Finding the large r behavior of the bowl potential v2 PhL 12.5.15

This is incorporated into Section 2.5 item 3 of bowl doc, all is well.

The potential is this

$$V(\xi,u) = (V_0/\pi)\ \{\ \pi - \cot^{-1}[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - cosu}}]\ +\ \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi - cos(2u_0 - u)}}\ \cot^{-1}[\frac{\sqrt{2}\cos(u_0 - u/2)}{\sqrt{ch\xi - cos(2u_0 - u)}}]\ \}\quad (1)$$

In the large r region we know from (1.3.16) that

$$\sqrt{ch\xi - cosu}\ \approx \sqrt{2}\ a/r$$

$$\xi << 1$$

$$u = 2\pi + \varepsilon \qquad \text{with } |\varepsilon| << 1$$

For our purposes, we can approximate many expressions in the above potential

$$\cos(u/2) \approx \cos(2\pi/2) = \cos(\pi) = -1$$

$$\cos(2u_0 - u) \approx \cos(2u_0 - 2\pi) = \cos(2u_0)$$

$$ch\xi - \cos(2u_0 - u)\ \approx 1 - \cos(2u_0) = 2\sin^2 u_0$$

$$\sqrt{ch\xi - \cos(2u_0 - u)}\ = \sqrt{2}\ \ |\sin u_0|$$

$$\cos(u_0 - u/2) \approx \cos(u_0 - \pi) = -\cos u_0$$

Then the ratios appearing in *** may be approximated as $u_0 = \beta\ \ \pi - u_0$

$$\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi - cosu}}\ \approx\ \frac{-\sqrt{2}}{\sqrt{2}\ a/r}\ = -\ r/a$$

$$\frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi - cos(2u_0 - u)}}\ \approx\ \frac{\sqrt{2}\ a/r}{\sqrt{2}\ |\sin u_0|}\ = R/r \qquad R = a\ |\sin u_0|$$

$$\frac{\sqrt{2}\cos(u_0 - u/2)}{\sqrt{ch\xi - cos(2u_0 - u)}}\ \approx \frac{-\sqrt{2}\cos u_0}{\sqrt{2}\ |\sin u_0|}\ = -\ \sigma\ \cot u_0 \qquad \sigma \equiv \text{sign}(\sin u_0)$$

The resulting potential is then

$$V(\xi,u) = (V_0/\pi)\ \{\ \pi - \cot^{-1}[-\ r/a]\ +\ (R/r)\ \cot^{-1}[\ -\ \sigma\ \cot u_0]\ \}$$

The first two terms are

$$\pi - \cot^{-1}[-r/a] = \cot^{-1}[r/a] = \tan^{-1}[a/r] \approx a/r = R\,|\sin u_0|\,(1/r)$$

The last term depends on σ.

$\sigma = -1$: $\cot^{-1}[\cot u_0] = u_0$

$\sigma = +1$: $\cot^{-1}[-\cot u_0] = \pi - \cot^{-1}[\cot u_0] = \pi - u_0$

Taking the case $\sigma = +1$ which means $0 \le u_0 \le \pi$ we find

$$V(\xi,u) = (V_0/\pi)\ \{\ R\,|\sin u_0|\,(1/r) + (R/r)(\pi - u_0)\ \}$$

$$= (V_0 R/\pi)\ \{\ \sin u_0 + (\pi - u_0)\ \}\ (1/r)$$

As shown in the upper part of Fig (1.3.18), in this case one has bowl angle $\beta = \pi - u_0$ and so

$$V(\xi,u) = (V_0 R/\pi)\ \{\ \sin\beta + \beta\ \}\ (1/r)$$

For $\sigma = -1$ the potential is instead

$$V(\xi,u) = (V_0/\pi)\ \{\ R\,|\sin u_0|\,(1/r) + (R/r)\,u_0\ \}$$

$$= (V_0 R/\pi)\ \{\ |\sin u_0| + u_0\ \}$$

As shown in the lower part of Fig (1.3.18), in this case one has bowl angle $\beta = u_0$ and so again,

$$V(\xi,u) = (V_0 R/\pi)\ \{\ \sin\beta + \beta\ \}\ (1/r)$$

Writing $V = Q/r = CV_0/r$ one reads off the bowl capacitance,

$$C = (R/\pi)\ \{\ \sin\beta + \beta\ \}$$

1. Find region where r is large

Consider

$$r = a\sqrt{sh^2\xi + \sin^2 u} \,/\, (ch\xi - \cos u) \qquad (2)$$

To get large r, we must have small ξ and cosu close to 1. This happens in range near $u = 2\pi$.

