Section between ********** lines is installed, do not edit here!

The stuff below the lower ****** involves integrating things over the toroid area. It also involves doing the two integrals which now appear in Appendix H.4. // 12.31.15

****

10.5 Surface charge density on a toroid

The toroid surface charge density may be obtained from the potential in this manner,

$$\sigma = +(1/4\pi)\,(1/h_{\xi_0})\,[\partial_\xi V(\xi,u)]|^{\xi=\xi_0} \qquad 1/h_{\xi_0} = (ch\xi_0 - cosu)/a\ . \tag{10.5.1}$$

An explanation is given below (4.1.2), and the sign here is + because ξ decreases moving outward from the toroid surface. The toroid potential was found in (10.1.11) to be

$$V(\xi,u) = V_0 \frac{\sqrt{2}}{\pi} \sqrt{ch\xi - cosu}\ \Sigma_{n=0}^{\infty} \varepsilon_n P_{n-1/2}(ch\xi) \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \cos(nu)\ . \quad \textit{(10.1.11)} \tag{10.5.2}$$

$$= V_0 \frac{\sqrt{2}}{\pi} \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \cos(nu)\,[\ \sqrt{ch\xi - cosu}\ P_{n-1/2}(ch\xi)\ ]$$

Therefore (10.5.10 reads,

$$\sigma = +\frac{1}{4\pi} V \frac{\sqrt{2}}{\pi} \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \cos(nu) * \frac{1}{h_{\xi_0}} \partial_\xi\,[\ \sqrt{ch\xi - cosu}\ P_{n-1/2}(ch\xi)\ ]\,|^{\xi=\xi_0} \tag{10.5.3}$$

We need then to compute,

$$\partial_\xi\,[\ \sqrt{ch\xi - cosu}\ P_{n-1/2}(ch\xi)\ ]$$

$$= \sqrt{ch\xi - cosu}\ [\ P'_{n-1/2}(ch\xi)\ sh\xi\ ] + [(1/2)\,(1/\sqrt{ch\xi - cosu}\ ) * sh\xi\ ]\ P_{n-1/2}(ch\xi)$$

$$= sh\xi\,[\ \sqrt{ch\xi - cosu}\ P'_{n-1/2}(ch\xi) + (1/2)\,(1/\sqrt{ch\xi - cosu}\ )\ P_{n-1/2}(ch\xi)\ ]\,|^{\xi=\xi_0}$$

$$= sh\xi_0\,[\ \sqrt{ch\xi_0 - cosu}\ P'_{n-1/2}(ch\xi_0) + (1/2)(1/\sqrt{ch\xi - cosu}\ )\ P_{n-1/2}(ch\xi_0)\ ]\ , \tag{10.5.4}$$

where $P'_\nu(z)$ means $\partial_z P_\nu(z)$. Then using $1/h_{\xi_0} = (ch\xi_0 - cosu)/a$,

$$(1/h_{\xi_0})\partial_\xi\,[\ \sqrt{ch\xi - cosu}\ P_{n-1/2}(ch\xi)\ ]\,|^{\xi=\xi_0}$$

$$= (sh\xi_0/a)\,(ch\xi_0 - cosu)\ \left[\ \sqrt{ch\xi_0 - cosu}\ P'_{n-1/2}(ch\xi_0) + (1/2)(1/\sqrt{ch\xi - cosu}\ )P_{n-1/2}(ch\xi_0)\ \right] .$$

$$= (sh\xi_0/a)\ \left[\ (ch\xi_0 - cosu)^{3/2}\ P'_{n-1/2}(ch\xi_0) + (1/2)\sqrt{ch\xi - cosu}\ P_{n-1/2}(ch\xi_0)\ \right] . \qquad (10.5.5)$$

Inserting this into (10.5.3) gives

$$\sigma = +\frac{1}{4\pi}\ V_0\ \frac{\sqrt{2}}{\pi}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\ cos(nu) * (sh\xi_0/a) *$$

$$\left[\ (ch\xi_0 - cosu)^{3/2}\ P'_{n-1/2}(ch\xi_0) + (1/2)\sqrt{ch\xi - cosu}\ P_{n-1/2}(ch\xi_0)\ \right] . \qquad (10.5.6)$$

We now write this as the sum of the two terms $\sigma = \sigma_1 + \sigma_2$ where

$$\sigma_1 = +\frac{1}{4\pi}\ V_0\ \frac{\sqrt{2}}{\pi}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\ cos(nu) * (sh\xi_0/a)\ \ (ch\xi_0 - cosu)^{3/2}\ \ P'_{n-1/2}(ch\xi_0)$$

$$\sigma_2 = \ \ \frac{1}{4\pi}\ V_0\ \frac{\sqrt{2}}{\pi}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\ cos(nu) * (sh\xi_0/a) *\ \sqrt{ch\xi - cosu}\ \ * \frac{1}{2}\ P_{n-1/2}(ch\xi_0)\ ]$$

which we then reorganize to get

$$\sigma_1 = \frac{1}{4\pi}\ V_0\ \frac{\sqrt{2}}{\pi}\ \ (sh\xi_0/a)\ \ (ch\xi_0 - cosu)^{3/2}\ \ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \ P'_{n-1/2}(ch\xi_0)\ \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\ \ cos(nu)$$

$$\sigma_2 = \frac{1}{4\pi}\ V_0\ \frac{\sqrt{2}}{\pi}\ \ (sh\xi_0/a)\ \sqrt{ch\xi - cosu}\ \ \frac{1}{2}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \ Q_{n-1/2}(ch\xi_0)\ cos(nu)\ \ . \qquad (10.5.7)$$

Recall (10.1.8a) with $a = ch\xi_0$ and $b = 1$ and $x = u$,

$$1 = \sqrt{ch\xi - cosu}\ \ \frac{\sqrt{2}}{\pi}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \ Q_{n-1/2}(\ ch\xi_0)\ cos(nu)\ . \qquad (10.5.8)$$

This "sum rule" greatly simplifies σ_2 so that now,

$$\sigma_2 = \frac{1}{4\pi}\ V_0\ (sh\xi_0/a)\ \frac{1}{2}\ \ = \frac{1}{4\pi}\ V_0\ \frac{\sqrt{2}}{\pi}\ \ (sh\xi_0/a)\ \ \{\ \frac{\pi}{2\sqrt{2}}\ \ \}\ \ . \qquad (10.5.9)$$

Reconstruct the sum $\sigma = \sigma_2 + \sigma_1$ to get

$$\sigma(u;\ \xi_0)\ = \frac{1}{4\pi}\ V_0\ \frac{\sqrt{2}}{\pi}\ \ (sh\xi_0/a)$$

$$\{\ \frac{\pi}{2\sqrt{2}}\ \ +\ (ch\xi_0 - cosu)^{3/2}\ \ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ \ P'_{n-1/2}(ch\xi_0)\ \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\ \ \ cos(nu)\ \}\ .$$

Next, extract a factor $\pi/\sqrt{2}$ from {...} and replace $(sh\xi_0/a) = 1/R$ from (1.3.7) to get,

$$\sigma(u;\xi_0) = \frac{V_0}{4\pi R}\left[\frac{1}{2} + \frac{\sqrt{2}}{\pi}(ch\xi_0 - cosu)^{3/2}\sum_{n=0}^{\infty}\varepsilon_n\, P'_{n-1/2}(ch\xi_0)\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\cos(nu)\right]. \tag{10.5.10}$$

This is our final result for the surface charge density on a toroid having label ξ_0 and tube radius R held at potential V_0. We have scanned all our known sources, but cannot find verification of this result, so we shall be extra attentive to doing "checks".

