

The Charged Bowl in Toroidal Coordinates
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1. Introduction.

This document was motivated by a "stub" section appearing in an informal 2002 paper by Kirk McDonald (see References) which reviews various solutions of the charged bowl problem. Here is that section in its entirety:

2.5 Solution in Toroidal Coordinates

The problem of a charged, conducting spherical bowl can also be solved in toroidal coordinates [10].

where [10] is our Lebedev reference. The present document is meant to fill in this stub section!

The phrase "charged bowl" refers to an isolated conducting spherical bowl, sometimes called a spherical cap, shell or segment. An arbitrary plane slicing through a full spherical shell divides that shell into two spherical bowls, each with the same circular lip. The "charged bowl problem" is this: put some charge Q on an isolated conducting bowl and determine the resulting electrostatic potential V everywhere as well as the charge densities σ on the inner and outer bowl surfaces. The reader versed in related problems (e.g., the charged disk) would not be surprised to find the charge densities to be divergent (but integrable) at the bowl edge and non-zero everywhere on the bowl. In practice, one normally assumes that the bowl is at some potential V_0 relative to infinity, and then the charge on the bowl is $Q = C_{\text{bowl}} V_0$. Of course finding C_{bowl} is part of the problem.

In 1869 Kelvin used the now-famous method of inversion (last seen lurking in green Jackson) to solve the charged bowl problem for the inner and outer charge densities. Kelvin's "bowl paper" requires great patience to read, and is outlined in McDonald's paper. Modern readers might have "forgotten" things like "the chord theorem" and other geometric properties of circles. We comment more on Kelvin's bowl paper in **Appendix C** and discuss how Kelvin showed the now-well-known fact that the inner and outer bowl charge densities differ by the constant $V_0/(4\pi R)$ which is independent of the bowl's lip angle and of the location on the bowl. Certainly this result is valid for a fully closed bowl since $\sigma = 0$ inside and on the outside $\sigma = Q/A = C_{\text{sphere}} V_0/A = R V_0/(4\pi R^2) = V_0/(4\pi R)$.

In **Appendix D** we mention a fascinating variation of the Kelvin inversion approach indicated by Smythe in a set of problems.

Besides inversion, various other methods exist to obtain an exact solution of this problem. Perhaps the most well known is the use of "dual" Legendre series equations in friendly spherical coordinates as

outlined by Sneddon and others. One series (Dirichlet) sets the potential to a constant on the bowl, while a radial derivative of that series (Neumann) sets the charge density to zero on the cap. The problem is then to find series coefficients which satisfy both equations, neither of which is invertible since it only covers a partial range of polar angle. In matrix language, these dual equations are $A\Psi = f$ and $B\Psi = g$ where Ψ is an infinite vector of unknown coefficients. **Appendix E** shows formally how such dual equations may be solved, and **Appendix G** solves the charged bowl problem using this method. A tool used in the dual equations approach is the Abel transform, a summary of which is given in **Appendix F**.

Our main concern, despite all these spherical coordinate appendices, is to obtain an exact solution of the charged bowl problem using the much less friendly toroidal coordinates. Surely this has been done many hundred times since the mid 19th century, but the author has not found much on the web -- certainly not much that is freely downloadable.

After doing the bowl, we turn our attention briefly to the problem of the **charged toroid**.

The general plan is shown in the Contents above. **Appendix A** provides simple Maple code to allow the reader to make plots of the bowl potential. **Appendix B** concerns a limit which seems to indicate that the capacitance of a degenerate toroid of unit tube radius is given by the strange number $C = 1.7413$ which can be compared to the capacitance $C = 2$ of its embedding sphere.

In all that follows, we shall refer to the spherical conducting bowl as "the bowl", and shall refer to the unoccupied remainder of the bowl's sphere as "the cap". The symbol ε_n stands for $2-\delta_{n,0}$, sometimes called Neumann's factor or number.

2. Solution of the Charged Bowl Problem in Toroidal Coordinates.

(a) Level curves, the charged ellipsoid problem and a dashed hope

As one sees scanning through the beautiful pictures in Moon & Spencer's strangely but correctly named *Field Theory Handbook*, each orthogonal coordinate system has its characteristic level surfaces. In spherical coordinates these are spheres, polar cones and azimuthal planes, whereas in toroidal coordinates they are "bowls and toroids" and azimuthal planes. In spherical coordinates a bowl has a hole in it since the polar angle runs only part of its range, but in toroidal coordinates there is no such "hole". That is to say, a bowl has a label (a value of one of the toroidal coordinates) and as the other two coordinates sweep their *full ranges*, a bowl is swept out. The fact that the bowl is a level surface in toroidals makes one at least interested in solving the charged bowl problem in this system.

In ellipsoidal coordinates the level surfaces are ellipsoids and asymmetric hyperboloids of one and two sheets, which certainly sounds foreboding. One can solve the "charged ellipsoid problem" and the result is shockingly simple (as Kelvin also showed) and σ on the ellipsoid is simple even in Cartesian coordinates (more magic geometry). A family of confocal ellipsoids can be described by the equation $x^2/(\xi_1^2 - a^2) + y^2/(\xi_1^2 - b^2) + z^2/(\xi_1^2) = 1$ where ξ_1 in $(0, \infty)$ is the "label" of an ellipsoid. It happens then that the label ξ_1 is also the largest semi-major axis of the ellipsoid, while $a > b$ are focal distances associated with the other two axes. Ellipsoidal coordinates are fully separable, and a Laplace-satisfying potential function which is constant on each of these ellipsoids and which vanishes at infinity is given by

$$\begin{aligned} V(\xi_1)/\text{const} &= F_0^0(\xi_1) E_0^0(\xi_2) E_0^0(\xi_3) = [(1/a) \text{sn}^{-1}(a/\xi_1, b/a)] * 1 * 1 = (1/a) \text{sn}^{-1}(a/\xi_1, b/a) \\ &= (1/a) F(\sin^{-1}(a/\xi_1), b/a) \end{aligned}$$

Here (ξ_1, ξ_2, ξ_3) are the three ellipsoidal coordinates in Morse & Feshbach notation, E and F are first and second kind Lamé functions, and a different F is the first kind elliptic integral. By setting $V = V_0$ on a *particular* ellipsoid $\xi_1 = c$, one then obtains the potential anywhere outside this charged ellipsoid,

$$V(\xi_1) = V_0 F(\sin^{-1}(a/\xi_1), k=b/a) / F(\sin^{-1}(a/c), k=b/a) \quad (2.1)$$

Although not immediately obvious, this expression is exactly the same if one swaps $a \leftrightarrow b$ and such a swap is necessary to show that the above form agrees with that of Kelvin. [There is some confusion in elliptic F notation to beware: $F(\phi, k) = F(\phi | k^2) = F(\phi \setminus \sin^{-1} k)$, the first notation being that of GR7 8.111.2.]

One might *hope* that the bowl potential in toroidals could be as simply stated as the ellipsoid potential in ellipsoidals, but alas it is not so, but only in the following sense. In toroidals a certain weight factor cross-links the bowl and toroid coordinates (labels), which makes the Laplace equation separable only in the form of three separated functions times the weight factor $\sqrt{\text{ch}\xi - \text{cosu}}$, where ξ is a toroid label and u a bowl label. This weaker kind of separability is called R-separability by Moon and Spencer. A solution $\sqrt{\text{ch}\xi - \text{cosu}} [A(\xi)=1]B(u)[C(\phi)=1]$ evaluated on the surface of bowl u_0 gives $V = \sqrt{\text{ch}\xi - \text{cosu}_0} B(u_0)$ which varies with ξ , not allowing $V = V_0$ on the bowl. *Nevertheless*, it will turn out that the charged bowl potential can be expressed in simple inverse trig functions, and is thus even simpler than the charged ellipsoid potential.

(b) Cartesian, spherical, toroidal and ellipsoidal atomic forms

"Atomic forms" or just "atoms" are the author's private phrases for "harmonics", which word means simple solutions of the Laplace equation which can be superposed to construct non-simple solutions. The Cartesian atoms illustrate the idea that if you curve toward axis in 2 dimensions, you must curve away from axis in the 3rd:

osc	osc	expo	
(1) $[\sin(k_x x), \cos(k_x x)]$,	$[\sin(k_y y), \cos(k_y y)]$,	$[\exp(\kappa_z z), \exp(-\kappa_z z)]$	$k_z = \text{imaginary} = i\kappa_z$
toward	toward	away	$\kappa_z = \sqrt{k_x^2 + k_y^2}$

For solving practical problems, 2 of the 3 coordinates have to be oscillatory to allow for functional completeness, here on a surface of x and y , so that expansions can be inverted and problems solved. For spherical atoms, two interesting atomic forms can be written in which azimuthal ϕ is oscillatory:

r in $(0, \infty)$	z in $(-1, 1)$	ϕ in $(0, 2\pi)$	
expo osc osc (1) $[r^n, r^{-n-1}] [P_n^m(z), Q_n^m(z)] [\sin(m\phi), \cos(m\phi)]$			$z = \cos\theta$
osc expo osc (2) $(1/\sqrt{r}) [r^{i\tau}, r^{-i\tau}] [P_{i\tau-1/2}^m(z), Q_{i\tau-1/2}^m(z)] [\sin(m\phi), \cos(m\phi)]$			$n = i\tau - 1/2$
osc expo osc $\sim (1/\sqrt{r}) [\sin(\tau \ln r), \cos(\tau \ln r)] [P_{i\tau-1/2}^m(z), Q_{i\tau-1/2}^m(z)] [\sin(m\phi), \cos(m\phi)]$			

The first is doubtless more familiar to the reader, but the second is appropriate for, say, a Dirichlet problem involving a cone, since the atomic form is oscillatory both ways across the surface of a cone, allowing a prescribed potential there to be inverted. The underlying fact is that each oscillatory coordinate becomes a 1D Sturm-Liouville problem with a complete set of eigenfunctions, and then these two sets provide a complete set of eigenfunctions for a 2D surface spanned by those coordinates. In passing, we note that the Legendre functions in form (2) above are called conical functions and have $|z| < 1$.