Let $u = 2\pi+\varepsilon$ so $\cos u = \cos\varepsilon \approx 1-\varepsilon^2/2$. Also $\sin u = \sin\varepsilon \approx \varepsilon$

$$\text{numer} = a\sqrt{sh^2\xi + \sin^2 u} = a\sqrt{\xi^2 + \varepsilon^2}$$

$$\text{denom} = ch\xi - \cos u = 1+\xi^2/2 - [\,1-\varepsilon^2/2] = (\xi^2 + \varepsilon^2)/2$$

Thus

$$r = a\sqrt{sh^2\xi + \sin^2 u} \,/\, (ch\xi - \cos u) = a\sqrt{\xi^2 + \varepsilon^2} \,/\, [(\xi^2 + \varepsilon^2)/2]$$

$$= 2a / \sqrt{\xi^2 + \varepsilon^2}$$

Now note that

$z/\rho = \sin u/sh\xi = \varepsilon/\xi \equiv 1/f$ could be any size!!!

$\xi = f\varepsilon$

Then $\sqrt{\xi^2 + \varepsilon^2} = \sqrt{f^2\varepsilon^2 + \varepsilon^2} = \sqrt{f^2 + 1}\,\varepsilon$

so that

$$r = 2a / \sqrt{\xi^2 + \varepsilon^2} = [2a/\sqrt{f^2 + 1}\,]\,(1/\varepsilon) \qquad (3)$$

$$1/r = \varepsilon\sqrt{f^2 + 1} \,/\, 2a \qquad (4)$$

Thus, r is large where ε is small.

2. Evaluate the potential

Now go through things as in the previous paradox document:

$$V(\xi,u) = (V_0/\pi)\ \{\,\pi - \underbrace{\cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}}\right]}_{R1} + \underbrace{\frac{\sqrt{ch\xi - \cos u}}{\sqrt{ch\xi-\cos(2u_0- u)}}\ \cot^{-1}\left[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0- u)}}\right]}_{R2\ \text{and}\ R3}\ \} \quad (1)$$

$\cos(u/2) = \cos(\pi+\varepsilon/2) = \cos(\varepsilon/2+\pi) = -\cos(\varepsilon/2) \approx -[\,1 - (\varepsilon/2)^2/2] = \varepsilon^2/8 - 1$

$\cos(u) = \cos(2\pi+\varepsilon) = \cos\varepsilon \approx 1 - \varepsilon^2/2$

$ch\xi - \cos u = (1 + \xi^2/2 - 1 + \varepsilon^2/2) = (\xi^2+\varepsilon^2)/2 = (f^2\varepsilon^2+\varepsilon^2)/2 = (1/2)(f^2+1)\varepsilon^2$

$\sqrt{ch\xi - \cos u} \approx (1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon$

$1/\sqrt{ch\xi - \cos u} = 1/[\,(1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon\,] = \sqrt{2}/(\sqrt{f^2+1}\,\varepsilon)$

$\cos(2u_0-u) = \cos(2u_0-[2\pi+\varepsilon]) = \cos(2u_0-\varepsilon-2\pi) = \cos(2u_0-\varepsilon)$

$\approx \cos(2u_0) - \varepsilon[\,-\sin(2u_0)\,] = \cos(2u_0) + \varepsilon\sin(2u_0)$

$ch\xi - \cos(2u_0-u) = 1+\xi^2/2 - \cos(2u_0-\varepsilon) = \xi^2/2 + [1-\cos(2u_0-\varepsilon)] = \xi^2/2 + 2\sin^2(u_0-\varepsilon/2)$

but $\sin(u_0-\varepsilon/2) \approx \sin(u_0) - (\varepsilon/2)\cos(u_0)$

$\sin^2(u_0-\varepsilon/2) \approx \sin^2(u_0) - \varepsilon\sin u_0\cos u_0$

Therefore

$ch\xi - \cos(2u_0-u) \approx \xi^2/2 + 2\sin^2(u_0-\varepsilon/2) = (f^2/2)\varepsilon^2 + 2[\,\sin^2(u_0) - \varepsilon\sin u_0\cos u_0]$

$= 2\sin^2u_0 - 2\varepsilon\sin u_0\cos u_0 + \text{order}(\varepsilon^2)$ which I will ignore