First Check: The thin-wire toroid limit

In the thin wire limit the parameter ξ_0 gets large as shown in Fig (1.2.3). The distance ρ_c to the tube center line approaches a, and the tube radius R approaches 0. Since $ch\xi_0$ gets large, we invoke the large-x limits of the P and Q functions from Appendix H.

Consider first the terms in the sum in (10.5.10) which have n > 0. With $x = ch\xi_0$, one finds

$$P'_{n-1/2}(x) \rightarrow x^{n-3/2} \qquad \textit{(H.6.3)}$$

$$\frac{Q_{n-1/2}(x)}{P_{n-1/2}(x)} \rightarrow x^{-2n} \qquad \textit{(H.6.7)}$$

$$(ch\xi_0 - cosu)^{3/2} \rightarrow x^{3/2}.$$

Therefore, the nth sum term goes as

$$x^{3/2}\, x^{n-3/2}\, x^{-2n} = x^{-n}. \tag{10.5.11}$$

For large x, these terms all decay away, relative to the constant term 1/2 appearing in (10.5.10).

The n = 0 term has different behavior. Again from Appendix H,

$$P'_{-1/2}(x) \rightarrow \frac{\sqrt{2}}{\pi}x^{-3/2}[1 - (1/2)\ln(8x)] \qquad \textit{(H.6.16)}$$

$$\frac{Q_{-1/2}(x)}{P_{-1/2}(x)} \rightarrow = (\pi^2/2)\, 1/\ln(8x) \qquad \textit{(H.6.15)}$$

$$(ch\xi_0 - cosu)^{3/2} \rightarrow x^{3/2}.$$

For large $x = ch\xi_0$ the n = 0 term in (10.5.1) is then

$$\frac{\sqrt{2}}{\pi} * x^{3/2} * 1 * \frac{\sqrt{2}}{\pi} x^{-3/2} [\, 1 - (1/2)\ln(8x)] * (\pi^2/2)\, 1/\ln(8x)$$

$$= [\, 1 - (1/2)\ln(8x)]\, (1/\ln(8x)) = 1/\ln(8x) - 1/2 \qquad (10.5.12)$$

This -1/2 cancels the +1/2 appearing in (10.5.10) and we end up with

$$\sigma(u; \xi_0) \approx \frac{V_0}{4\pi R} \frac{1}{\ln(8x)} \,. \qquad (10.5.13)$$

We keep in mind that $R \to 0$ in our limit, but we maintain R for a while longer. Meanwhile,

$x = ch\xi_0 \approx sh\xi_0 = a/R$ // (1.3.7)

so

$\ln(8x) = \ln(8a/R) = \ln(a) + \ln(8/R) \approx \ln(8/R)$ as $R \to 0$

Then we find

$$\sigma(u; \xi_0) \approx \frac{V_0}{4\pi R} \frac{1}{\ln(8/R)} \qquad (10.5.14)$$

This charge density is uniform in u, as one would expect since a piece of the thin ring thinks it is a piece of straight wire with uniform σ. The curvature radius $\rho_c = a$ is huge compared to the wire radius R. Thus, to find the total charge on the toroid, we multiply σ by the area of a toroid,

$A = 4\pi^2 R\rho_c$ // toroid area

to get

$$Q = \sigma A = \frac{V_0}{4\pi R} \frac{1}{\ln(8/R)}\, 4\pi^2 R\rho_c = V_0\, \pi\rho_c\, \frac{1}{\ln(8/R)} = CV_0$$

Setting $\rho_c = 1$ as in Section 10.4 below, one gets

$$C = \frac{\pi}{\ln(8/R)} \qquad (10.5.15)$$

which agrees with (10.2.12) below as the capacitance of the toroid in the thin-wire limit. We therefore regard our σ result (10.5.10) as being correct in the thin-wire limit.

Second Check: Integrating the surface charge

This task is anything but simple and is carried out in Appendix L with support from other Appendices. The integration of course is performed directly in toroidal coordinates. We outline the main steps here.

1. Start with the charge density (10.5.10),

$$\sigma(u;\xi_0) = \frac{V_0}{4\pi R}\left[\frac{1}{2} + \frac{\sqrt{2}}{\pi}(ch\xi_0 - \cos u)^{3/2}\sum_{n=0}^{\infty}\varepsilon_n\, P'_{n-1/2}(ch\xi_0)\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\cos(nu)\right]. \qquad (L.2.1)$$

2. Integrate over the toroidal surface to get the total charge Q,

$$Q = \frac{V_0}{2R}\, a^2\, sh\xi_0\left\{\pi\,(ch\xi_0)(1/sh^3\xi_0) + \frac{4}{\pi}\sum_{n=0}^{\infty}\varepsilon_n\, P'_{n-1/2}(ch\xi_0)\frac{[Q_{n-1/2}(ch\xi_0)]^2}{P_{n-1/2}(ch\xi_0)}\right\}. \qquad (L.2.6)$$

3. Use the Wronskian $1/(1-z^2) = P_\nu(z)\,Q'_\nu(z) - P'_\nu(z)\,Q_\nu(z)$ to rewrite the above sum as

$$\sum_{n=0}^{\infty}\varepsilon_n\, P'_{n-1/2}(ch\xi_0)\frac{[Q_{n-1/2}(ch\xi_0)]^2}{P_{n-1/2}(ch\xi_0)} = (1/sh\xi_0)^2\sum_{n=0}^{\infty}\varepsilon_n\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \qquad (L.2.9)$$

$$+\sum_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(ch\xi_0)\, Q'_{n-1/2}(ch\xi_0). \qquad (L.2.10)$$

4. Evaluate the second sum using these two facts, where the second is the derivative of the first,

$$\sum_{n=0}^{\infty}\varepsilon_n\,[Q_{n-1/2}(z)]^2 = (\pi^2/2)\frac{1}{\sqrt{z^2-1}} \qquad (K.4.1)$$

$$\sum_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(z)\, Q'_{n-1/2}(z) = -(\pi^2/4)\, z\,(z^2-1)^{-3/2}. \qquad (K.4.2)$$

The QQ' sum term in item 3 exactly cancels the first term in item 2 above, giving this result

$$Q = V_0\,\frac{2a}{\pi}\sum_{n=0}^{\infty}\varepsilon_n\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \qquad (10.5.16)$$

which implies that the toroid has capacitance

$$C = \frac{2a}{\pi}\sum_{n=0}^{\infty}\varepsilon_n\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}. \qquad (10.5.17)$$

This agrees with the result (10.2.3) which was found completely independently by taking the far-away limit of the potential (10.1.11).

