u_0]\}]$$

$$= \pi - \cot^{-1}[\{ \text{cotu}_0 + \varepsilon [1/2 + \cot^2 u_0]\}]$$

There is a correction term here. We have

$$\cot^{-1}(a + b\varepsilon) = \cot^{-1}(a) + b\varepsilon [-1/(1+a^2)]$$

Now set $a = \text{cotu}_0$ and $b = (1/2 + \cot^2 u_0)$ // did a check on this, it seems right

so then

$$\cot^{-1}[R3] = \pi - \cot^{-1}[\{ \text{cotu}_0 + \varepsilon [1/2 + \cot^2 u_0]\}]$$

$$= \pi - \{ \cot^{-1}(\text{cotu}_0) + (1/2 + \cot^2 u_0) (-1/(1+\cot^2 u_0)) \varepsilon \}$$

OK, I computed this extra term, but I can see that it is not going to do anything, so we have

$$\cot^{-1}[R3] = (\pi - u_0)$$

Next we have

$$R2 \cot^{-1}[R3] = R2 (\pi - u_0) = (1/\sqrt{2}) \sqrt{f^2+1}\, \varepsilon / (\sqrt{2} |\text{sinu}_0|) * (\pi - u_0)$$

So my final result is

$$V(\xi,u) = (V_0/\pi) \{ \pi - \cot^{-1}[R1] + R2 \cot^{-1}[R3] \}$$

$$= (V_0/\pi) \{ 0 + \sqrt{f^2+1}\, \varepsilon /2 + (1/\sqrt{2}) \sqrt{f^2+1}\, \varepsilon / (\sqrt{2} |\text{sinu}_0|) * (\pi - u_0) \}$$

$$= (V_0/\pi) \{ \sqrt{f^2+1}\, \varepsilon /2 + \sqrt{f^2+1}\, (\varepsilon/2) / (|\text{sinu}_0|) * (\pi - u_0) \}$$

doing corrected from here on, absorbing the red correction term :

$$= (V_0/\pi) \sqrt{f^2+1}\, \varepsilon /2 \{ 1 + 1 / (|\text{sinu}_0|) * (\pi - u_0) \}$$

$$= (V_0/\pi) \sqrt{f^2+1}\, \varepsilon /2 \{ |\text{sinu}_0| + (\pi - u_0) \} / |\text{sinu}_0|$$

$$= (V_0/\pi) [\sqrt{f^2+1}\, (\varepsilon /2a)] \{ |\text{sinu}_0| + (\pi - u_0) \} * (a/|\text{sinu}_0|)$$

$$= (V_0/\pi) [1/r] \{ |\text{sinu}_0| + (\pi - u_0) \} * R$$

$$= (RV_0/\pi) [1/r] \{ |\text{sinu}_0| + (\pi - u_0) \}$$

So my conclusion is that

$$V(\xi,u) = (RV_0/\pi)\{\ (\pi-u_0) + |\sin u_0|\ \}\ \ *(1/r)$$

This is the correct answer now that I have the missing term.

To get the correct answer, note that

$$C = (R/\pi)\ \{\ (\pi- u_0) + \sin(u_0)\ \}$$

$$Q = CV_0 \qquad V = Q/r\ =\ (CV_0)/r\ =\ (V_0R/\pi)\ \{\ (\pi- u_0) + \sin(u_0)\ \}/\ r$$

whereas I got

$$(RV_0/\pi)\{\ (\pi-u_0) + |\sin u_0|\ \}\ \ *(1/r)$$

So as in my last attempt, I get the first term but I have somehow lost the second term!!!

Note: I see looking back at "capacitance of the toroidal bowl v1.doc" that I had this same missing term problem there on 1/25/11. I penciled in the missing term in red at that time, but I don't seem to show where it should come from! I state in the intro that I never resolved the mystery.

Is a resolution in the v2.doc or v3.doc of this series? Well, I try this again in v2.doc and there I get a different kind of wrong answer! There I am just missing a π. Don't think the secret is there. In the version 3 I actually get it right for the first time. So what IS the secret here??? I will have to study that document and compare to this.

3. Where is my missing term???

I have now later found the missing term. So how did I miss it below in my careful "analysis"?

The missing term is this

$$\Delta V = (V_0R/\pi)\sin(u_0)\ /\ r = (V_0/\pi)a\ /\ r\ = (V_0/\pi)a\ \varepsilon\ \sqrt{f^2\ +1}\ \ /\ 2a$$

$$= (V_0/\pi)\ (\varepsilon/2)\ \sqrt{f^2\ +1}$$

Where might this come from?? I have

$$V(\xi,u) = (V_0/\pi)\ \ \{\ \pi\ -\ \cot^{-1}[R1]\ \ +\ R2\ \cot^{-1}[R3]\ \ \}$$

So let's start the search:

Fact: The following missing term must be hiding somewhere

$\pi - \cot^{-1}[R1] + R2\cot^{-1}[R3]$ must include hidden term $(\varepsilon/2)\sqrt{f^2+1}$

This is a linear term in ε, so make sure you don't throw out linear terms! Recall

$R1 = -2/[\sqrt{f^2+1}\,](1/\varepsilon) + (1/4\sqrt{f^2+1}\,)\,\varepsilon + \text{order}(\varepsilon^3)$ < 0 always!!!!