$\approx 2\sin^2u_0 - 2\varepsilon\sin u_0\cos u_0$

$= 2\sin u_0(\sin u_0 - 2\varepsilon\cos u_0) = 2\sin^2u_0 - 4\varepsilon\sin u_0\cos u_0 = 2\sin^2u_0[\,1 - 2\varepsilon\cot u_0]$

$[ch\xi - \cos(2u_0-u)]^{-1} = [\,2\sin^2u_0[\,1 - 2\varepsilon\cot u_0]\,]^{-1} = (1 + 2\varepsilon\cot u_0)/(2\sin^2u_0)$

$1/\sqrt{ch\xi - \cos(2u_0-u)} = (1 + \varepsilon\cot u_0)/(\sqrt{2}\,|\sin u_0|)$

$\sqrt{ch\xi - \cos(2u_0-u)} = \sqrt{2}\,|\sin u_0|\sqrt{1 - 2\varepsilon\cot u_0} = \sqrt{2}\,|\sin u_0|(1 - \varepsilon\cot u_0)$

the above agrees with my old v3 doc

$1/\sqrt{ch\xi - \cos(2u_0-u)} = 1/[\sqrt{2}\,|\sin u_0|(1 - \varepsilon\cot u_0)] = (1+\varepsilon\cot u_0)/(\sqrt{2}\,|\sin u_0|)$

$\cos(u_0-u/2) = \cos(u_0 - \pi - \varepsilon/2) = -\cos(u_0-\varepsilon/2)$

$\approx -[\,\cos(u_0) + (\varepsilon/2)\sin u_0]$

So here is a **summary** of the above results

$\cos(u/2) = -1 + \varepsilon^2/8$ ok

$\cos(u) = 1 - \varepsilon^2/2$ ok

$ch\xi - \cos u = (1/2)(f^2+1)\varepsilon^2$ ok

$\sqrt{ch\xi - \cos u} = (1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon$ ok

$\cos(2u_0-u) = \cos(2u_0) + \varepsilon \sin(2u_0)$ ok

$1/\sqrt{ch\xi - \cos(2u_0-u)} = (1+\varepsilon \cot u_0)/(\sqrt{2}\,|\sin u_0|)$ ok

$\cos(u_0 - u/2) = -[\cos(u_0) + (\varepsilon/2)\sin u_0]$. ok

in v3 doc I get $|\sqrt{2}\cos(u_0 - u/2)| = |\sqrt{1+\cos(2u_0-u)}| = \sqrt{2}\,|\cos(u_0)|$

so this does not confirm my sign here.

Now write again

$$V(\xi,u) = (V_0/\pi)\ \{\pi - \underbrace{\cot^{-1}\left[\frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}}\right]}_{R1} + \underbrace{\frac{\sqrt{ch\xi - \cos u}}{\sqrt{ch\xi-\cos(2u_0-u)}}\ \cot^{-1}\left[\frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0-u)}}\right]}_{R2\ \ R3}\ \}\quad (1)$$

$$R1 = \frac{\sqrt{2}\cos(u/2)}{\sqrt{ch\xi-\cos u}} = \frac{\sqrt{2}\,(-1+\varepsilon^2/8)}{(1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon} = \frac{2(-1+\varepsilon^2/8)}{\sqrt{f^2+1}\,\varepsilon} = -2/[\sqrt{f^2+1}\,\varepsilon] + (1/4\sqrt{f^2+1})\,\varepsilon$$ ok

$$R2 = \frac{\sqrt{ch\xi - \cos u}}{\sqrt{ch\xi-\cos(2u_0-u)}} = (1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon\ *\ (1+\varepsilon\cot u_0)/(\sqrt{2}\,|\sin u_0|)$$

$$\approx (1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon\,/\,(\sqrt{2}\,|\sin u_0|) + \text{order}(\varepsilon^2)$$ ok

$$R3 = \frac{\sqrt{2}\cos(u_0-u/2)}{\sqrt{ch\xi-\cos(2u_0-u)}} = -\sqrt{2}\ [\cos(u_0) + (\varepsilon/2)\sin u_0] * (1+\varepsilon\cot u_0)/(\sqrt{2}\,|\sin u_0|)$$

$$= -[\cos(u_0) + (\varepsilon/2)\sin u_0]\,[(1+\varepsilon\cot u_0]/|\sin u_0|$$

$$= -\,\text{sign}(\sin u_0)\,[\cot(u_0) + (\varepsilon/2)]\,[1+\varepsilon\cot u_0]$$

$$= -\,\text{sign}(\sin u_0)\,\{\cot u_0 + \varepsilon\,[1/2 + \cot^2 u_0]\} + \text{order}(\varepsilon^2)$$ ok