****

Question: Suppose I try the degenerate toroid limit here. What will happen? I have $ch\xi_0 = \rho_c/R$ and this does approach 1. We have above,

$$\sigma(u;\xi_0) = \frac{V_0}{4\pi R}\left[\frac{1}{2} + \frac{\sqrt{2}}{\pi}(ch\xi_0 - cosu)^{3/2}\sum_{n=0}^{\infty}\varepsilon_n P'_{n-1/2}(ch\xi_0)\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\cos(nu)\right].$$

I need to know about P' and so I need to know about P near 1. I recently did this somewhere.

$$P_{n-1/2}(z) \approx 1 + (1/2)[n^2 - 1/4](z-1)$$

$$P'_{n-1/2}(z) = (1/2)[n^2 - 1/4] \qquad \text{valid for all n}$$

Let's try to build the limit!

$$\sigma(u;\xi_0) = \frac{V_0}{4\pi R}\left[\frac{1}{2} + \frac{\sqrt{2}}{\pi}(1 - cosu)^{3/2}\sum_{n=0}^{\infty}\varepsilon_n P'_{n-1/2}(ch\xi_0)\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\cos(nu)\right].$$

$$\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \approx -\frac{1}{2\Gamma(n+1/2)}\ln(x-1) \qquad \text{valid for all n}$$

so we seem to get

$$\sigma(u;\xi_0) = -\frac{V_0}{4\pi R}\left[\frac{1}{2} + \frac{\sqrt{2}}{\pi}(1 - cosu)^{3/2}\sum_{n=0}^{\infty}\varepsilon_n (1/2)[n^2 - 1/4]\frac{1}{2\Gamma(n+1/2)}\ln(x-1)\cos(nu)\right].$$

$$= -\frac{V_0}{4\pi R}\left[\frac{1}{2} + \frac{\sqrt{2}}{\pi}(1 - cosu)^{3/2}(1/2)\ln(x-1)\sum_{n=0}^{\infty}\varepsilon_n [n^2 - 1/4]\frac{1}{2\Gamma(n+1/2)}\cos(nu)\right]$$

Does that n sum do anything interesting? Ask Maple. The sum approaches 8.28 and seems to converge. It does not go to 0.

So how do I explain a divergent σ in the horn toroid limit? Well at least when u = 0 we get (1 - cosu) = 0 and perhaps we can argue this kills the divergence. But for any other u, the series is finite and ln(x-1) blows up!

Well, here is my defense. The limit is not uniformly convergent $z \to 1$ here as well as in the T(z) case, so you cannot take the $z \to 1$ limit through the infinite sum, and that action is what creates the ln(x-1) problem which does not really exist. It is wrong just as saying T(1) = 0 is wrong.

Probably numerically you would find that there is some finite limit of σ at the horn toroid limit. It would be very good to plot σ for something near this case to see what happens. I have done really no plots and this needs doing tomorrow!

Not sure where to put the check that integral is right.

****

The good news: this exactly agrees with what I published in the original bowl doc!

As a check on this result, we integrate it over the toroid using this differential area,

$$dA = (h_u du)(h_\varphi d\varphi) = a/(ch\xi_0 - cosu) * a\, sh\xi_0/(ch\xi_0 - cosu) * du d\varphi$$

$$= a^2\, sh\xi_0\, (ch\xi_0 - cosu)^{-2} du d\varphi$$

Insert test: Test this to see if you can generate the toroid area!!

$$\int\int dA = 2\pi\, a^2\, sh\xi_0 \int_0^{2\pi} du\, (ch\xi_0 - cosu)^{-2}$$

$$= 2\pi a^2\, sh\xi_0\, 2\pi\, \frac{1}{ch\xi}\frac{1}{sh\xi} = 4\pi^2 a^2 / ch\xi_0$$

The result is supposed to be

$$4\pi^2 R\rho_c$$

and I know that $\rho_c = a/th\xi$ and $R_t = a/sh\xi$ so

$$4\pi^2 R\rho_c = 4\pi^2\ a/sh\xi_0 * a/th\xi_0 = 4\pi^2 a^2/$$ STOP, something is wrong.

Then

$$Q = \int\int \sigma\, dA = a^2\, sh\xi_0 \left[\int_0^{2\pi} d\varphi \right] \left[\int_0^{2\pi} du\, (ch\xi_0 - cosu)^{-2}\ \sigma(u; \xi_0) \right]$$

$$= 2\pi a^2\, sh\xi_0 \int_0^{2\pi} du\, (ch\xi_0 - cosu)^{-2}\ \sigma(u; \xi_0)$$

We are then faced with these two integrals

$$\int_0^{2\pi} du\, cos(nu)(ch\xi_0 - cosu)^{-2}$$

$$\int_0^{2\pi} du\, cos(nu)\, (ch\xi_0 - cosu)^{-1/2} = 2\sqrt{2}\, Q_{n-1/2}(ch\xi_0) \quad \text{// from (K.3.8)}$$

On scratch I show that

$$\int_0^{2\pi} du\, cos(nu)(a - cosu)^{-2}$$

This does not look good, but at least I have a tentative result for this integral.

Another integral

$$\int_0^{2\pi} du\ (z - cosu)^{-2} = 2\pi \frac{1}{ch\xi}\frac{1}{sh\xi} \quad \text{// wrong}$$

Out of steam at 8:30 PM.

Next day 12.19.15.

Let's try a simpler case. Compute the circumference of the toroid.