Without further ado, we can write two similar atomic forms for toroidal coordinates,

$$\begin{aligned}
 & \xi \text{ in } (0, \infty) \quad u \text{ in } (0, 2\pi) \quad \phi \text{ in } (0, 2\pi) \\
 (1) \quad & \sqrt{\text{ch}\xi - \cos u} \quad \begin{matrix} \text{expo} \\ [P_{n-1/2}^m(\text{ch}\xi), Q_{n-1/2}^m(\text{ch}\xi)] \end{matrix} \quad \begin{matrix} \text{osc} \\ [\sin(nu), \cos(nu)] \end{matrix} \quad \begin{matrix} \text{osc} \\ [\sin(m\phi), \cos(m\phi)] \end{matrix} \\
 (2) \quad & \sqrt{\text{ch}\xi - \cos u} \quad \begin{matrix} \text{osc} \\ [P_{i\tau-1/2}^m(\text{ch}\xi), Q_{i\tau-1/2}^m(\text{ch}\xi)] \end{matrix} \quad \begin{matrix} \text{expo} \\ [\exp(\tau u), \exp(-\tau u)] \end{matrix} \quad \begin{matrix} \text{osc} \\ [\sin(m\phi), \cos(m\phi)] \end{matrix}
 \end{aligned}$$

To solve bowl problems, we need the two coordinates other than the bowl label u to be oscillatory, and that means we must use form (2). In the first form, if a problem has a full azimuth, the parameter m gets quantized to integers, and this indirectly causes parameter n on P to be quantized to integers, so the spectra of both Sturm-Liouville problems are discrete. In form (2) even if m is quantized, parameter τ remains unquantized and, since $P_v = P_{-v-1}$, the τ spectrum can be restricted to the positive real axis.

Again in passing, the Legendre functions in system (1) are sometimes called toroidal functions or ring functions, argument > 1 , whereas those appearing in (2) are called Mehler functions, again with argument > 1 . At least that is the author's terminology.

In reference to our earlier charged ellipsoid comments, we just mention the atomic form for ellipsoidal coordinates,

$$[E_m^P(\xi_1), F_m^P(\xi_1)] [E_m^P(\xi_2), F_m^P(\xi_2)] [E_m^P(\xi_3), F_m^P(\xi_3)]$$

where $0 < \xi_3 < b < \xi_2 < a < \xi_1$ with $a > b$ the confocal ellipsoid focal distances. As noted earlier, E and F are the first and second kind Lamé functions. The ξ_i are the three roots of the cubic equation which is the equation of the ellipsoid given above. In this system there is no azimuthal coordinate, and the separated solutions are triply cross-linked in that all three functions bear the same quantum numbers m and p (separation constants arising when the Laplace equation is separated). This system, though quite complicated, is well explained in full detail in the last chapter of Hobson's book.

(c) Smythian forms and one for the bowl

Please forgive the author's predilection for strange phrases. A Smythian form refers to a linear combination (sum and/or integral) of atomic forms, having some to-be-determined "coefficients", which provides a candidate solution to some problem. Such a form usually "builds in" certain boundary conditions, such as continuity between two regions of space on whose boundary a Green's point charge lies in a Green's Function problem. Smythe's book makes excellent use of such forms, and one might even

refer to "the method of Smythian forms" in a list of effective methods of solving boundary value problems.

So, we propose the following Smythian form for the potential of our charged bowl (since things are azimuthally symmetric, we have only $m = 0$ atoms)

$$V(\xi, u) = \sqrt{\text{ch}\xi - \text{cosu}} \int_0^\infty d\tau P_{i\tau-1/2}(\text{ch}\xi) [A(\tau)\text{ch}(u\tau) + B(\tau)\text{sh}(u\tau)] \quad (2.2)$$

where A and B are to-be-determined coefficient functions. Although it is not immediately obvious, this form builds in the boundary condition that the potential vanish at infinity, and for that reason the Q function has been rejected. Since the Mehler P functions form a complete orthogonal set, it is possible to set in some prescribed potential $V = f(\xi, u)$ on the surface of a bowl, and invert to find the coefficient functions. At first it seems odd that there are two functions A and B to be found, and only one boundary condition, but the dilemma is quickly resolved by realizing that we apply the boundary condition separately on each surface of the bowl, so really there are two boundary conditions and two unknown coefficient functions. For our charged bowl problem, the prescribed Dirichlet potential is $f(\xi, u) = V_0$.

(d) Solution to the charged bowl potential problem: the Mehler-Fock Transform

The generalized Mehler-Fock (Mehler-Fok) transform (Oberhettinger and Higgins page 2) is this,

$$\begin{aligned} g(y) &= \int_0^\infty d\tau P_{-1/2+i\tau}^m(y) f(\tau) && // \text{ expansion} \\ f(\tau) &= (\tau/\pi) \text{sh}(\pi\tau) \Gamma(1/2-m+i\tau) \Gamma(1/2-m-i\tau) \int_1^\infty dy P_{-1/2+i\tau}^m(y) g(y) && // \text{ projection} \end{aligned}$$

and for $m = 0$, using $\Gamma(1/2+i\tau)\Gamma(1/2-i\tau) = \pi/\cosh(\pi\tau)$, one obtains the regular Mehler-Fock transform,

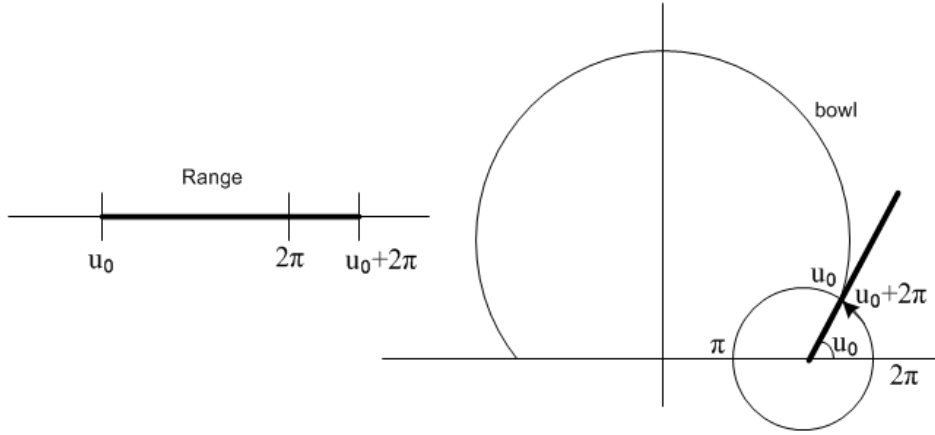
$$\begin{aligned} g(y) &= \int_0^\infty d\tau P_{-1/2+i\tau}(y) f(\tau) && // \text{ expansion} \\ f(\tau) &= \tau \text{th}(\pi\tau) \int_1^\infty dy P_{-1/2+i\tau}(y) g(y) && // \text{ projection} \end{aligned}$$

which can also be written as

$$\begin{aligned} G(\xi) &= \int_0^\infty d\tau P_{-1/2+i\tau}(\text{ch}\xi) f(\tau) && // \text{ expansion} \\ f(\tau) &= \tau \text{th}(\pi\tau) \int_0^\infty d\xi \text{sh}\xi P_{-1/2+i\tau}(\text{ch}\xi) G(\xi) && // \text{ projection} \end{aligned} \quad (2.3)$$

For azimuthally symmetric bowl problems, this transform is the one associated with the Sturm Liouville problem in the ξ coordinate. Every reasonable Sturm-Liouville problem defines a complete set of functions and therefore defines a transform (there are thousands of them), and this happens to be the transform for our oscillatory ξ coordinate. Admittedly, this transform is more sparsely found in the literature than, say, the Fourier Integral Cosine Transform.