To summarize:

$R1 = -\,2/[\sqrt{f^2+1}\,\varepsilon] + (1/4\sqrt{f^2+1}\,)\,\varepsilon$ ok

$R2 = (1/\sqrt{2}\,)\,\sqrt{f^2+1}\,\varepsilon\,/\,(\sqrt{2}\,|\sin u_0|)$ ok

$R3 = -\,\text{sign}(\sin u_0)\,\{\cot u_0 + \varepsilon\,[\,1/2 + \cot^2 u_0]\}$ ok

The potential is then

$$V(\xi,u) = (V_0/\pi)\;\{\,\pi - \cot^{-1}[R1] \;+\; R2\,\cot^{-1}[R3]\;\}$$

So let's now look at the individual terms. First

$$\pi - \cot^{-1}[R1] = \cot^{-1}[-R1] = \cot^{-1}\,[\,2/[\sqrt{f^2+1}\,\varepsilon] - (1/4\sqrt{f^2+1}\,)\,\varepsilon]$$

As $\varepsilon \to 0$, the first term becomes $+\infty$ and the second term does not matter at all, So

$\pi - \cot^{-1}[R1] = \cot^{-1}(+\infty) = 0$ that term is completely gone!!! (wrong!!!!)

STOP!!! Consider this alternative evaluation of the first term!

$$\pi - \cot^{-1}[R1] = \cot^{-1}[-R1] = \cot^{-1}\,[\,2/[\sqrt{f^2+1}\,\varepsilon] - (1/4\sqrt{f^2+1}\,)\,\varepsilon]$$

$$\approx \cot^{-1} 2/[\sqrt{f^2+1}\,\varepsilon] \approx \tan^{-1}\,(\sqrt{f^2+1}\,\varepsilon/2) \approx \sqrt{f^2+1}\,\varepsilon/2$$

and there suddenly we have a new linear ε term that was not there before! So I will add this "new term" in red below.

Next, consider

$$\cot^{-1}[R3] = \cot^{-1}[-\,\text{sign}(\sin u_0)\,\{\cot u_0 + \varepsilon\,[\,1/2 + \cot^2 u_0]\}\,]$$

Let's now assume that $0 \le u_0 < \pi$. This is the normal range to describe any "upper bowl" over the full range from full sphere to disk! Then $\text{sign}(\sin u_0) = +1$ and we have

$$\cot^{-1}[R3] = \cot^{-1}[-\,\{\cot u_0 + \varepsilon\,[\,1/2 + \cot^2 u_0]\}\,]$$

$$= \pi - \cot^{-1}[\,\{\cot u_0 + \varepsilon\,[\,1/2 + \cot^2 u_0]\}\,]$$

There is a correction term here. We have

$$\cot^{-1}(a + b\varepsilon) = \cot^{-1}(a) + b\varepsilon\,[\,-1/(1+a^2)\,]$$

Now set $a = \cot u_0$ and $b = (1/2 + \cot^2 u_0)$ // did a check on this, it seems right

so then

$$\cot^{-1}[R3] = \pi - \cot^{-1}[\{ \cot u_0 + \varepsilon [1/2 + \cot^2 u_0]\}]$$

$$= \pi - \{ \cot^{-1}(\cot u_0) + (1/2 + \cot^2 u_0) (-1/(1+\cot^2 u_0)) \varepsilon \}$$

OK, I computed this extra term, but I can see that it is not going to do anything, so we have

$$\cot^{-1}[R3] = (\pi - u_0)$$

Next we have

$$R2 \cot^{-1}[R3] = R2 (\pi - u_0) = (1/\sqrt{2}) \sqrt{f^2+1}\, \varepsilon / (\sqrt{2} |\sin u_0|) * (\pi - u_0)$$