$ds = (h_u du) = a/(ch\xi_0 - cosu) * du$

$$circ = a \int_0^{2\pi} du\ (\alpha - cosu)^{-1} = a \int_0^{2\pi} du \frac{1}{\alpha - cosu}\frac{b}{b} \qquad \alpha = ch\xi_0 \quad b = \alpha^{-1}$$

$$= ab \int_0^{2\pi} du \frac{1}{1 - bcosu}$$

Let's go with this integral from GR7

3.613

$$1.^6 \quad \int_0^{\pi} \frac{\cos nx\, dx}{1 + a\cos x} = \frac{\pi}{\sqrt{1-a^2}}\left(\frac{\sqrt{1-a^2}-1}{a}\right)^n \qquad [a^2 < 1, \quad n \geq 0] \qquad \text{BI (64)(12)}$$

Use a = -b, n = 0 and result is

$$\int_0^{2\pi} du \frac{1}{1 - bcosu} = \frac{2\pi}{\sqrt{1-b^2}}$$

so then

$$circ = \frac{2\pi ab}{\sqrt{1-b^2}} = \frac{2\pi ab\alpha}{\sqrt{\alpha^2-1}} = \frac{2\pi a}{\sqrt{\alpha^2-1}} = 2\pi a/sh\xi_0$$

But $R = a/sh\xi_0$ so we get $2\pi R$ which is the correct answer.

Idea for area integral. Start with this integral

$$\int_0^{\pi} \frac{\cos nx\, dx}{1 + a\cos x} = \frac{\pi}{\sqrt{1-a^2}} \left(\frac{\sqrt{1-a^2} - 1}{a} \right)^n$$

Check this with Maple numeric:

a = 0.7	n = 0	ok
a = 0.7	n = 1	ok
a = - 0.7	n = 1	ok

and replace a with a = 1/b everywhere , where ab = 1. We have three pieces to compute:

$$J = \int_0^{\pi} dx \cos(nx) \frac{1}{1+a\cos x} = \int_0^{\pi} dx \cos(nx) \frac{1}{1+a\cos x} \frac{b}{b} = b \int_0^{\pi} dx \cos(nx) \frac{1}{b+\cos x}$$

$$\frac{\pi}{\sqrt{1-a^2}} \frac{b}{b} = b \frac{\pi}{\sqrt{b^2-1}}$$

$$\frac{\sqrt{1-a^2}-1}{a} \frac{b}{b} = (\sqrt{b^2-1} - b) \quad .$$

Then we know that

$$J = b \int_0^{\pi} dx \cos(nx) \frac{1}{b+\cos x} = b \frac{\pi}{\sqrt{b^2-1}} (\sqrt{b^2-1} - b)^n$$

or

$$I(b) \equiv \int_0^{\pi} dx \cos(nx) \frac{1}{b+\cos x} = \frac{\pi}{\sqrt{b^2-1}} (\sqrt{b^2-1} - b)^n \qquad b>1 \qquad b = |b|$$

I think that if b < 1, you have to replace $\sqrt{b^2-1} \rightarrow -\sqrt{b^2-1}$ if I draw a phasor diagram. Then

$$I(b) \equiv \int_0^{\pi} dx \cos(nx) \frac{1}{b+\cos x} = - \frac{\pi}{\sqrt{b^2-1}} (-\sqrt{b^2-1} - b)^n \qquad b<-1$$

$$= - \frac{\pi}{\sqrt{b^2-1}} (-1)^n \frac{\pi}{\sqrt{b^2-1}} (\sqrt{b^2-1} + b)^n$$

$$= (-1)^{n+1} \frac{\pi}{\sqrt{b^2-1}} (\sqrt{b^2-1} + b)^n \qquad b <-1 \qquad b = -|b|$$

I can combine these to get

$$I(b) = \frac{\pi}{\sqrt{b^2-1}} (\sqrt{b^2-1} - |b|)^n \qquad b > 1$$

$$I(b) = (-1)^{n+1} \frac{\pi}{\sqrt{b^2-1}} (\sqrt{b^2-1} - |b|)^n \qquad b < -1$$

Combine again to get

$$I(b) = (\text{sign } b)^{n+1} \frac{\pi}{\sqrt{b^2-1}} (\sqrt{b^2-1} - |b|)^n$$

This now checks out in Maple:

b = 1.7 n = 2
b = -1.7 n = 2
b = 1.7 n = 3
b = -1.7 n = 3 all four are now OK

Now compute

$$\partial_b I(b) = -\int_0^\pi dx \cos(nx)(b+\cos x)^{-2} = \partial_b \left[\frac{\pi}{\sqrt{b^2-1}} (\sqrt{b^2-1} - b)^n\right]$$

and this at least gives a path for computing our integral of interest. Now for n = 0 we get

$$I(b) \equiv \int_0^\pi dx \frac{1}{b+\cos x} = \frac{\pi}{\sqrt{b^2-1}}$$

$$\partial_b I(b) = -\int_0^\pi dx (b+\cos x)^{-2} = \partial_b \left[\frac{\pi}{\sqrt{b^2-1}}\right] = \pi \partial_b (b^2-1)^{-1/2}$$

$$= \pi (-1/2) (b^2-1)^{-3/2} 2b = -\pi b (b^2-1)^{-3/2}$$

and therefore we have a new integral to add to the list

$$\int_0^\pi dx (b+\cos x)^{-2} = \pi |b| (b^2-1)^{-3/2} . \qquad |b| > 1$$

Notice that both sides are positive, and that there is no pole inside the implied contour.

Application: compute the toroid area:

$$dA = (h_u du)(h_\varphi d\varphi) = a/(\text{ch}\xi_0 - \cos u) * a \,\text{sh}\xi_0/(\text{ch}\xi_0 - \cos u) * du d\varphi$$

$$= a^2 \,\text{sh}\xi_0 (\text{ch}\xi_0 - \cos u)^{-2} du d\varphi$$

$$= a^2 \,\text{sh}\xi_0 (-\text{ch}\xi_0 + \cos x)^{-2} dx d\varphi \qquad u = x$$

The dφ integral is 2π so we end up with

$$A = 2\pi\ a^2\ sh\xi_0\ 2\ \int_0^{\pi} dx\ (b+\cos x)^{-2} \qquad \text{where } b = -ch\xi_0 \qquad |b| > 1 \text{ check}$$

$$= 2\pi\ a^2\ sh\xi_0\ 2\ \pi|b|\ (b^2-1)^{-3/2}$$

$$= 4\pi^2 a^2 sh\xi_0\ b\ (b^2-1)^{-3/2} = 4\pi^2 a^2 sh\xi_0\ [ch\xi_0]\ (sh^2\xi_0)^{-3/2}$$

$$= 4\pi^2 a^2 sh\xi_0\ [ch\xi_0]\ (sh\xi_0)^{-3} = 4\pi^2 a^2\ ch\xi_0/sh^2\xi_0$$

$$= 4\pi^2\ (a/sh\xi_0)(a/th\xi_0) = 4\pi^2 R\rho_c$$

which is the correct result for toroid area. Maybe put these integrals into H.4 !

