Various kinds of Legendre Functions PhL 12.30.15

AS2010 have a huge array of P and Q functions. I will try to describe them all.

0. The Legendre functions are

$P_\nu(x)$ $Q_\nu(x)$ $P_\nu(z)$ $Q_\nu(z)$

Standard solutions: $\mathsf{P}_\nu(\pm x)$, $\mathsf{Q}_\nu(\pm x)$, $\mathsf{Q}_{-\nu-1}(\pm x)$, $P_\nu(\pm x)$, $Q_\nu(\pm x)$, $Q_{-\nu-1}(\pm x)$. $\mathsf{P}_\nu(x)$ and $\mathsf{Q}_\nu(x)$ are real when $\nu \in \mathbb{R}$ and $x \in (-1, 1)$, and $P_\nu(x)$ and $Q_\nu(x)$ are real when $\nu \in \mathbb{R}$ and $x \in (1, \infty)$.

1. Ferrers Functions (Legendre functions on the cut)

The term "on the cut" is ambiguous, what it means is on the interval (-1,1).

Ferrers Function of the First Kind

14.3.1
$$\mathsf{P}^\mu_\nu(x) = \left(\frac{1+x}{1-x}\right)^{\mu/2} \mathbf{F}\left(\nu+1, -\nu; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2}x\right).$$

Ferrers Function of the Second Kind

14.3.2
$$\mathsf{Q}^\mu_\nu(x) = \frac{\pi}{2\sin(\mu\pi)} \left(\cos(\mu\pi) \left(\frac{1+x}{1-x}\right)^{\mu/2} \mathbf{F}\left(\nu+1, -\nu; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2}x\right) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \left(\frac{1-x}{1+x}\right)^{\mu/2} \mathbf{F}\left(\nu+1, -\nu; 1+\mu; \tfrac{1}{2} - \tfrac{1}{2}x\right)\right.$$

Here and elsewhere in this chapter

14.3.3
$$\mathbf{F}(a, b; c; x) = \frac{1}{\Gamma(c)} F(a, b; c; x)$$

These P and Q are non-italic and the A has a straight bar and beware the bold F ! The first of these agrees with Bateman p 124 (14), where Bateman uses the non-bold F function. In other words, both use this

Ferr Bate

$\mathsf{P}^{\mu}_{\nu}(x) = P^{\mu}_{\nu}(x) = \left(\frac{1+x}{1-x}\right)^{\mu/2} F(\nu+1, -\nu; 1-\mu; 1/2 - x/2)$

In the Q equation, we can use

$\Gamma(1+\mu)\Gamma(-\mu) = -\pi/\sin(\mu\pi)$.

Neither term agrees with Bateman! They are close, but not the same due to $\cos(\mu\pi)$ and $e^{-i\mu\pi}$ issues. So the Q differs from Bateman. See below for more detail.

Unless stated otherwise in §§14.2–14.20 it is assumed that the arguments of the functions $\mathsf{P}^{\mu}_{\nu}(x)$ and $\mathsf{Q}^{\mu}_{\nu}(x)$ lie in the interval $(-1,1)$, and the arguments of the functions $P^{\mu}_{\nu}(x)$, $Q^{\mu}_{\nu}(x)$, and $\boldsymbol{Q}^{\mu}_{\nu}(x)$ lie in the interval $(1,\infty)$. For extensions to complex arguments see §§14.21–14.28.

So the Ferrers functions are geared to (-1,1) while the Assoc Legendre to (1,∞).

2. Associated Legendre Functions

14.3.6

$$P^{\mu}_{\nu}(x)=\left(\frac{x+1}{x-1}\right)^{\mu/2}\mathbf{F}\left(\nu+1,-\nu;1-\mu;\tfrac{1}{2}-\tfrac{1}{2}x\right).$$

14.3.7

$$Q^{\mu}_{\nu}(x)=e^{\mu\pi i}\frac{\pi^{1/2}\,\Gamma(\nu+\mu+1)\left(x^2-1\right)^{\mu/2}}{2^{\nu+1}x^{\nu+\mu+1}}\mathbf{F}\left(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+1,\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+\tfrac{1}{2};\nu+\tfrac{3}{2};\frac{1}{x^2}\right)$$

Notice the italic font and the curly Q bar. As before, bold F functions.