$R2 = (1/\sqrt{2}\,)\sqrt{f^2+1}\,\varepsilon / (\sqrt{2}\,|\sin u_0|) + \text{order}(\varepsilon^2)$

$R3 = -\{\cot u_0 + \varepsilon[\,1/2 + \cot^2 u_0]\} + \text{order}(\varepsilon^2)$

Warning: $\cot u_0 > 0$ only for $0 \le u_0 < \pi/2$, so I will just assume that for now.

Write this as

$R1 = -A/\varepsilon + B\varepsilon$	$A,B > 0$	$A = 2/[\sqrt{f^2+1}\,]$, $B = (1/4\sqrt{f^2+1}\,)$
$R2 = C\varepsilon$	$C > 0$	$C = \sqrt{f^2+1}\,/(2\lvert\sin u_0\rvert)$
$R3 = -D - \varepsilon E$	$D,E > 0$	$D = \cot u_0$, $E = [\,1/2 + \cot^2 u_0]$

Then we have

$\pi - \cot^{-1}[R1] + R2\cot^{-1}[R3]$

$= \pi - \cot^{-1}[-A/\varepsilon + B\varepsilon] + C\varepsilon\cot^{-1}[-D - \varepsilon E]$

$= \cot^{-1}[A/\varepsilon - B\varepsilon] + C\varepsilon\cot^{-1}[-D - \varepsilon E]$

The missing term is in one of these two terms, or maybe in both! We know that

$\partial_x \cot^{-1}(x) = -1(1+x^2)$

$\cot^{-1}(x+\alpha) = \cot^{-1}(x) - \alpha/(1+x^2)$ α = small

For the first term, let $x = A/\varepsilon$ and $\alpha = -B\varepsilon$. Then

$\cot^{-1}[A/\varepsilon - B\varepsilon] = \cot^{-1}(A/\varepsilon) + B\varepsilon/(1+(A/\varepsilon)^2)$

$= \cot^{-1}(A/\varepsilon) + B\,\varepsilon^3/(\varepsilon^2+A^2) \approx \cot^{-1}(A/\varepsilon) + B\,\varepsilon^3/A^2$ // this is all OK

Now as $\varepsilon \to 0$, the first term is zero, and the correction term is order ε^3!!! [wrong!!!] That cannot be it. Assuming I made no algebra mistakes prior.

Here is what you then failed to do

$\cot^{-1}(A/\varepsilon) = \tan^{-1}(\varepsilon/A) \approx \varepsilon/A$ and there is the missing linear term!!!

Fact: Them missing term is NOT in the first term of V.

Now for the second term

$\cot^{-1}[-D - \varepsilon E] = \pi - \cot^{-1}[D + \varepsilon E]$

$= \pi - \{ \cot^{-1}(D) - \varepsilon E/(1+D^2) \}$

$= \pi - \cot^{-1}(D) - \varepsilon E/(1+D^2)$ $\quad D = \cot u_0$

$= \pi - u_0 - \varepsilon [1/2 + \cot^2 u_0] / (1+\cot^2 u_0) = \pi - u_0 - \varepsilon Q$

$= \pi - u_0 - \varepsilon [1/2 + \cot^2 u_0] * \sin^2 u_0$

I am suspicious of the 1/2, but I checked it and could not find anything wrong with it. So here at least is an "extra term" of sorts. We then have

$R2\ \cot^{-1}[R3] = C\varepsilon[\pi - u_0 - \varepsilon Q]$

$= C\varepsilon(\pi - u_0) + K\varepsilon^2$

But an ε^2 term in the second term cannot supply my missing term which has to be linear in ε.

Fact: Them missing term is NOT in the scond term of V.

So for the fourth time, I cannot see where this term is lurking!

The missing term within {.....} must have the form $K\varepsilon$, that is, it must be linear in ε.

Conclusion: There must be an algebra error earlier on of a major sort.

Comment: I don't really know that my $V(\xi,u)$ is correct. I know that in Maple it gives what looks like a 1/r drop off, but I don't know if the constant is correct or not.

So do I have any *confirmation* on any of my potential forms??

$$V(\xi,u) = V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ \frac{ch[(\pi - u_0)\tau]\ ch[(\pi + u_0 - u)\tau]}{ch^2(\pi\tau)} \quad . \qquad (2.4.13)$$

$$V(\xi,u) = (V_0/\pi)\ \{ \cot^{-1}[\frac{\sqrt{2}\cos(\pi-u/2)}{\sqrt{ch\xi-cosu}}] + \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi-cos(2u_0- u)}}\ \cot^{-1}[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-cos(2u_0- u)}}]\ \}$$

I have no Lebedev result to look at. Check his book again. No, it is not there. He has a section on toroidal coordinates. How about Smythe? No, he gets the capacitance.

I claim that if I use the first form above I get the right capacitance. Maybe I should process that section now to make sure my claim is correct!