What other integrals might I need to check the total charge?

$$\sigma(u;\xi_0) = \frac{V_0}{4\pi R}\ [\ \frac{1}{2}\ +\ \frac{\sqrt{2}}{\pi}\ (ch\xi_0 - \cos u)^{3/2}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ P'_{n-1/2}(ch\xi_0)\ \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\ \cos(nu)\ ]$$

I will need two integrals. Recall from above that, after the dφ integration

$$dA = 2\pi\ a^2\ sh\xi_0\ (ch\xi_0 - \cos u)^{-2} du$$

So I will need these two integrals:

$$J1 = \int_0^{\pi} du\ \cos(nu)(ch\xi_0 - \cos u)^{-2}$$

$$J2 = \int_0^{\pi} du\ \cos(nu)(ch\xi_0 - \cos u)^{-2}(ch\xi_0 - \cos u)^{3/2}$$

$$(1/2) \int_0^{2\pi} du\ \cos(nu)\ (ch\xi_0 - \cos u)^{-1/2} = \sqrt{2}\ Q_{n-1/2}(ch\xi_0) \qquad \text{// from (K.3.8)}$$

So only the first integral remains. But I have a path to do that integral from above, where recall.

$$\partial_b I(b) = -\ \int_0^{\pi} dx\ \cos(nx)(b+\cos x)^{-2} = \partial_b\ [\frac{\pi}{\sqrt{b^2-1}}\ (\sqrt{b^2-1}\ - b)^n\ ]$$

where $b = -\ ch\xi_0$

That is to say,

$$J1 = \int_0^{\pi} du\ \cos(nu)(ch\xi_0 - \cos u)^{-2} = \int_0^{\pi} dx\ \cos(nx)(-ch\xi_0 + \cos x)^{-2}$$

$$= - \{ \partial_b [\frac{\pi}{\sqrt{b^2-1}} (\sqrt{b^2-1} - b)^n] \} |^{b = -ch\xi 0}$$

Now I need to preserve analyticity with signs. So here is what Maple says

```
f :=  (Pi/sqrt(b^2-1))*(sqrt(b^2-1) - b)^n;
```

$$f := \frac{\pi (\sqrt{b^2-1} - b)^n}{\sqrt{b^2-1}}$$

```
-diff(f,b):simplify(%);
```

$$\frac{(b + n\sqrt{b^2-1})(\sqrt{b^2-1} - b)^n \pi}{\left(\frac{3}{2}\right)}$$
$$(b^2-1)$$

But now I have to consider the two cases b > 1 and b < -1. The result is correct as is for b > 1. I think this is the result for b < - 1

$$f = (b - n\sqrt{b^2-1})(-\sqrt{b^2-1} - b)^n \pi / [- \sqrt{b^2-1}^{\,3}]$$

$$= (-1)^{n+1} (b - n\sqrt{b^2-1})(\sqrt{b^2-1} + b)^n \pi (b^2-1)^{-3/2} \qquad b < - 1$$

$$f = (b + n\sqrt{b^2-1})(\sqrt{b^2-1} - b)^n \pi (b^2-1)^{-3/2} \qquad b > 1$$

So here is side by side:

$$f = \pi (b + n\sqrt{b^2-1})(\sqrt{b^2-1} - b)^n (b^2-1)^{-3/2} \qquad b > 1 \qquad |b| = b$$

$$f = (-1)^{n+1} \pi (b - n\sqrt{b^2-1})(\sqrt{b^2-1} + b)^n (b^2-1)^{-3/2} \qquad b < - 1$$

Now rewrite the second as

$$f = \pi (b + n\sqrt{b^2-1})(\sqrt{b^2-1} - b)^n (b^2-1)^{-3/2} \qquad b > 1 \qquad |b| = b$$

$$f = (-1)^n \pi (-b + n\sqrt{b^2-1})(\sqrt{b^2-1} + b)^n (b^2-1)^{-3/2} \qquad b < - 1 \qquad |b| = -b$$

I can combine these to get

$$f = (sign(b))^n \pi (|b| + n\sqrt{b^2-1})(\sqrt{b^2-1} - |b|)^n (b^2-1)^{-3/2}$$

So here is my claim now:

$$\int_0^{\pi} dx\, \cos(nx)(b+\cos x)^{-2} = (\text{sign}(b))^n\ \pi\ (|b| + n\sqrt{b^2-1}\ )(\sqrt{b^2-1} - |b|)^n\ (b^2-1)^{-3/2}$$

Take it out for a Maple spin please:

b = 1.7	n = 2	ok
b = -1.7	n = 2	ok
b = 1.7	n = 3	ok
b = -1.7	n = 3	ok

NOW lets take b = -chξ, so sign(b) = -1. The result is then

$$\int_0^{\pi} dx\, \cos(nx)(-ch\xi+\cos x)^{-2}$$

$$= (-1)^n\ \pi\ (ch\xi + nsh\xi)(sh\xi - ch\xi)^n\ (1/sh^3\xi) \qquad \xi > 0$$

Now $sh\xi - ch\xi = -e^{-\xi}$ so write as

$$= (-1)^n\ \pi\ (ch\xi + nsh\xi)(-e^{-\xi})^n\ (1/sh^3\xi) \qquad \xi > 0$$

$$= \pi\ (ch\xi + nsh\xi)e^{-n\xi}\ (1/sh^3\xi) \qquad \xi > 0$$

Now do a Maple test on this result

ξ = 1.7	n = 2	ok
ξ = 1.7	n = 3	ok

So here is my conclusion:

$$\int_0^{\pi} dx\, \cos(nx)(-ch\xi+\cos x)^{-2} = \pi\ (ch\xi + nsh\xi)e^{-n\xi}\ (1/sh^3\xi) \qquad \xi > 0$$

$$\int_0^{\pi} du\, \cos(nu)(ch\xi-\cos u)^{-2} = \pi\ (ch\xi + nsh\xi)e^{-n\xi}\ (1/sh^3\xi) \qquad \xi > 0$$

Now go back to the "two integrals needed" point:

$$\sigma(u;\xi_0) = \frac{V_0}{4\pi R}\ [\ \frac{1}{2}\ +\ \frac{\sqrt{2}}{\pi}\ (ch\xi_0 - \cos u)^{3/2}\ \Sigma_{n=0}^{\infty}\ \varepsilon_n\ P'_{n-1/2}(ch\xi_0)\ \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\ \cos(nu)\ ]$$

So I will need these two integrals:

$$J1 = \int_0^{\pi} du\, \cos(nu)(ch\xi_0 - \cos u)^{-2} = \pi\ (ch\xi_0 + nsh\xi_0)e^{-n\xi_0}\ (1/sh^3\xi_0)$$

$$J2 = \int_0^{\pi} du\, \cos(nu)(ch\xi_0 - cosu)^{-2}(ch\xi_0 - cosu)^{3/2} = \sqrt{2}\, Q_{n-1/2}(ch\xi_0) \qquad \text{// from (K.3.8)}$$