Fact: The P here is identical to the Ferrers P and is **identical to Bateman** (14).

Fact: The Q function here is **identical to Bateman** (41). So these are the functions I want to deal with!

3. Bold Q function

To confuse things more, AS2010 also defines

$$\boldsymbol{Q}^{\mu}_{\nu}(x)=e^{-\mu\pi i}\frac{Q^{\mu}_{\nu}(x)}{\Gamma(\nu+\mu+1)}$$

so this is a bolded italic Q. Notice phase and gamma. Thing on right is assoc Leg fnctn.

4.

(14.2.2) reduces to (14.2.1) when $\mu=0$. Ferrers functions and the associated Legendre functions are related to the Legendre functions by the equations $\mathsf{P}^{0}_{\nu}(x)=\mathsf{P}_{\nu}(x)$, $\mathsf{Q}^{0}_{\nu}(x)=\mathsf{Q}_{\nu}(x)$, $P^{0}_{\nu}(x)=P_{\nu}(x)$, $Q^{0}_{\nu}(x)=Q_{\nu}(x)$, $\boldsymbol{Q}^{0}_{\nu}(x)=\boldsymbol{Q}_{\nu}(x)=Q_{\nu}(x)/\Gamma(\nu+1)$.

$\mathsf{P}^{\mu}_{\nu}(x)$, $\mathsf{P}^{\mu}_{-\frac{1}{2}+i\tau}(x)$, and $\mathsf{Q}^{\mu}_{\nu}(x)$ are real when ν, μ, and $\tau\in\mathbb{R}$, and $x\in(-1,1)$; $P^{\mu}_{\nu}(x)$, $Q^{\mu}_{\nu}(x)$, and $\boldsymbol{Q}^{\mu}_{\nu}(x)$ are real when ν and $\mu\in\mathbb{R}$, and $x\in(1,\infty)$.

OK, in *all cases* we have $X_\nu^0 = X_\nu$ so there are no hidden surprises when you set $\mu = 0$ on the upper index position.

3. Back up to Ferrers Q and Compare to Bateman Q

Ferrers Function of the Second Kind

14.3.2

$$Q_\nu^\mu(x) = \frac{\pi}{2\sin(\mu\pi)}\left(\cos(\mu\pi)\left(\frac{1+x}{1-x}\right)^{\mu/2}\mathbf{F}(\nu+1,-\nu;1-\mu;\tfrac{1}{2}-\tfrac{1}{2}x)\right.$$
$$\left. - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{1-x}{1+x}\right)^{\mu/2}\mathbf{F}(\nu+1,-\nu;1+\mu;\tfrac{1}{2}-\tfrac{1}{2}x)\right)$$

First use

$$-\Gamma(1+\mu)\Gamma(-\mu) = \frac{\pi}{\sin(\mu\pi)}$$

Then Ferrer says

$$Q^\mu_\nu(x) = -(1/2)\Gamma(1+\mu)\Gamma(-\mu)\ \cos(\mu\pi)\left(\frac{1+x}{1-x}\right)^{\mu/2} F(\nu+1,-\nu;1-\mu;1/2-x/2)/\Gamma(1-\mu)$$
$$+ (1/2)\Gamma(1+\mu)\Gamma(-\mu)\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{1-x}{1+x}\right)^{\mu/2} F(\nu+1,-\nu;1+\mu;1/2-x/2)/\Gamma(1+\mu)$$

$$= -(1/2)[\Gamma(1+\mu)\Gamma(-\mu)/\Gamma(1-\mu)]\ \cos(\mu\pi)\left(\frac{1+x}{1-x}\right)^{\mu/2} F(\nu+1,-\nu;1-\mu;1/2-x/2)$$
$$+ (1/2)\Gamma(-\mu)\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{1-x}{1+x}\right)^{\mu/2} F(\nu+1,-\nu;1+\mu;1/2-x/2)$$

$$= (1/2)\Gamma(\mu)\cos(\mu\pi)\left(\frac{1+x}{1-x}\right)^{\mu/2} F(\nu+1,-\nu;1-\mu;1/2-x/2) \qquad // -[\Gamma(1+\mu)\Gamma(-\mu)/\Gamma(1-\mu)] = \Gamma(\mu)$$
$$+ (1/2)\Gamma(-\mu)\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{1-x}{1+x}\right)^{\mu/2} F(\nu+1,-\nu;1+\mu;1/2-x/2)$$