So my final result is

$$V(\xi,u) = (V_0/\pi) \{ \pi - \cot^{-1}[R1] + R2 \cot^{-1}[R3] \}$$

$$= (V_0/\pi) \{ 0 + \sqrt{f^2+1}\, \varepsilon /2 + (1/\sqrt{2}) \sqrt{f^2+1}\, \varepsilon / (\sqrt{2} |\sin u_0|) * (\pi - u_0) \}$$

$$= (V_0/\pi) \{ \sqrt{f^2+1}\, \varepsilon /2 + \sqrt{f^2+1}\, (\varepsilon/2) / (|\sin u_0|) * (\pi - u_0) \}$$

doing corrected from here on, absorbing the red correction term :

$$= (V_0/\pi) \sqrt{f^2+1}\, \varepsilon /2 \{ 1 + 1 / (|\sin u_0|) * (\pi - u_0) \}$$

$$= (V_0/\pi) \sqrt{f^2+1}\, \varepsilon /2 \{ |\sin u_0| + (\pi - u_0) \}/ |\sin u_0|$$

$$= (V_0/\pi) [\sqrt{f^2+1}\, (\varepsilon /2a)] \{ |\sin u_0| + (\pi - u_0) \} * (a/|\sin u_0|)$$

$$= (V_0/\pi) [1/r] \{ |\sin u_0| + (\pi - u_0) \} * R$$

$$= (RV_0/\pi) [1/r] \{ |\sin u_0| + (\pi - u_0) \}$$

So my conclusion is that

$$V(\xi,u) = (RV_0/\pi)\{ (\pi-u_0) + |\sin u_0| \} *(1/r)$$

This is the correct answer now that I have the missing term.

To get the correct answer, note that

$$C = (R/\pi) \{ (\pi- u_0) + \sin(u_0) \}$$

$Q = CV_0 \quad V = Q/r = (CV_0)/r = (V_0R/\pi)\{(\pi - u_0) + \sin(u_0)\}/r$

whereas I got

$$(RV_0/\pi)\{(\pi - u_0) + |\sin u_0|\} * (1/r)$$

So as in my last attempt, I get the first term but I have somehow lost the second term!!!

Note: I see looking back at "capacitance of the toroidal bowl v1.doc" that I had this same missing term problem there on 1/25/11. I penciled in the missing term in red at that time, but I don't seem to show where it should come from! I state in the intro that I never resolved the mystery.

Is a resolution in the v2.doc or v3.doc of this series? Well, I try this again in v2.doc and there I get a different kind of wrong answer! There I am just missing a π. Don't think the secret is there. In the version 3 I actually get it right for the first time. So what IS the secret here??? I will have to study that document and compare to this.

3. Where is my missing term???

I have now later found the missing term. So how did I miss it below in my careful "analysis"?

The missing term is this

$$\Delta V = (V_0R/\pi)\sin(u_0)/r = (V_0/\pi)a/r = (V_0/\pi)a\,\varepsilon\sqrt{f^2+1}/2a$$

$$= (V_0/\pi)(\varepsilon/2)\sqrt{f^2+1}$$

Where might this come from?? I have

$$V(\xi,u) = (V_0/\pi)\{\pi - \cot^{-1}[R1] + R2\cot^{-1}[R3]\}$$

So let's start the search:

Fact: The following missing term must be hiding somewhere

$$\pi - \cot^{-1}[R1] + R2\cot^{-1}[R3] \quad \text{must include hidden term} \quad (\varepsilon/2)\sqrt{f^2+1}$$

This is a linear term in ε, so make sure you don't throw out linear terms! Recall

$$R1 = -2/[\sqrt{f^2+1}](1/\varepsilon) + (1/4\sqrt{f^2+1})\varepsilon + \text{order}(\varepsilon^3) \qquad < 0 \text{ always!!!!}$$

$$R2 = (1/\sqrt{2})\sqrt{f^2+1}\,\varepsilon/(\sqrt{2}\,|\sin u_0|) + \text{order}(\varepsilon^2)$$

$$R3 = -\{\cot u_0 + \varepsilon[1/2 + \cot^2 u_0]\} + \text{order}(\varepsilon^2)$$

Warning: $\cot u_0 > 0$ only for $0 \le u_0 < \pi/2$, so I will just assume that for now.