So now I have both the required integrals, so maybe now I can compute the total charge on the toroid.

$$dA = 2\pi a^2\, sh\xi_0\, (ch\xi_0 - cosu)^{-2} du \qquad \text{after } d\varphi \text{ integration}$$

Then

$$\int \sigma(u;\xi_0)\, dA = 2\pi a^2\, sh\xi_0 \int \sigma(u;\xi_0)\, (ch\xi_0 - cosu)^{-2} du$$

$$= \frac{V_0}{4\pi R}\, 2\pi a^2\, sh\xi_0 \left\{ \frac{1}{2} \int_0^{2\pi} du\, (ch\xi_0 - cosu)^{-2} \right.$$

$$\left. + \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \varepsilon_n\, P'_{n-1/2}(ch\xi_0) \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \int_0^{2\pi} du\, \cos(nu)\, (ch\xi_0 - cosu)^{-1/2} \right\}$$

Now recall that

$$\int_0^{2\pi} du\, (ch\xi_0 - cosu)^{-2} = 2\pi\, (ch\xi)(1/sh^3\xi) \qquad \text{since } n = 0$$

$$\int_0^{2\pi} du\, \cos(nu)\, (ch\xi_0 - cosu)^{-1/2} = 2\sqrt{2}\, Q_{n-1/2}(ch\xi_0)$$

Here then is my claimed answer (which looks very wrong)

$$Q = \frac{V_0}{4\pi R}\, 2\pi a^2\, sh\xi_0 \left\{ \frac{1}{2}\, 2\pi\, (ch\xi)(1/sh^3\xi) \right.$$

$$\left. + \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \varepsilon_n\, P'_{n-1/2}(ch\xi_0) \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\, 2\sqrt{2}\, Q_{n-1/2}(ch\xi_0) \right\}$$

or

$$Q = \frac{V_0}{2R}\, a^2\, sh\xi_0 \left\{ \frac{1}{2}\, 2\pi\, (ch\xi_0)(1/sh^3\xi_0) + \frac{4}{\pi} \sum_{n=0}^{\infty} \varepsilon_n\, P'_{n-1/2}(ch\xi_0) \frac{Q^2_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \right\}$$

But this looks nothing like the Snow capacitance formula I expect from $Q = CV_0$. I so know this however,

$$1/(1-z^2) = P_\nu(z)\, Q_\nu'(z) - P_\nu'(z)\, Q_\nu(z)$$

That could get rid of P', but then you are stuck with a Q'.

Problem: The σ is going to contain P' no matter what! So there must be some way to get rid of it! Look at the properties. Here are some derivative formulas I might use to get rid of P'

14.10.4 $$\left(1-x^2\right)\frac{d\mathsf{P}_\nu^\mu(x)}{dx} = (\mu-\nu-1)\,\mathsf{P}_{\nu+1}^\mu(x)+(\nu+1)x\,\mathsf{P}_\nu^\mu(x),$$

14.10.5 $$\left(1-x^2\right)\frac{d\mathsf{P}_\nu^\mu(x)}{dx} = (\nu+\mu)\,\mathsf{P}_{\nu-1}^\mu(x)-\nu x\,\mathsf{P}_\nu^\mu(x).$$

These say

$(1-x^2)P'_\nu(x) = (-\nu-1)P_{\nu+1}(x) + (\nu+1)xP_\nu(x)$

$(1-x^2)P'_\nu(x) = \nu P_{\nu-1}(x) - \nu xP_\nu(x)$

I write again with $\nu = n-1/2$

$(1-x^2)P'_{n-1/2}(x) = -(n+1/2)P_{n+1/2}(x) + (n+1/2)xP_{n-1/2}(x)$

$(1-x^2)P'_{n-1/2}(x) = (n-1/2)P_{n-3/2}(x) - (n-1/2)xP_{n-1/2}(x)$

and says Bateman

$(1-x^2)Q'_{n-1/2}(x) = -(n+1/2)Q_{n+1/2}(x) + (n+1/2)xQ_{n-1/2}(x)$

$(1-x^2)Q'_{n-1/2}(x) = (n-1/2)Q_{n-3/2}(x) - (n-1/2)xQ_{n-1/2}(x)$

Now my charge formula seems to have this action,

$$\Sigma_{n=0}^{\infty}\varepsilon_n\, P'_{n-1/2}(ch\xi_0)\frac{Q^2_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

whereas the capacitance has this action

$$\Sigma_{n=0}^{\infty}\varepsilon_n\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

Just for fun, use the wronskian in the first sum

$1/(1-z^2) = P_\nu(z)\,Q'_\nu(z) - P'_\nu(z)\,Q_\nu(z)$

$$\Sigma_{n=0}^{\infty}\varepsilon_n\, P'_{n-1/2}(ch\xi_0)\frac{Q^2_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

$$= \Sigma_{n=0}^{\infty}\varepsilon_n\, P'_{n-1/2}(ch\xi_0)Q_{n-1/2}(ch\xi_0)\frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

$$= \Sigma_{n=0}^{\infty} \varepsilon_n \left[P_{n-1/2}(ch\xi_0) Q'_{n-1/2}(ch\xi_0) - 1/(1-z^2) \right] \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

This just doesn't do anything for me.

So something is wrong somewhere!!

Suppose for some reason this is true:

$$\Sigma_{n=0}^{\infty} \varepsilon_n \left[P_{n-1/2}(ch\xi_0) Q'_{n-1/2}(ch\xi_0) \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \right] = f(\xi_0) = \text{a simple function}$$

Then I would have

$$\Sigma_{n=0}^{\infty} \varepsilon_n \, P'_{n-1/2}(ch\xi_0) \frac{Q^2_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} = f(\xi_0) - 1/(1-z^2) * \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

$$= f(\xi_0) - (1/sh\xi_0)^2 \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} .$$

Then I would have

$$Q = \frac{V_0}{2R} a^2 sh\xi_0 \left\{ \frac{1}{2} 2\pi (ch\xi_0)(1/sh^3\xi_0) + \frac{4}{\pi} \Sigma_{n=0}^{\infty} \varepsilon_n \, P'_{n-1/2}(ch\xi_0) \frac{Q^2_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \right\}$$

$$= \frac{V_0}{2R} a^2 sh\xi_0 \left\{ \frac{1}{2} 2\pi (ch\xi_0)(1/sh^3\xi_0) + \frac{4}{\pi} \left[f(\xi_0) + (1/sh\xi_0)^2 \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \right] \right\}$$

Now suppose $f(\xi_0)$ cancels the first term. Then we get

$$Q = \frac{V_0}{2R} a^2 sh\xi_0 \left\{ + \frac{4}{\pi} \left[+ (1/sh\xi_0)^2 \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \right] \right\}$$

$$= \frac{V_0}{2R} a^2 (1/ sh\xi_0) \frac{4}{\pi} \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