$$= F_1 + F_2$$

```
GAMMA(1+mu)*GAMMA(-mu);
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$$\Gamma(1+\mu)\Gamma(-\mu)$$
```
simplify(%);
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$$\frac{\pi}{\sin(\pi(1+\mu))}$$
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GAMMA(1+mu)*GAMMA(-mu)/GAMMA(1-mu);
```
$$\frac{\Gamma(1+\mu)\Gamma(-\mu)}{\Gamma(1-\mu)}$$
```
simplify(%);
```
$$-\Gamma(\mu)$$

In contrast, here is what Bateman (32) says

$$Q^{\mu}_{\nu}(x) = e^{i\pi\mu}\ (1/2)\,\Gamma(\mu)\,\left(\frac{1+x}{1-x}\right)^{\mu/2} F(\nu+1,-\nu;1-\mu;1/2-x/2)$$

$$+ (1/2)\, e^{i\pi\mu}\,\Gamma(-\mu)\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{1-x}{1+x}\right)^{\mu/2} F(\nu+1,-\nu;\,1+\mu;\,1/2-x/2)$$

$$= \left(e^{i\pi\mu} / \cos\mu\pi\right) F_1 \ + \ e^{i\pi\mu} F_2$$

and that seems very strange to me. I guess F_1 and F_2 solve the Legendre equation and here we are just talking about coefficients, so all these equations solve Legendre.

Now what happens when $\mu = 0$?

$$Q_{\nu}(x) = Q^{0}_{\nu}(x) = F_1 + F_2 = Q^{0}_{\nu}(x) = Q_{\nu}(x)$$

So AS failed to state this, but:

Conclusion: Ferrer and Bateman (32) for Q agree with $\mu = 0$.

Examples: According to the above bold thing, we would have

$$\boldsymbol{Q}_{\nu}(x) = Q_{\nu}(x) / \Gamma(\nu+1)$$

Then

$$\boldsymbol{Q}_0(x) = Q_0(x) / \Gamma(1) = Q_0(x) = Q_0(x)$$

$$\boldsymbol{Q}_1(x) = Q_1(x) / \Gamma(2) = Q_1(x) = Q_1(x)$$

This is confirmed on page 359

14.5.5 $$\mathsf{P}_0(x) = P_0(x) = 1,$$

14.5.6 $$\mathsf{P}_1(x) = P_1(x) = x.$$

14.5.7 $$\mathsf{Q}_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$$

14.5.8 $$\mathsf{Q}_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1.$$

14.5.9 $$Q_0(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),$$

14.5.10 $$Q_1(x) = \frac{x}{2}\ln\left(\frac{x+1}{x-1}\right) - 1.$$

3. How does M&F define the P function?

$$P_n^m(z) = = \frac{(n+m)!(1-z^2)^{\frac{1}{2}m}}{2^m m!(n-m)!} F\left(m-n, m+n+1 | m+1 | \frac{1-z}{2}\right)$$

I will translate this to say (assuming we have the same F)

$$\mathsf{P}_\nu^\mu(z) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} (1-z^2)^{\mu/2}\ 2^{-\mu}\ \frac{1}{\Gamma(\mu+1)}\ F(\mu-\nu, \mu+\nu+1; \mu+1; 1/2-z/2)$$

This is completely different from both Ferrer and Bateman who said

$$\mathsf{P}^\mu_\nu(x) = P^\mu_\nu(x) = \left(\frac{1+x}{1-x}\right)^{\mu/2} F(\nu+1,-\nu; 1-\mu; 1/2- x/2)$$

The M&F thing fits NONE of the Bateman forms. Lets try Kummer (1) on page 105

$$F(a,b;c,z) = (1-z)^{c-a-b} F(c-a,c-b;c;z)$$

Use this with the Bateman result with