Write this as

$R1 = -A/\varepsilon + B\varepsilon$	$A,B > 0$	$A = 2/[\sqrt{f^2+1}\,]$, $B = (1/4\sqrt{f^2+1}\,)$
$R2 = C\varepsilon$	$C > 0$	$C = \sqrt{f^2+1}\ / (2\lvert\sin u_0\rvert)$
$R3 = -D - \varepsilon E$	$D,E > 0$	$D = \cot u_0$, $E = [\ 1/2 + \cot^2 u_0]$

Then we have

$$\pi - \cot^{-1}[R1] + R2\cot^{-1}[R3]$$

$$= \pi - \cot^{-1}[\,-A/\varepsilon + B\varepsilon] + C\varepsilon\cot^{-1}[\,-D - \varepsilon E]$$

$$= \cot^{-1}[A/\varepsilon - B\varepsilon] + C\varepsilon\cot^{-1}[\,-D - \varepsilon E]$$

The missing term is in one of these two terms, or maybe in both! We know that

$$\partial_x \cot^{-1}(x) = -1(1+x^2)$$

$$\cot^{-1}(x+\alpha) = \cot^{-1}(x) - \alpha/(1+x^2) \qquad \alpha = \text{small}$$

For the first term, let $x = A/\varepsilon$ and $\alpha = -B\varepsilon$. Then

$$\cot^{-1}[A/\varepsilon - B\varepsilon] = \cot^{-1}(A/\varepsilon) + B\varepsilon\,/\,(1+(A/\varepsilon)^2)$$

$$= \cot^{-1}(A/\varepsilon) + B\,\varepsilon^3/(\varepsilon^2+A^2) \approx \cot^{-1}(A/\varepsilon) + B\,\varepsilon^3/A^2 \qquad \text{// this is all OK}$$

Now as $\varepsilon \to 0$, the first term is zero, and the correction term is order ε^3!!! [wrong!!!] That cannot be it. Assuming I made no algebra mistakes prior.

Here is what you then failed to do

$\cot^{-1}(A/\varepsilon) = \tan^{-1}(\varepsilon/A) \approx \varepsilon/A$ and there is the missing linear term!!!

Fact: Them missing term is NOT in the first term of V.

Now for the second term

$$\cot^{-1}[\,-D - \varepsilon E] = \pi - \cot^{-1}[\,D + \varepsilon E]$$

$= \pi - \{ \cot^{-1}(D) - \varepsilon E/(1+D^2) \}$

$= \pi - \cot^{-1}(D) - \varepsilon E/(1+D^2)$ $\qquad D = \cot u_0$

$= \pi - u_0 - \varepsilon [1/2 + \cot^2 u_0] / (1+\cot^2 u_0) = \pi - u_0 - \varepsilon Q$

$= \pi - u_0 - \varepsilon [1/2 + \cot^2 u_0] * \sin^2 u_0$

I am suspicious of the 1/2, but I checked it and could not find anything wrong with it. So here at least is an "extra term" of sorts. We then have

$R2 \cot^{-1}[R3] = C\varepsilon[\pi - u_0 - \varepsilon Q]$

$= C\varepsilon(\pi - u_0) + K\varepsilon^2$

But an ε^2 term in the second term cannot supply my missing term which has to be linear in ε.

Fact: Them missing term is NOT in the scond term of V.

So for the fourth time, I cannot see where this term is lurking!

The missing term within {.....} must have the form $K\varepsilon$, that is, it must be linear in ε.

Conclusion: There must be an algebra error earlier on of a major sort.

Comment: I don't really know that my $V(\xi,u)$ is correct. I know that in Maple it gives what looks like a 1/r drop off, but I don't know if the constant is correct or not.

So do I have any *confirmation* on any of my potential forms??

$$V(\xi,u) = V_0 \sqrt{2}\ \sqrt{ch\xi - cosu}\ \int_0^\infty d\tau\ P_{i\tau-1/2}(ch\xi)\ \frac{ch[(\pi - u_0)\tau]\ ch[(\pi + u_0 - u)\tau]}{ch^2(\pi\tau)} \quad . \qquad (2.4.13)$$

$$V(\xi,u) = (V_0/\pi)\ \{ \cot^{-1}[\frac{\sqrt{2}\ \cos(\pi - u/2)}{\sqrt{ch\xi - cosu}}]\ +\ \frac{\sqrt{ch\xi - cosu}}{\sqrt{ch\xi - \cos(2u_0 - u)}}\ \cot^{-1}[\frac{\sqrt{2}\ \cos(u_0 - u/2)}{\sqrt{ch\xi - \cos(2u_0 - u)}}]\ \}$$

I have no Lebedev result to look at. Check his book again. No, it is not there. He has a section on toroidal coordinates. How about Smythe? No, he gets the capacitance.

I claim that if I use the first form above I get the right capacitance. Maybe I should process that section now to make sure my claim is correct!