Before using this transform to invert the two boundary condition equations, we have to first know a little more about the range of the bowl label coordinate u . To that end, here is a picture:



This shows in cross section one up-side-down bowl having label u_0 . Following Lebedev *et. al.*'s excellent suggestion, instead of running the bowl label in the usual $(0, 2\pi)$ range, we shall have it lie in $(u_0, u_0+2\pi)$ so it can be continuous away from the two sides of the bowl. As we start off with u_0 as the angle shown, we can imagine a set of mathematical bowl surfaces attached to the two "focal points" on the x axis, whose labels run from u_0 up to $u_0 + 2\pi$. We will explain all this below, but it is perhaps useful to solve the problem first, then see the coordinate details later. The inside surface of the bowl lies at $u = u_0$ while the outside surface is at $u = u_0+2\pi$. It is convenient to define $u_0' \equiv u_0 + 2\pi$ and to note then that $\cos(u_0') = \cos(u_0)$. Here then are the two boundary conditions:

$$V_0 = \sqrt{\text{ch}\xi - \cos u_0} \int_0^\infty d\tau P_{i\tau-1/2}(\text{ch}\xi) [A(\tau)\text{ch}(u_0\tau) + B(\tau)\text{sh}(u_0\tau)] \quad (2.4)$$

$$V_0 = \sqrt{\text{ch}\xi - \cos u_0'} \int_0^\infty d\tau P_{i\tau-1/2}(\text{ch}\xi) [A(\tau)\text{ch}(u_0'\tau) + B(\tau)\text{sh}(u_0'\tau)] \quad u_0' \equiv u_0 + 2\pi$$

Application of the Mehler-Fock transform (2.3) gives

$$[A(\tau)\text{ch}(u_0\tau) + B(\tau)\text{sh}(u_0\tau)] = V_0 \tau \text{th}(\pi\tau) \int_0^\infty d\xi \text{sh}(\xi) P_{i\tau-1/2}(\text{ch}\xi) / \sqrt{\text{ch}\xi - \cos u_0} \quad (2.5)$$

$$[A(\tau)\text{ch}(u_0'\tau) + B(\tau)\text{sh}(u_0'\tau)] = V_0 \tau \text{th}(\pi\tau) \int_0^\infty d\xi \text{sh}(\xi) P_{i\tau-1/2}(\text{ch}\xi) / \sqrt{\text{ch}\xi - \cos u_0}$$

Then using this integral (derived in section (e) below)

$$\int_1^\infty dx P_{i\tau-1/2}(x) / \sqrt{x - \cos u_0} = \sqrt{2} \text{ch}[\tau(u_0-\pi)] / (\tau \text{sh}(\pi\tau)) \quad 0 \leq u_0 \leq 2\pi \quad (2.6)$$

one can do a simple Cramer's Rule solution to find the coefficient functions

$$\begin{aligned} A(\tau) &= V_0 \sqrt{2} \text{ch}[(\pi-u_0)\tau] \text{ch}[(\pi+u_0)\tau] / \text{ch}^2(\pi\tau) \\ B(\tau) &= -V_0 \sqrt{2} \text{ch}[(\pi-u_0)\tau] \text{sh}[(\pi+u_0)\tau] / \text{ch}^2(\pi\tau) \end{aligned} \quad (2.7)$$

Inserting these into our Smythian form (2.2) then gives the potential of the charged bowl of label u_0

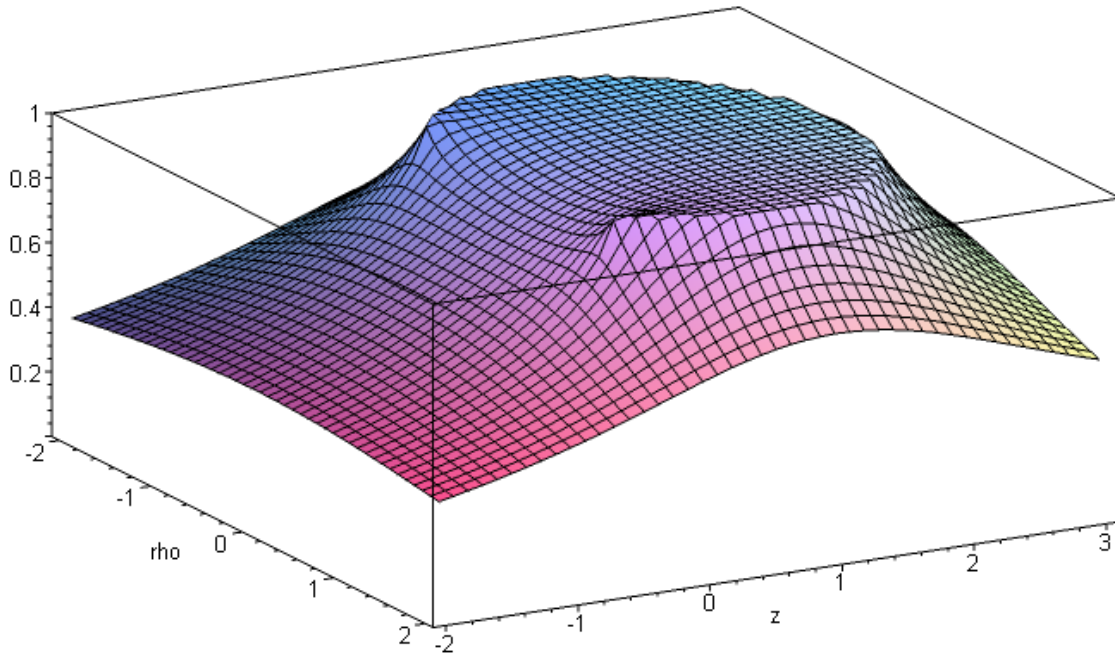
$$V(\xi, u) = V_0 \sqrt{2} \sqrt{\text{ch}\xi - \cos u} \int_0^\infty d\tau P_{i\tau-1/2}(\text{ch}\xi) \text{ch}[(\pi-u_0)\tau] \text{ch}[(\pi+u_0-u)\tau] / \text{ch}^2(\pi\tau) \quad (2.8)$$

We are all done! It is shown below that this integral can in fact be evaluated into the following set of elementary functions

$$V(\xi, u) = (V_0/\pi) \sqrt{\text{ch}\xi - \cos u} * \left\{ \frac{(1/\sqrt{\text{ch}\xi - \cos u}) \cot^{-1}[(-1)^{\eta_1} \sqrt{1+\cos u} / \sqrt{\text{ch}\xi - \cos u}]}{+ (1/\sqrt{\text{ch}\xi - \cos(2u_0-u)}) \cot^{-1}[(-1)^{\eta_2} \sqrt{1+\cos(2u_0-u)} / \sqrt{\text{ch}\xi - \cos(2u_0-u)}]} \right\} \quad (2.9)$$

$$\begin{array}{lll} \text{where} & \eta_1 = \text{floor}[(3\pi-u)/2\pi] & u_0 \text{ in } (0, \pi) \\ & \eta_2 = \text{floor}[(2u_0-u+\pi)/2\pi] & u \text{ in } (u_0, u_0+2\pi) \end{array}$$

where the strange η exponents are related to analytic continuation as described later. We can throw this equation into Maple and obtain very nice plots of the charged bowl potential across a symmetric slice of the bowl. Here is the plot for a bowl with $u_0 = \pi/4$. Along the bowl edge the potential is constant at the value $V_0 = 1$, and outside the region shown it drops off to 0 at infinity. Kelvin would have liked Maple.



(e) Derivation of an integral

We wish to verify integral (2.6), which does not seem to appear as such in the usual sources,

$$\int_1^\infty dx P_{i\tau-1/2}(x) / \sqrt{x - \cos u_0} = \sqrt{2} \operatorname{ch}[\tau(u_0 - \pi)] / (\tau \operatorname{sh}(\tau\pi)) \quad 0 \leq u_0 \leq 2\pi \quad (2.6)$$

Start with the left hand side (LHS) and expand part of the integrand using (10.3) below

$$1/\sqrt{x - \cos u_0} = (1/\pi) \sqrt{2} \sum_{n=0}^{\infty} \varepsilon_n Q_{n-1/2}(x) \cos(nu_0)$$

so that

$$\text{LHS} = (\sqrt{2}/\pi) \sum_{n=0}^{\infty} \varepsilon_n \cos(nu_0) \int_1^\infty dx P_{i\tau-1/2}(x) Q_{n-1/2}(x)$$

But according to GR7 page 770 7.114.1 the PQ integral above is just $1/(n^2 + \tau^2)$ so that

$$\text{LHS} = (\sqrt{2}/\pi) \sum_{n=0}^{\infty} \varepsilon_n \cos(nu_0) / (n^2 + \tau^2)$$

Then use GR7 page 47 1.445.2, which can be written as

$$\sum_{n=0}^{\infty} \varepsilon_n \cos(nu_0) / (n^2 + \tau^2) = (\pi/\tau) \operatorname{ch}[\tau(\pi - u_0)] / \operatorname{sh}(\tau\pi) \quad 0 \leq u_0 \leq 2\pi$$

to conclude that

$$\text{LHS} = \sqrt{2} \operatorname{ch}[\tau(\pi - u_0)] / [\tau \operatorname{sh}(\tau\pi)]$$