$z = (1-x)/2$ $a = \nu+1$ $b = -\nu$ $c = 1-\mu$

$c - a - b = 1-\mu - \nu-1 +\nu = -\mu$ $c-a = (1-\mu)-(\nu+1) = -\mu-\nu$ $c-b = 1-\mu+\nu$

$1-z = 1 - (1-x)/2 = 1 - 1/2 + x/2 = 1/2+x/2 = (1+x) * (1/2)$

$(1-z)^{c-a-b} = [(1+x) * (1/2)]^{-\mu} = 2^\mu (1+x)^{-\mu}$

The Bateman result then becomes

$$\left(\frac{1+x}{1-x}\right)^{\mu/2} * 2^{\mu} (1+x)^{-\mu} * F(-\mu-\nu,\ 1-\mu+\nu;\ 1-\mu;\ (1-x)/2)$$

$$= (1+x)^{-\mu/2} (1-x)^{-\mu/2} 2^{\mu} F(-\mu-\nu,\ 1-\mu+\nu;\ 1-\mu;\ (1-x)/2)$$

$$= (1-x^2)^{-\mu/2} 2^{\mu} F(-\mu-\nu,\ 1-\mu+\nu;\ 1-\mu;\ (1-x)/2)$$

So I have shown that

$$P^{\mu}_{\nu}(x) = (1-x^2)^{-\mu/2} 2^{\mu} F(-\mu-\nu,\ 1-\mu+\nu;\ 1-\mu;\ (1-x)/2)$$

But then

$$P^{-\mu}_{\nu}(x) = (1-x^2)^{\mu/2} 2^{-\mu} F(\mu-\nu,\ 1+\mu+\nu;\ 1+\mu;\ (1-x)/2)$$

and here is the M&F function

$${}_{MF}P_{\nu}^{\mu}(z) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} 2^{-\mu}(1-z^2)^{\mu/2} \frac{1}{\Gamma(\mu+1)} F(\mu-\nu,\ \mu+\nu+1;\ \mu+1;\ 1/2-z/2)$$

$$= \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \frac{1}{\Gamma(\mu+1)} P^{-\mu}_{\nu}(x)$$

So this is pretty wacko, beware anything that uses these functions. For $\mu = 0$ we get

$${}_{MF}P_{\nu}(z) = P_{\nu}(x)$$

so at least M&F agree with Bateman on the P_{ν} functions.

4. How does M&F define the Q function?

From page 1327

$$Q_n^m = (-1)^m (z^2-1)^{\frac{1}{2}m} \frac{d^m}{dz^m} Q_n(z) = (-1)^m (z^2-1)^{\frac{1}{2}m} V_{n-m}^m(z)$$

$$= (z^2-1)^{\frac{1}{2}m} \frac{2^m \Gamma(\frac{1}{2})(n+m)!}{\Gamma(n+\frac{3}{2})(2z)^{n+m+1}} F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2} \middle| n+\tfrac{3}{2}\middle| \frac{1}{z^2}\right)$$

$$Q_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} \left[(z^2-1)^n \ln\left(\frac{z+1}{z-1}\right)\right] - \tfrac{1}{2} P_n(z) \ln\left(\frac{z+1}{z-1}\right)$$

The Q^m_n agrees exactly with Bateman except Bateman gives the above times e^{inm}. So

$e^{-i\mu\pi}\, Q_\nu^\mu(z) \;=\; {}_{MF}Q_\nu^\mu(z)$

so these are pretty close.

Now, is there any gimmick in the Q_n function? Is it the same as his Q_n^0 ?? I don't see anything like the rule he quotes above. So maybe he gives some special cases? Not without an upper 0 !

Their **toroidals** are different still, here is page 1329:

$$Q^m_{n-\frac{1}{2}}(\cosh\mu) \;=\; (-1)^m \sinh^m(\mu)\, V^m_{n-m-\frac{1}{2}}(\cosh\mu)$$
$$= \frac{\sqrt{\pi}\,\Gamma(n+m+\frac{1}{2})}{2^{n+\frac{1}{2}}n!}\,\frac{\coth^m\mu}{\cosh^{n+\frac{1}{2}}\mu}\cdot$$
$$\cdot F\left(\frac{n-m+\frac{1}{2}}{2}, \frac{n-m+\frac{3}{2}}{2}\,|n+1|\,\operatorname{sech}^2\mu\right)$$