And then I would get

$$C = \frac{1}{2R} a^2 (1/ sh\xi_0) \frac{4}{\pi} \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

Now $R = a/sh\xi_0$ so we then get

$$C = \frac{1}{2}\, a \frac{4}{\pi} \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

$$= \frac{2a}{\pi} \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

My earlier calculation was

$$C = (2a/\pi)\, \Sigma_{n=0}^{\infty} \varepsilon_n \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}$$

and then this makes things work! So I have at least a theory. I have then to show that

$$\Sigma_{n=0}^{\infty} \varepsilon_n \left[P_{n-1/2}(ch\xi_0) Q'_{n-1/2}(ch\xi_0) \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)} \right] = f(\xi_0)$$

$$\frac{1}{2}\, 2\pi\, (ch\xi_0)(1/sh^3\xi_0) + \frac{4}{\pi} \left[f(\xi_0) \right] = 0$$

$$\pi\, (ch\xi_0)(1/sh^3\xi_0) + \frac{4}{\pi} f(\xi_0) = 0$$

$$-\, (\pi^2/4)\, (ch\xi_0)(1/sh^3\xi_0) = f(\xi_0)$$

So I would have to show that

$$\Sigma_{n=0}^{\infty} \varepsilon_n \; Q_{n-1/2}(ch\xi_0) Q'_{n-1/2}(ch\xi_0) = -\, (\pi^2/4)\, (ch\xi_0)(1/sh^3\xi_0)$$

Good luck!!! Note that

$$f(x)\, f'(x) = (1/2)\, \partial_x\, (f(x))^2$$

Then the above would read

$$\Sigma_{n=0}^{\infty} \varepsilon_n \; Q_{n-1/2}(z) Q'_{n-1/2}(z) = -\, (\pi^2/4)\, z\, (z^2-1)^{-3/2}$$

$$(1/2)\, \Sigma_{n=0}^{\infty} \varepsilon_n \; \partial_z\, Q^2_{n-1/2}(z) = -\, (\pi^2/4)\, z\, (z^2-1)^{-3/2}$$

$$(1/2) \partial_z \left[\Sigma_{n=0}^{\infty} \varepsilon_n \left[Q_{n-1/2}(z) \right]^2 \right] = -\, (\pi^2/4)\, z\, (z^2-1)^{-3/2}$$

$$\partial_z \left[\Sigma_{n=0}^{\infty} \varepsilon_n \left[Q_{n-1/2}(z) \right]^2 \right] = -\, (\pi^2/2)\, z\, (z^2-1)^{-3/2} \qquad \text{// hope for result}$$

At least this looks like a simpler series to add up. So what is

$$g(z) = \Sigma_{n=0}^{\infty} \varepsilon_n [Q_{n-1/2}(z)]^2 \qquad ?$$

The only tool I really have is this:

$$Q_{n-1/2}(z) = (1/\sqrt{2}) \int_0^{\pi} dt \cos(nt)/\sqrt{z - \cos t}$$

Then I could say

$$\Sigma_{n=0}^{\infty} \varepsilon_n [Q_{n-1/2}(z)]^2 = (1/2) \Sigma_{n=0}^{\infty} \varepsilon_n \int_0^{\pi} dt \cos(nt)/\sqrt{z - \cos t} \int_0^{\pi} dt' \cos(nt')/\sqrt{z - \cos t'}$$

$$= (1/2) \int_0^{\pi} dt \int_0^{\pi} dt' \frac{1}{\sqrt{z - \cos t}\sqrt{z - \cos t'}} \Sigma_{n=0}^{\infty} \varepsilon_n \cos(nt)\cos(nt')$$

But recall from Appendix K that

$$\Sigma_{n=0}^{\infty} (\varepsilon_n/\pi) \cos(n\theta)\cos(n\theta) = \delta(\theta-\theta')$$
$$\Sigma_{n=0}^{\infty} (\varepsilon_n/\pi) \cos(nt)\cos(nt') = \delta(t-t')$$

Then we get

$$\Sigma_{n=0}^{\infty} \varepsilon_n [Q_{n-1/2}(z)]^2 = (1/2) \int_0^{\pi} dt \int_0^{\pi} dt' \frac{1}{\sqrt{z - \cos t}\sqrt{z - \cos t'}} \pi \delta(t-t')$$

$$= \pi/2 \int_0^{\pi} dt \frac{1}{z-\cos t}$$

But this is an integral I think I know from above. Above I showed that

$$\int_0^{\pi} dx \cos(nx) \frac{1}{b+\cos x} = (\text{sign } b)^{n+1} \frac{\pi}{\sqrt{b^2-1}} (\sqrt{b^2-1} - |b|)^n$$

and for n = 0 this says

$$\int_0^{\pi} dx \frac{1}{b+\cos x} = (\text{sign } b) \frac{\pi}{\sqrt{b^2-1}}$$

Now write

$$\int_0^{\pi} dt \frac{1}{z-\cos t} = - \int_0^{\pi} dt \frac{1}{-z+\cos t} = - [(-1) \frac{\pi}{\sqrt{z^2-1}}] = \frac{\pi}{\sqrt{z^2-1}} = \pi/\text{sh}\xi$$

Then I have shown that

$$\Sigma_{n=0}^{\infty} \varepsilon_n [Q_{n-1/2}(z)]^2 = (\pi/2) \int_0^{\pi} dt \frac{1}{z\text{-}\cos t} = (\pi/2) \frac{\pi}{\sqrt{z^2-1}} = (\pi^2/2) \frac{1}{\sqrt{z^2-1}} = (\pi^2/2)(z^2-1)^{-1/2}$$

which would be astonishing to say the least.

Maple: A plot confirms that $\Sigma_{n=0}^{\infty} \varepsilon_n [Q_{n-1/2}(z)]^2 = (\pi^2/2)(z^2-1)^{-1/2}$!!!!!

I was hoping to get

$$\partial_z [\Sigma_{n=0}^{\infty} \varepsilon_n [Q_{n-1/2}(z)]^2] = - (\pi^2/2) z (z^2-1)^{-3/2}$$

Is this what I get? Well here is what I got

$$\partial_z [\Sigma_{n=0}^{\infty} \varepsilon_n [Q_{n-1/2}(z)]^2] = \partial_z (\pi^2/2)(z^2-1)^{-1/2}$$

$$= (\pi^2/2) \partial_z (z^2-1)^{-1/2} = -(\pi^2/2)(1/2)(z^2-1)^{-3/2} 2z = - (\pi^2/2) z(z^2-1)^{-3/2}$$

Done!!!! I have confirmed that the σ integrates to Q !!