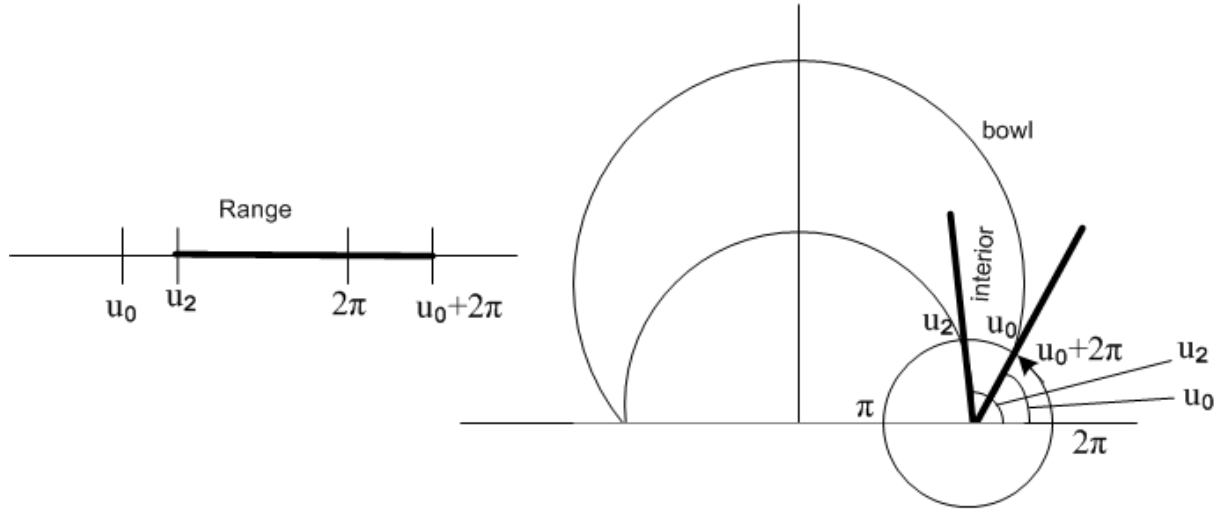
and therefore (2.6) is validated

3. Solution of the Charged Double-Bowl Problem in Toroidal Coordinates.

(a) The general double bowl solution

This seems a good place to address this related problem. Sneddon deals with the problem of two separated bowls having a common symmetry axis. *That* problem seems to have no closed form solution and Sneddon reduces its solution to solving one or more second-kind Fredholm integral equations. *Our* double-bowl problem is much simpler in that both bowls have a common lip and the same potential V_0 . The 3D shapes formed in this way can vary from a sort of 3D lune, to a bowl with a flat lid on it, to a "lens" composed of two spherical surfaces (two facing attached bowls). If the two bowls of the lens have the same size, the solution to this problem describes the evaporation from the surface of a liquid drop sitting on a flat surface, which drop then is half the lens shape. See Hu and Larson regarding such "sessile droplets". Quantity $\pi - u$ is the "contact angle" of such a drop. It is nice to see that solutions to 19th century electrostatic and heat flow potential theory problems have 21st century applications.

The solution to the double bowl problem follows exactly that of the single-bowl problem. For obscure reasons, we defined the two bowl labels to be u_0 and u_2 rather than the more reasonable u_0 and u_1 , forgiveness is requested. Here is the picture showing a lune shaped cross section



We use the exact same Smythian form as for the single bowl,

$$V(\xi, u) = \sqrt{\text{ch}\xi - \cos u} \int_0^\infty d\tau P_{i\tau-1/2}(\text{ch}\xi) [A(\tau)\text{ch}(u\tau) + B(\tau)\text{sh}(u\tau)] \quad (3.1)$$

As before we define $u_0' \equiv u_0 + 2\pi$ and note that $\cos(u_0') = \cos(u_0)$. Here then are the boundary conditions for the double-bowl problem, where for the moment we use V_0 and V_2 :

$$V_2 = \sqrt{\text{ch}\xi - \cos u_2} \int_0^\infty d\tau P_{i\tau-1/2}(\text{ch}\xi) [A(\tau)\text{ch}(u_2\tau) + B(\tau)\text{sh}(u_2\tau)] \quad (3.2)$$

$$V_0 = \sqrt{\text{ch}\xi - \cos u_0'} \int_0^\infty d\tau P_{i\tau-1/2}(\text{ch}\xi) [A(\tau)\text{ch}(u_0'\tau) + B(\tau)\text{sh}(u_0'\tau)] \quad u_0' \equiv u_0 + 2\pi$$

These conditions are for the "exterior problem", we want the potential outside the lune. Comparison with (2.4) shows that we set $0 \rightarrow 2$ in the first condition and the second condition is the same. The coefficients this time are a little more complicated:

$$\begin{aligned} \text{ch}(\pi\tau) A(\tau) &= V_0 \sqrt{2} Q\text{ch}[\tau(\pi-u_0)] / [C \text{ch}(u_0'\tau) + D \text{sh}(u_0'\tau)] \\ \text{ch}(\pi\tau) B(\tau) &= V_0 \sqrt{2} P\text{ch}[\tau(\pi-u_0)] / [C \text{ch}(u_0'\tau) + D \text{sh}(u_0'\tau)] \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} C &= \{ V_0 \text{sh}(u_2\tau) \text{ch}[\tau(\pi-u_0)] - V_2 \text{sh}(u_0'\tau) \text{ch}[\tau(\pi-u_2)] \} \\ D &= \{ V_2 \text{ch}(u_0'\tau) \text{ch}[\tau(\pi-u_2)] - V_0 \text{ch}(u_2\tau) \text{ch}[\tau(\pi-u_0)] \} \end{aligned}$$

Inserting $A(\tau)$ and $B(\tau)$ into the Smythian form (3.1), and setting $u_0' \equiv u_0 + 2\pi$ and $V_2 = V_0$, we find after some algebra this relatively simple result for the potential of the charged double-bowl,

$$V(\xi, u) = V_0 \sqrt{2} \sqrt{\text{ch}\xi - \cos u} \int_0^\infty d\tau P_{i\tau-1/2}(\text{ch}\xi) * \frac{\{ \text{ch}[(\pi-u_0)\tau] \text{sh}[(u-u_2)\tau] + \text{ch}[(\pi-u_2)\tau] \text{sh}[(2\pi+u_0-u)\tau] \}}{\{ \text{ch}(\pi\tau) \text{sh}[(2\pi+u_0-u_2)\tau] \}} \quad (3.4)$$

This result appears in the Hu paper mentioned above in this form, where $\beta, \beta_1, \beta_2 = u, u_0, u_2$ and $\alpha = \xi$.

$$u = V\sqrt{2 \cosh \alpha - 2 \cos \beta} \int_0^\infty \frac{\cosh[(\pi - \beta_1)\tau] \sinh[(\beta - \beta_2)\tau] + \cosh[(\pi - \beta_2)\tau] \sinh[(2\pi + \beta_1 - \beta)\tau]}{\cosh(\pi\tau) \sinh[(2\pi + \beta_1 - \beta_2)\tau]} P_{-(1/2)+i\tau}(\cosh \alpha) d\tau \quad (22)$$

If one takes the limit $u_2 \rightarrow u_0$, the solution to the single-bowl problem is recovered. We have not made any attempt to evaluate this integral but it may yield an expression involving elementary functions.

(b) the special case of a bowl with a flat lid.

Taking $u_2 = \pi$ to get the flat lid, the solution (3.4) reduces to the following, where $\xi = \alpha$, $\beta = u$, and $u_0 = \beta_1$,

$$V(\alpha, \beta) = V_0 \sqrt{2 \sqrt{\cosh \alpha - \cos \beta}} \int_0^\infty d\tau P_{i\tau-1/2}(\cosh \xi) * \{ \text{sh}[(2\pi + \beta_1 - \beta)\tau] - \text{ch}[(\pi - \beta_1)\tau] \text{sh}[(\pi - \beta)\tau] \} / \{ \text{sh}[(\pi + \beta_1)\tau] \text{ch}(\pi\tau) \} \quad (3.5)$$

which may be compared with the result quoted in Lebedev et. al. Problem 503 (page 241) with $\beta_0 = \beta_1$

$$u(\alpha, \beta) = V\sqrt{2 \cosh \alpha - 2 \cos \beta} \times \int_0^\infty [\sinh(2\pi + \beta_0 - \beta)\tau - \cosh(\pi - \beta_0)\tau \times \sinh(\pi - \beta)\tau] \frac{P_{-1/2+i\tau}(\cosh \alpha)}{\sinh(\pi + \beta_0)\tau \cosh \pi\tau} d\tau,$$

(c) The sessile drop (lens shaped double-bowl)

This time we take

$$\begin{aligned} u_0 = \beta_1 &= [\pi - \theta] \\ u_2 = \beta_2 &= [\pi + \theta] \end{aligned}$$

where θ is the small "contact angle" for the drop. Potential (3.4) then becomes

$$V(\alpha, \beta) = V_0 \sqrt{2 \sqrt{\cosh \alpha - \cos \beta}} \int_0^\infty d\tau P_{i\tau-1/2}(\cosh \xi) \text{ch}(\theta\tau) \text{ch}[(2\pi - \beta)\tau] / \{ \text{ch}(\pi\tau) \text{ch}[(\pi - \theta)\tau] \} \quad (3.6)$$

which appears in the Hu paper as

$$\frac{c - c_\infty}{c_v - c_\infty} = \sqrt{2 \cosh \alpha - 2 \cos \beta} \int_0^\infty \frac{\cosh(\theta\tau) \cosh[(2\pi - \beta)\tau]}{\cosh(\pi\tau) \cosh[(\pi - \theta)\tau]} P_{-(1/2)+\pi}(\cosh \alpha) d\tau \quad (26)$$

4. The Single Bowl Surface Charge Densities and Capacitance

(a) Bowl Charge Densities.