Is this compatible with his earlier Bateman-like claim? I will convert their earlier Q result right here, which earlier result is page 1327

$$= (z^2-1)^{\frac{1}{2}m}\,\frac{2^m\Gamma(\frac{1}{2})(n+m)!}{\Gamma(n+\frac{3}{2})(2z)^{n+m+1}}\,F\left(\frac{n+m+1}{2}, \frac{n+m+2}{2}\,|n+\tfrac{3}{2}|\,\frac{1}{z^2}\right)$$

$= sh^m\xi \;\; 2^m \sqrt{\pi}\,\Gamma(n\text{-}1/2\text{+}m\text{+}1)\; 2^{-(n-1/2)-m-1}\; z^{-(n-1/2)-m-1} \,/\, \Gamma(n\text{+}1) * F$

$= sh^m\xi \;\; 2^m \sqrt{\pi}\,\Gamma(n\text{-}1/2\text{+}m\text{+}1)\; 2^{-n-m-1/2}\; z^{-n-m-1/2} \,/\, \Gamma(n\text{+}1) * F$

$= sh^m\xi \;\; 2^{-n-1/2} \sqrt{\pi}\,\Gamma(n\text{+}m\text{+}1/2)\;\; (ch\xi)^{-n-m-1/2} \,/\, \Gamma(n\text{+}1) * F$

$= sh^m\xi \;\; 2^{-n-1/2} \sqrt{\pi}\,\Gamma(n\text{+}m\text{+}1/2)\;\; (ch\xi)^{-n-m-1/2} \,/\, \Gamma(n\text{+}1) * F$

$= (\tanh\xi)^m\, (ch\xi)^{-n-1/2}\;\; 2^{-n-1/2} \sqrt{\pi}\,\Gamma(n\text{+}m\text{+}1/2) \,/\, \Gamma(n\text{+}1) * F$

$= \sqrt{\pi}\,\Gamma(n\text{+}m\text{+}1/2)\; 2^{-n-1/2}\, (\tanh\xi)^m\, (ch\xi)^{-n-1/2} \,/\, n! * F\left(\frac{n+m+1/2}{2}, \frac{n+m+3/2}{2}; n+1; 1/z^2\right)$

So once again they are inconsistent. So I have shown that their first Q function gives the thing below. So therefore

${}_{MF}Q_{n-1/2}{}^{-m}(z)$

$$= \sqrt{\pi}\,\Gamma(n-m+1/2)\,2^{-n-1/2}\,(\tanh\xi)^{-m}\,(ch\xi)^{-n-1/2}\ /\ n!\ *\ F\left(\frac{n-m+1/2}{2}, \frac{n-m+3/2}{2}; n+1; 1/z^2\right)$$

$$= \frac{\Gamma(n-m+1/2)}{\Gamma(n+m+1/2)}\ *$$

$$\sqrt{\pi}\ \Gamma(n+m+1/2)\,2^{-n-1/2}\,(\tanh\xi)^{-m}\,(ch\xi)^{-n-1/2}\ /\ n!\ *\ F\left(\frac{n-m+1/2}{2}, \frac{n-m+3/2}{2}; n+1; 1/z^2\right)$$

$$= \frac{\Gamma(n-m+1/2)}{\Gamma(n+m+1/2)}\ {}_{MFtor}Q_{n-1/2}^{m}(z)$$

Conclusion: for m = 0 these are all the same

$${}_{MFtor}Q_{n-1/2}(z) = {}_{MF}Q_{n-1/2}(z) = Q_{n-1/2}(z) = \text{Bateman}$$

so he has changed from m to - m and added a factor.

My conclusion is that for $Q_\nu(z)$, M&F are the same as Bateman. I would like to see some examples,

Here is another M&F declaration

$$Q_\alpha(z) = \frac{1}{2^{\alpha+1}} \int_{-1}^{1} \frac{(1-t^2)^\alpha}{(z-t)^{\alpha+1}}\,dt \qquad (5.3.29)$$

But my main interest today is how M&F define their "toroidal function"/

From page 598 vol 1 M&F say,

$$Q_0(z) = \tfrac{1}{2}\ln[(z+1)/(z-1)]; \quad Q_1(z) = \tfrac{1}{2}z\ln[(z+1)/(z-1)] - 1; \quad \text{etc.}$$

These are the two that I wanted. They are in agreement with the quote above. So there is I think no magic difference between Q_0 and Q_1 , but the issue is really $Q_{-1/2}$ versus the other $Q_{-1/2+n}$