Let's try a Maple check on our Q sum!

This Q series is so simple, it must be in the books. Here is one AS series that is nice

14.28.2

$$\sum_{n=0}^{\infty} (2n+1) Q_n(z_1) P_n(z_2) = \frac{1}{z_1 - z_2}, \quad z_1 \in \mathcal{E}_1, \, z_2 \in \mathcal{E}_2,$$

No, not in AS or GR7 or Bate. Why can't I do the capacitance sum that way?

It's the denominator! What do you do with 1/P ? I don't know any integral reps for that.

Now that I know how this all works, maybe there is a better way to write σ.

****

Charge density in terms of θ instead of u.

I know from (L.1.4) that

$$dA = a^2 \, sh\xi_0 \, (ch\xi_0 - \cos u)^{-2} du d\varphi$$

The charge in this area is

$$dQ = \sigma dA = \sigma [a^2 \, sh\xi_0 \, (ch\xi_0 - \cos u)^{-2}] \, du \, d\varphi = \sigma [R^2 \, sh^3\xi_0 \, (ch\xi_0 - \cos u)^{-2}] \, du \, d\varphi$$

Suppose I want to change from toroidal parameter u to the "circle angle θ" shown Bipolar (7.1). Then

$$dQ = \sigma \left[a^2 \, sh\xi_0 \frac{1}{(ch\xi_0 - cosu)^2} \right] \left[\frac{du}{d\theta} \right] d\theta \, d\varphi \qquad \frac{1}{(ch\xi_0 - cosu)^2}$$

Now I can use certain Bipolar facts,

$$\frac{1}{ch\xi - cosu} = \frac{ch\xi + cos\theta}{sh^2\xi} \qquad \frac{du}{d\theta} = \frac{sh\xi}{ch\xi + cos\theta} = \frac{ch\xi - cosu}{sh\xi}$$

Then

$$dQ = \sigma \left[a^2 \, sh\xi_0 \frac{1}{(ch\xi_0 - cosu)^2} \right] \left[\frac{du}{d\theta} \right] d\theta \, d\varphi$$

$$= \sigma \left[a^2 \, sh\xi_0 \left(\frac{ch\xi + cos\theta}{sh^2\xi}\right)^2 \right] \left[\frac{sh\xi}{ch\xi + cos\theta} \right] d\theta \, d\varphi$$

$$= \sigma \, a^2 \, (ch\xi + cos\theta) \frac{1}{sh^2\xi} \, d\theta \, d\varphi$$

$$= \sigma \, a^2 \frac{1}{sh^2\xi} \, (ch\xi + cos\theta) \, d\theta \, d\varphi = \sigma \, R^2 \, (ch\xi + cos\theta) \, d\theta \, d\varphi$$

and then the thing you want to plot is

$$dQ/d\varphi = \sigma \, a^2 \frac{1}{sh^2\xi_0} \, (ch\xi_0 + cos\theta) \, d\theta = \sigma \, R^2 \, (ch\xi_0 + cos\theta) \, d\theta$$

and then this is a task for Maple. What happens in the thin wire limit ? $ch\xi_0$ is large so we get

$$dQ/d\varphi = \sigma \, a^2 \, ch\xi_0 / sh^2\xi_0 \, d\theta \approx \sigma / sh\xi_0$$

But we know from (10.5.14) that

$$\sigma(u; \xi_0) \approx \frac{V_0}{4\pi R} \frac{1}{\ln(8/R)} = \text{a constant for fixed small R}$$

Then

$$dQ/d\varphi \approx a^2 \frac{V_0}{4\pi R} \frac{1}{\ln(8/R)} (1/sh\xi_0) d\theta$$

As expected, there is no θ dependence going around the circle. Thus

$$Q \text{ per } d\varphi = a^2 \frac{V_0}{4\pi R} \frac{1}{\ln(8/R)} (1/sh\xi_0)\ 2\pi$$

The total charge on the right is then

$$Q = a^2 \frac{V_0}{4\pi R} \frac{1}{\ln(8/R)} (1/sh\xi_0)\ 2\pi * 2\pi$$

which says the capacitance is

$$C = a^2 \frac{1}{4\pi R} \frac{1}{\ln(8/R)} (1/sh\xi_0)\ 2\pi * 2\pi$$

$$= a^2(\pi/R)(1/sh\xi_0) \frac{1}{\ln(8/R)} \approx$$

$$\approx a\pi \frac{1}{\ln(8/R)} \qquad\qquad a = \rho_c = 1$$

$C(R) \approx \pi / \ln(8/R) \equiv C_0(R)$.

OK, so here are the plotting instructions:

$$dQ/d\varphi = \sigma a^2 \frac{1}{sh^2\xi_0} (ch\xi_0 + \cos\theta)\, d\theta = \sigma R^2 (ch\xi_0 + \cos\theta)\, d\theta$$

But now we have to convert σ:

$$\sigma(u; \xi_0) = \frac{V_0}{4\pi R} \left[\frac{1}{2} + \frac{\sqrt{2}}{\pi} (ch\xi_0 - cosu)^{3/2} \sum_{n=0}^{\infty} \varepsilon_n\ P'_{n-1/2}(ch\xi_0) \frac{Q_{n-1/2}(ch\xi_0)}{P_{n-1/2}(ch\xi_0)}\ \cos(nu) \right] .$$

Look up in Bipolar,

$$\frac{1}{ch\xi - cosu} = \frac{ch\xi + \cos\theta}{sh^2\xi}$$

Now what about cos(nu) ? That could be a problem.

$$sinu = \frac{|sh\xi|\ \sin\theta}{ch\xi + \cos\theta}$$

$$cosu = \frac{ch\xi\ \cos\theta + 1}{ch\xi + \cos\theta}$$

First time I realized this difficulty. Let's try this idea

$\cos u + i \sin u = e^{iu} = [ch\xi \cos\theta + 1 + i\,|sh\xi|\,\sin\theta]/[ch\xi + \cos\theta\}$

Plan B: Let's just do it all numerically somehow. New Maple mws file please.

OK, I am entering this now. What do I do about P' ?

$P_\nu'(z) = \partial_z P_\nu(z) = dP_\nu(z)/dz = dP_\nu(ch\xi)/d(ch\xi) = dP_\nu(ch\xi)/[sh\xi\, d\xi]$

$= (1/sh\xi)\, \partial_\xi P_\nu(ch\xi) \equiv dP_\nu(ch\xi)$

So then I define

$dP(\nu,\xi) = (1/sh\xi)\, \partial_\xi P(\nu,\xi)$

What will Maple do when I try this? Well, luckily it knows how to do this, so I just enter it as shown.

```
dP := (nu,xi) -> (1/shξ) diff(P(nu,xi),xi):
```