We start with the potential as stated above in (2.8) and set up to compute the charge densities,

$$V(\xi, u) = V_0 \sqrt{2} \sqrt{\cosh \xi - \cos u} \int_0^\infty d\tau P_{i\tau-1/2}(\cosh \xi) \operatorname{ch}[(\pi - u_0)\tau] \operatorname{ch}[(\pi + u_0 - u)\tau] / \cosh^2(\pi\tau) \quad (2.8)$$

$$\begin{aligned} \sigma_+ &= -(1/4\pi) (1/h_u) \partial_u V(\xi, u)|^{u=u_0} & 1/h_u &= (\cosh \xi - \cos u)/a \\ \sigma_- &= +(1/4\pi) (1/h_u) \partial_u V(\xi, u)|^{u=u_0+2\pi} \end{aligned} \quad (4.1)$$

where the sign difference arises from the direction of approach to the surface relative to increasing coordinate u . Quantity h_u is the curvilinear scale factor $\sqrt{g_{uu}}$ in the u direction, and a is the radius of the circular lip of the bowl. Defining the above integral to be $f(u)$, we find that $\partial_u f|^{u_0} = \mp X$ at the surfaces, where

$$X = \int_0^\infty d\tau P_{i\tau-1/2}(\cosh \xi) \operatorname{ch}[(\pi - u_0)\tau] \tau \operatorname{sh}(\pi\tau) / \cosh^2(\pi\tau) \quad (4.2)$$

and this time the two signs arise from evaluating $\operatorname{sh}[(\pi + u_0 - u)\tau] = \pm \operatorname{sh}(\pi\tau)$ on the two surfaces. Evaluation of this integral (see section 7 (b) below) gives

$$\begin{aligned} X &= (1/\pi) [\sin(u_0/2)/B^2] [1 + (A/B) \tan^{-1}(A/B)] \\ \text{where } A &= \sqrt{2} \cos(u_0/2) \text{ and } B = \sqrt{\cosh \xi - \cos u_0} \end{aligned} \quad (4.3)$$

Meanwhile, from (2.8) above we have

$$\partial_u V = V_0 \sqrt{2} B (\mp X) + (1/2) \sin u \{ V_0 \sqrt{2} (1/B) f(u) \} \quad (4.4)$$

and we finally end up with

$$\begin{aligned} \sigma_+ = \sigma_{\text{in}} &= (V_0/4\pi^2 R) [(B/A) - \tan^{-1}(B/A)] \\ \sigma_- = \sigma_{\text{out}} &= (V_0/4\pi^2 R) [\pi + (B/A) - \tan^{-1}(B/A)] \end{aligned} \quad (4.5)$$

Therefore

and this makes the Lebedev and Kelvin results agree. In the Lebedev form, the bowl edge lies at $\xi = \infty$, while in the Kelvin form it lies at $\theta = \alpha$, and in either case σ (the first term) diverges at the edge, as noted in the Introduction.

Here is Kelvin's actual statement of his result

$$\rho = \frac{V}{2\pi^2 f} \left\{ \sqrt{\frac{\cos \alpha + 1}{\cos \eta - \cos \alpha}} - \tan^{-1} \sqrt{\frac{\cos \alpha + 1}{\cos \eta - \cos \alpha}} \right\} \dots (18), \quad (4.11)$$

where $f = 2R$, $\alpha(\text{Kelvin}) = \pi - \alpha$, and $\eta = \pi - \theta$, causing all cosines to negate.

(b) Bowl Capacitance

We start yet again with our toroidal charged bowl potential

$$V(\xi, u) = V_0 \sqrt{2} \sqrt{\text{ch} \xi - \cos u} \int_0^\infty d\tau P_{i\tau-1/2}(\text{ch} \xi) \text{ch}[(\pi - u_0)\tau] \text{ch}[(\pi + u_0 - u)\tau] / \text{ch}^2(\pi\tau) \quad (2.8)$$

The large- r limit is given in toroidal coordinates by $\xi \rightarrow 0$ and $u \rightarrow 2\pi$, so we set $P_{i\tau-1/2}(\text{ch} \xi) = 1$. Also, we can show from our equations below that, in this limit,

$$\sqrt{\text{ch} \xi - \cos u} \approx \sqrt{2} a / r. \quad (4.12)$$

The residual integral is then (after some work)

$$I \equiv \int_0^\infty d\tau \text{ch}[(\pi - u_0)\tau] \text{ch}[(u_0 - \pi)\tau] / \text{ch}^2(\pi\tau) = (1/2\pi) [\pi b / \sin(\pi b) + 1] \quad b = 1 - u_0 / \pi \quad (4.13)$$

So

$$V(\xi, u) \approx V_0 \sqrt{2} \sqrt{\text{ch} \xi - \cos u} (1/2\pi) [\pi b / \sin(\pi b) + 1] \approx (a V_0 / r) [b + \sin \pi b / \pi] / \sin(\pi b) \quad (4.14)$$

Thinking of this as $V = Q/r$ and using $Q = C V_0$ one finds

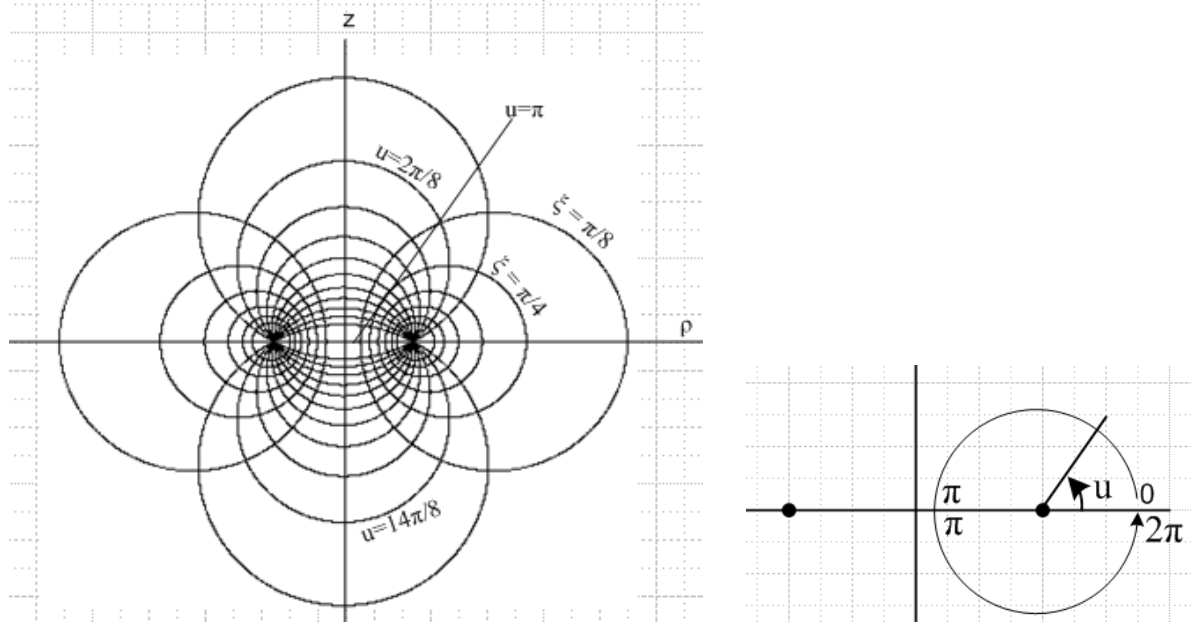
$$\begin{aligned} Q &= (a V_0) [b / \sin(\pi b) + 1 / \pi] \\ C &= (a) \{ b / \sin(\pi b) + 1 / \pi \} & b &= 1 - u_0 / \pi \\ &= R \sin(u_0) \{ (1 - u_0 / \pi) / [\sin(u_0)] + 1 / \pi \} & a &= R \sin(u_0) \\ &= (R / \pi) \{ \pi - u_0 + \sin(u_0) \} \end{aligned} \quad (4.15)$$

which is indeed the known capacitance of a spherical bowl. In the limit $u_0 \rightarrow 0$ the formula gives $C = R$ which is the capacitance of a sphere of radius R , while the limit $u_0 \rightarrow \pi$ ($b \rightarrow 0$) gives $C = 2a/\pi$ which is the capacitance of a disk of radius a .

5. Toroidal coordinates

(a) Verbal description

Here is the basic toroidal picture as a slice in any azimuthal plane, where ρ and z are the usual cylindrical coordinates,



In this picture, the level surfaces of u appear to be vertical set of circles, but each circle is really truncated at the plane $z=0$ [since $\text{sign}(z) = \text{sign}(\sin u)$ according to 5.1 below] so we have a set of upper bowls and a set of lower bowls. In this traditional picture, a huge upper bowl has as its limit an iris just above the $z=0$ plane which is labeled $u=0$. As these bowls shrink, we pass through the $u = 2\pi/8$ bowl shown, and eventually reach very shallow bowls with the limit of a disk of radius a at $u = \pi$. Continuing u we cover all the lower bowls and end up with an iris just below the $z=0$ plane at $u = 2\pi$. There is then a discontinuity in u at this iris value, as shown on the right. All these u bowls have a common circular lip in the $z=0$ plane of radius a , so the distance between the two focal points in the picture above is $2a$. One can interpret u as the external bowl lip angle relative to the $z=0$ plane, but that fact is not very obvious from the above pictures.

Meanwhile the horizontal circles show slices of the toroids labeled by ξ . In the limit $\xi \rightarrow 0$ one gets an exceedingly fat toroid with no hole whose inner surface sweeps up against the z axis. Conversely, as $\xi \rightarrow \infty$, the toroids shrink to a thin circular wire matching the bowl lips. It might be noted that these horizontal circles are the famous ones of Apollonius where $|\mathbf{r} - \mathbf{a}|/|\mathbf{r} + \mathbf{a}| = \text{constant}$, and in this case that constant is $e^{-\xi}$.

Toroidal coordinates start life as 2D bipolar coordinates with the same picture as above, but the left side circles have $\xi < 0$. For 3D toroidal coordinates, one takes just the right side of the above picture with $\xi > 0$ and rotates this around the z axis to get the 3D bowls and toroids.

Lest it be overlooked, the two sets of circles are orthogonal at all intersections, and are both orthogonal to the azimuthal unit vector at any point. The toroidal system is an orthogonal one with a diagonal metric tensor.

(b) Equations

The defining equations are these for toroidal coordinates (toroid,bowl,plane) = (ξ, u, φ) :

$$\begin{aligned} x &= a \cos \varphi \operatorname{sh} \xi / (\operatorname{ch} \xi - \cos u) & \rho &= a \operatorname{sh} \xi / (\operatorname{ch} \xi - \cos u) & \rho^2 &= x^2 + y^2 \\ y &= a \sin \varphi \operatorname{sh} \xi / (\operatorname{ch} \xi - \cos u) \\ z &= a \sin u / (\operatorname{ch} \xi - \cos u) \end{aligned} \quad (5.1)$$

Here then is a summary of all the basic facts of interest

$$\begin{aligned} \rho &= a \operatorname{sh} \xi / (\operatorname{ch} \xi - \cos u) \quad \rho > 0 & h_\xi &= h_u = a / (\operatorname{ch} \xi - \cos u) & h_\varphi &= \operatorname{sh} \xi \, h_\xi \\ z &= a \sin u / (\operatorname{ch} \xi - \cos u) & \rho^2 &= x^2 + y^2 & z/\rho &= \sin u / \operatorname{sh} \xi \end{aligned} \quad (5.2)$$

$$\begin{aligned} \xi &= \tanh^{-1} [2ap / (a^2 + \rho^2 + z^2)] & \text{metric tensor} &= \operatorname{diag}(h_\xi^2, h_u^2, h_\varphi^2) \\ \tan u &= [-2az / (a^2 - \rho^2 - z^2)] & // & \text{use algorithm to find } u \text{ (see later)} \end{aligned} \quad (5.3)$$

$$r^2 = a^2 (\operatorname{sh}^2 \xi + \sin^2 u) / (\operatorname{ch} \xi - \cos u)^2 = \rho^2 + z^2 \quad \tan \varphi \equiv z/\rho = \sin u / \operatorname{sh} \xi \quad (5.4)$$

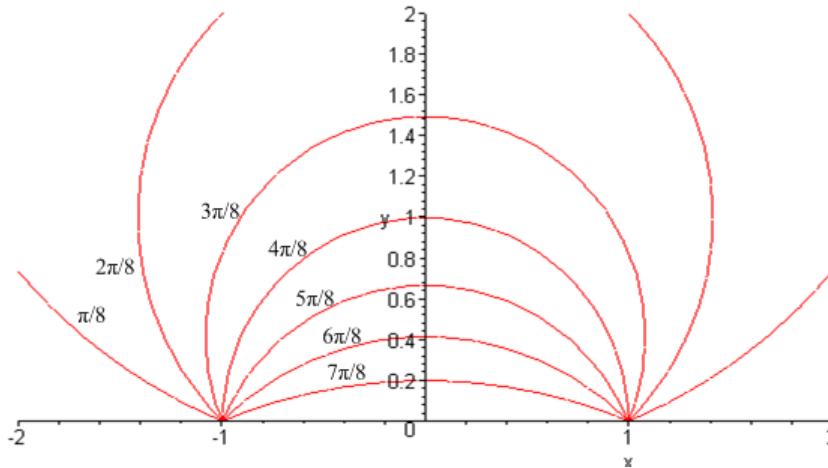
$$\begin{aligned} \rho^2 + (z - a \cot u)^2 &= a^2 / \sin^2 u & z_c &= a \cot(u) & \text{radius} &= a/|\sin u| & // & \text{vertical circles} \\ (\rho - a \coth \xi)^2 + z^2 &= a^2 / \operatorname{sh}^2 \xi & \rho_c &= a \coth \xi & \text{radius} &= a/|\operatorname{sh} \xi| & // & \text{horizontal circles} \end{aligned} \quad (5.5)$$

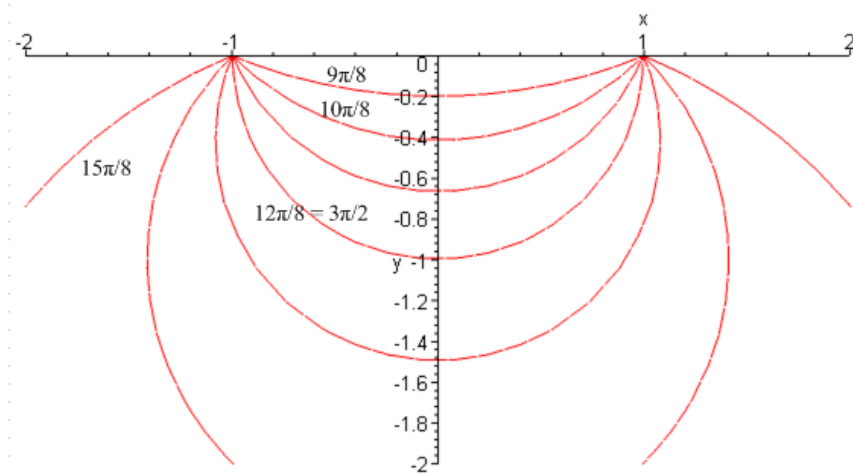
$$\text{ranges:} \quad \xi \text{ in } (0, \infty) \quad u \text{ in } (0, 2\pi) \quad \varphi \text{ in } (0, 2\pi)$$

Notice that the radius of the sphere on which a bowl u_0 lies is $R = a/|\sin u_0|$.

(c) More pictures

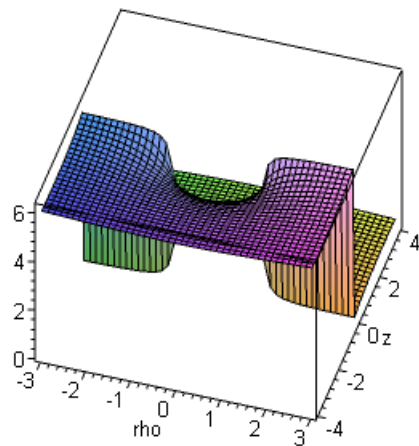
The following detailed Maple plots of u contours have proved helpful to the author:



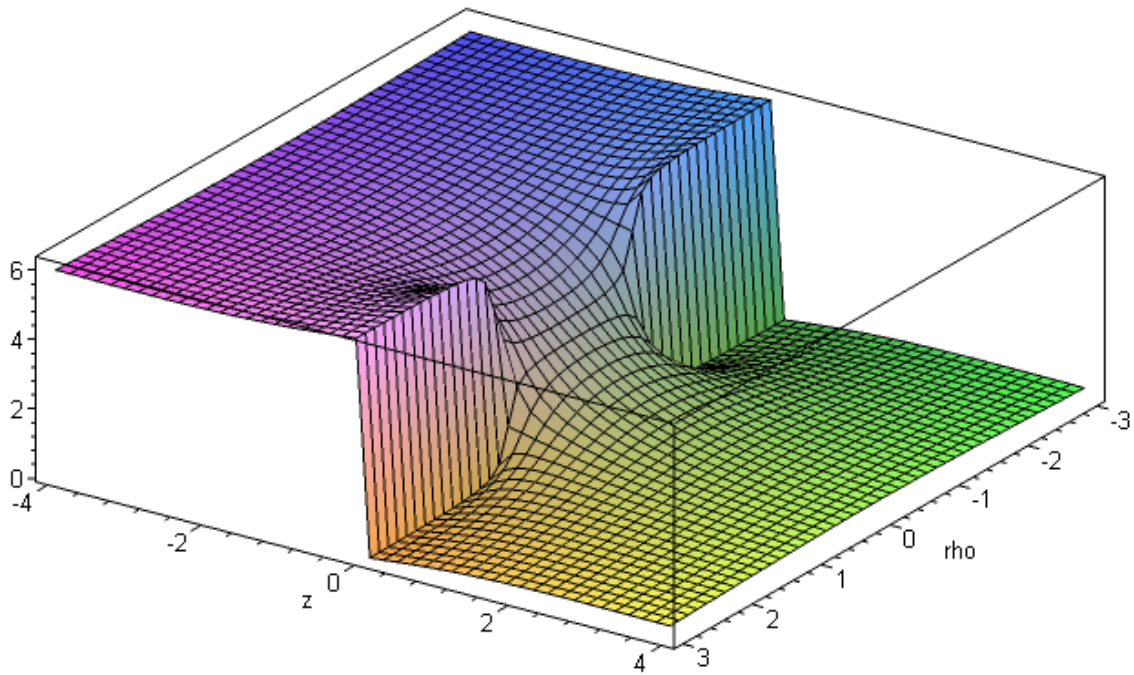


(d) plot of the bowl coordinate u versus coordinates ρ and z

Here is a plot of u oriented roughly to match the pictures in Section (a) above,



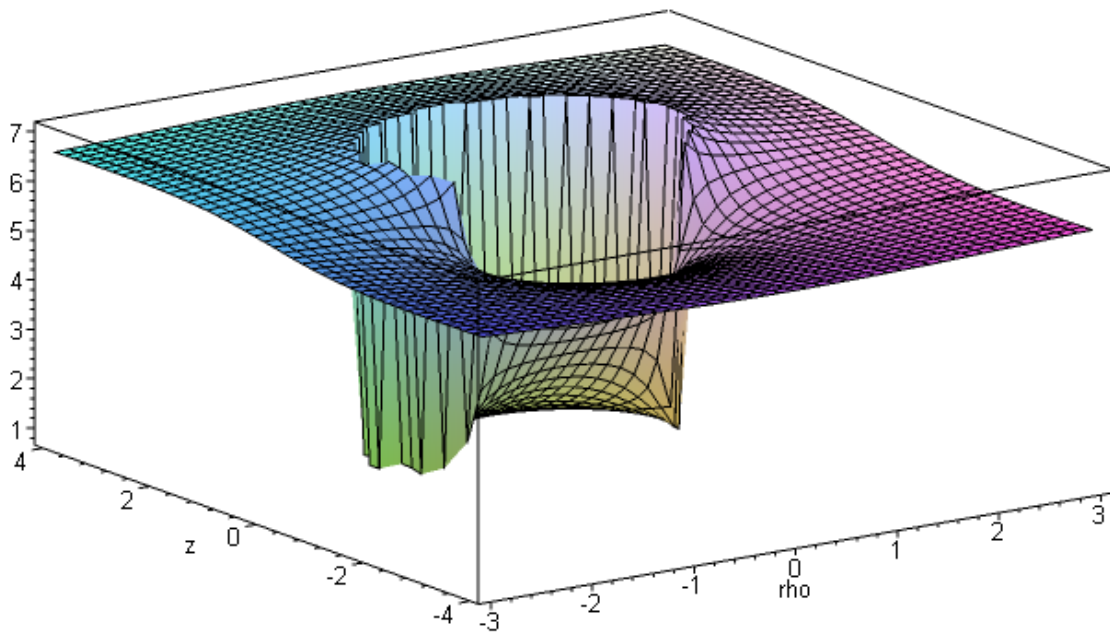
At the upper right (yellow, near "4") we start on the lower plane at $u = 0$. As we rotate CCW about the right focal point we go up the hill and arrive at the top right on the $u = 2\pi$ upper plane. Unfortunately this orientation does not show the hill very well, so we rotate things to get a better view



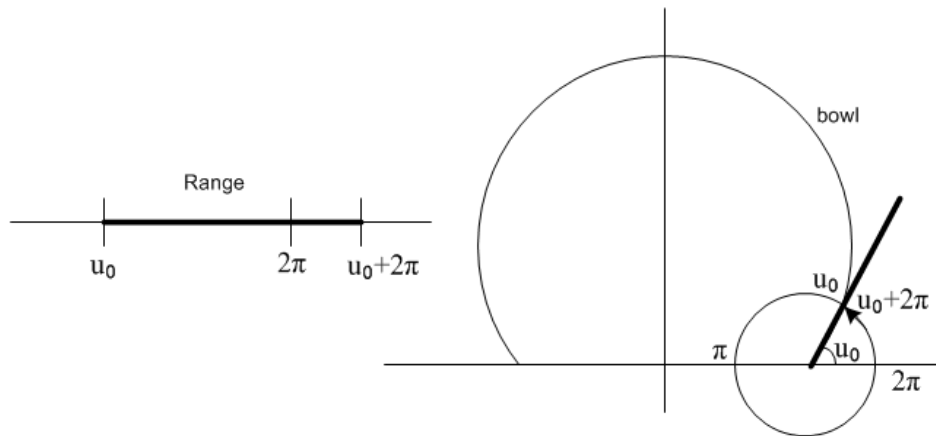
Here the vertical headwall is the discontinuity noted above.

(e) plot of u versus coordinates ρ and z after Lebedev's adjustment

If we adjust things so the u coordinate ranges from u_0 to $u_0 + 2\pi$, the above plot changes. Here is the plot for $u_0 = \pi/4 = .78$:



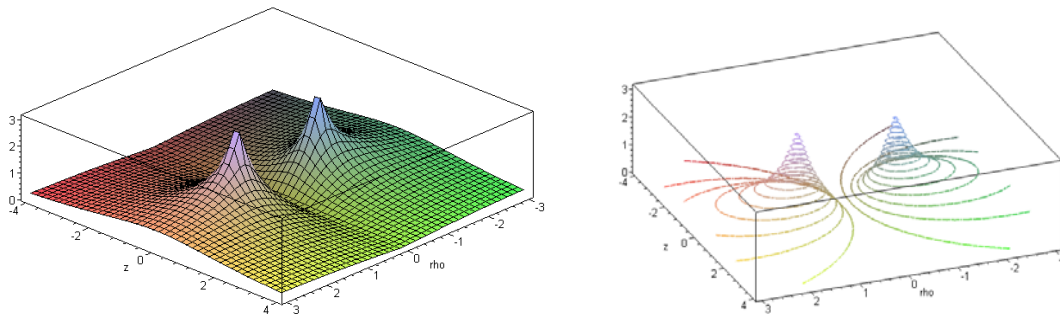
Now the spiral voyage begins inside the bowl at the foot of a vertical curved Niagara Falls which marks the inner boundary of our bowl. We climb up the hill and arrive at the upper plane and walk over to the falls and gaze down from the outer surface of the bowl. Here again is our bowl picture from above



We show all these pictures to make the point that toroidal coordinates are a bit "unfriendly" compared to say spherical coordinates. They have to do violent things to be able to have bowls as level surfaces.

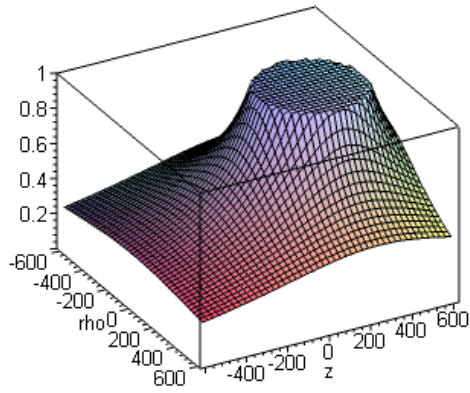
(f) plot of the toroid coordinate ξ versus coordinates ρ and z

This coordinate is much better behaved, though still a bit unusual. The two mountain peaks go up to infinity at the focal points for $\xi = \infty$, and there is a grounded crease along the ρ axis where $\xi = 0$. The contour plot on the right is perhaps better -- a top view of it would show our horizontal circle set.



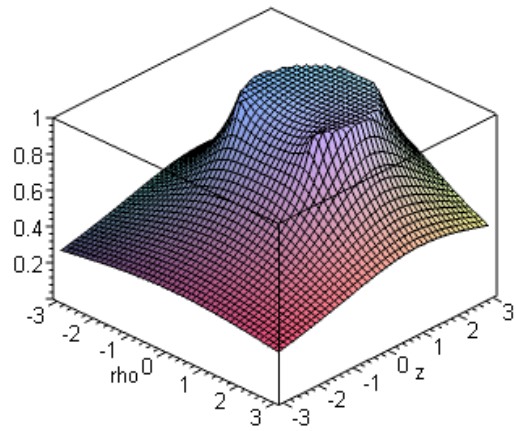
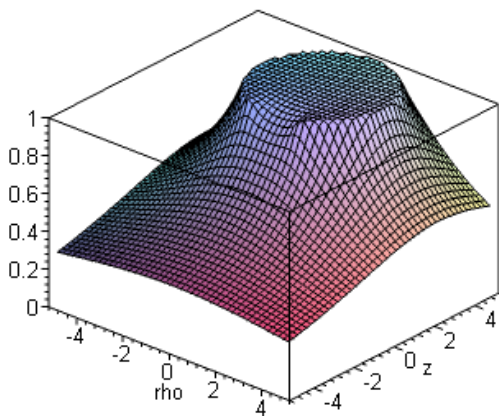
6. A selection of charged bowl potential plots

$u_0 = .01\pi/8$: (basically, $u_0 = 0$)



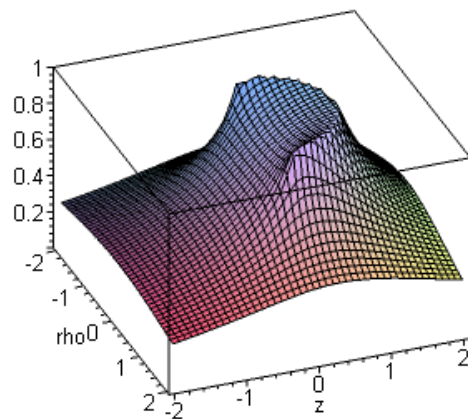
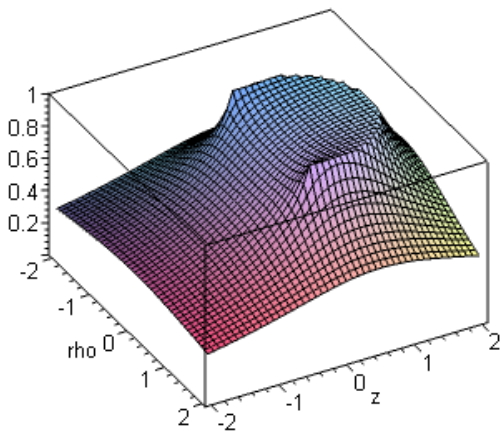
$u_0 = \pi/8$: a bowl that is almost a complete sphere

$u_0 = 2\pi/8$:



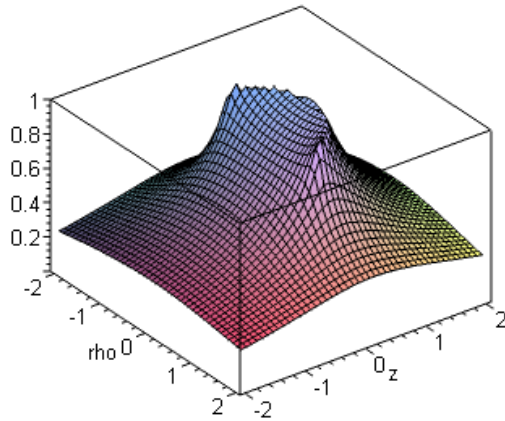
$u_0 = 3\pi/8$:

$u_0 = 4\pi/8$, the hemispherical bowl

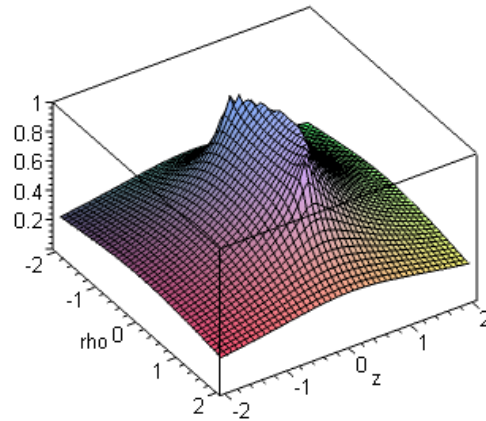


$u_0 = 5\pi/8$

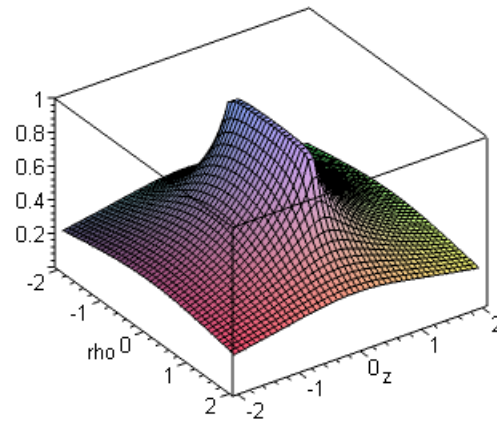
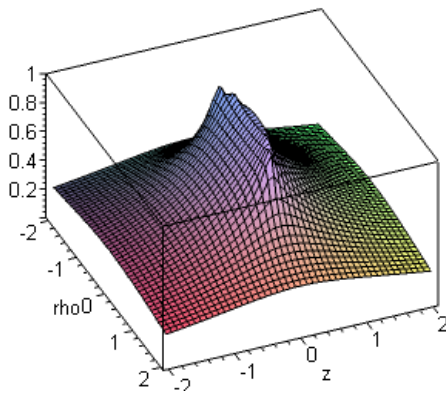
$u_0 = 6\pi/8$



$u_0 = 7\pi/8$, a very shallow bowl



$u_0 = \pi$, potential of a disk



When $u_0 = \pi$, the bowl slice becomes the line between the two foci, and the bowl becomes a perfect disk of radius a . So built into this problem solution is an exact plot of the disk potential. You can see, for example, that if you move 1" off the edge of the disk, V drops faster than if you move 1" out from the disk center.

7. Doing Mehler integrals.

(a) List of useful Mehler integrals

Mehler integration is the seamy underside of using toroidal coordinates in the Mehler function atomic form stated earlier, and is worth some comment. The general form of a "Mehler integral" is this:

$$G(y) = \int_0^\infty d\tau P_{i\tau-1/2}(y) f(\tau) \quad \text{normally with } y \geq 1$$

About the best we can do for P is a form like

$$P_{-1/2+i\tau}(ch\alpha) = e^{-(1/2)\alpha} e^{i\alpha\tau} F(1/2-i\tau, 1/2; 1; 1 - e^{-2\alpha})$$

which at least isolates the variable τ into just one of the hypergeometric function's parameter arguments. But even so, there is very little literature available on such integrals, so changing from P to F really buys little. There are some integrals of $P_{i\tau-1/2}(y) f(\tau)$ appearing in the literature (see below), but often has to do one's own integrals, especially if tables have typos as noted below.

The starting point for doing a Mehler integral is to elevate P into one of these integral representations (see p 962 GR7 8.715.1 and p 961 8.713.3 setting $\mu=0$)

$$P_{-1/2+i\tau}(y) = (\sqrt{2}/\pi) \int_0^a dx \cos(\tau x) / \sqrt{y - chx} \quad y=cha \quad (7.1)$$

$$P_{-1/2+i\tau}(y) = (\sqrt{2}/\pi) \operatorname{ch}(\pi\tau) \int_0^\infty dx \cos(\tau x) / \sqrt{y+chx} \quad (7.2)$$

which exposes a simple τ dependence. The τ integration can then usually be done, and one is faced with the final x integration which can hopefully be looked up somewhere. Here are some integrals the author has calculated in this manner:

$$\int_0^\infty d\tau P_{i\tau-1/2}(y) = (1/\sqrt{2}) (1/\sqrt{y-1}) \quad (7.3)$$

$$\int_0^\infty d\tau P_{i\tau-1/2}(y) \cos(a\tau) = (1/\sqrt{2}) (1/\sqrt{y-cha}) \operatorname{Heaviside}(y-cha) \quad (7.4)$$

$$\int_0^\infty d\tau P_{i\tau-1/2}(y) \cos(a\tau) \operatorname{sech}(\pi\tau) = (1/\sqrt{2}) 1/\sqrt{y+cha} \quad (7.5)$$

$$\int_0^\infty d\tau P_{i\tau-1/2}(y) \cos(a\tau) \operatorname{sech}^2(\pi\tau) = (\sqrt{2}/\pi) (1/\sqrt{y-cha}) \tan^{-1}[\sqrt{y-cha}/\sqrt{1+cha}] \quad (7.6)$$

$$\int_0^\infty d\tau P_{i\tau-1/2}(y) \operatorname{ch}(b\tau) \operatorname{sech}^2(\pi\tau) = (\sqrt{2}/\pi) (1/\sqrt{y-\cos b}) \cot^{-1}[\sqrt{1+\cos b}/\sqrt{y-\cos b}] \quad (7.7)$$

$$\int_0^\infty d\tau P_{i\tau-1/2}(y) \operatorname{sh}(b\tau) \operatorname{th}(\pi\tau) \operatorname{sech}(\pi\tau) = (\sqrt{2}/\pi) (1/\sqrt{y+\cos b}) \tan^{-1}[\sqrt{1-\cos b}/\sqrt{y+\cos b}] \quad (7.8)$$

and this set includes all those needed to obtain results given earlier in this paper. Along the way, the following facts are useful (the first is a corrected GR7 3.197.2)

$$\int_u^\infty dx x^{-\lambda} (x+\beta)^\nu (x-u)^{\mu-1} = u^{-\lambda} (\beta+u)^{\mu+\nu} B(\lambda-\mu-\nu, \mu) F(\lambda, \mu; \lambda-\nu; -\beta/u) \quad (7.9)$$

$$F(1, 1/2; 3/2; z) = [\tanh^{-1}\sqrt{z}] / \sqrt{z} \quad (7.10)$$

(b) The integral X appearing in Section 4(a)

As a Mehler integral example, the integral (4.2) above has this form,

$$X = \int_0^\infty d\tau P_{i\tau-1/2}(y) \operatorname{ch}(b\tau) \tau \operatorname{th}(\pi\tau) \operatorname{sech}(\pi\tau) \quad // y = \operatorname{ch}\xi \text{ and } b = \pi - u_0$$

which has an unpleasant τ factor in the integrand. This integral can be done as $X = \partial_b Y$ where

$$Y = \int_0^\infty d\tau P_{i\tau-1/2}(y) \operatorname{sh}(b\tau) \operatorname{th}(\pi\tau) \operatorname{sech}(\pi\tau)$$

$$X = \partial_b Y = \partial_b \left\{ (\sqrt{2}/\pi) (1/\sqrt{y+\operatorname{cosh} b}) \tan^{-1}[\sqrt{1-\operatorname{cosh} b}/\sqrt{y+\operatorname{cosh} b}] \right\} \quad // \text{ from (7.8) above}$$

Setting $y = \operatorname{ch}\xi$ and $b = \pi - u_0$ we then find (after doing the above ∂_b and some algebra) that

$$\begin{aligned} X &\equiv \int_0^\infty d\tau P_{i\tau-1/2}(\operatorname{ch}\xi) \operatorname{ch}([\pi - u_0]\tau) \tau \operatorname{th}(\pi\tau) \operatorname{sech}(\pi\tau) \\ &= (2/\pi) [\sin(u_0/2) / (2\operatorname{ch}\xi - 2\cos u_0)] * \\ &\quad [1 + (\sqrt{2} \cos(u_0/2) / \sqrt{\operatorname{ch}\xi - \cos u_0}) \tan^{-1}(\sqrt{2} \cos(u_0/2) / \sqrt{\operatorname{ch}\xi - \cos u_0})] \end{aligned}$$

which can be compared with the mysterious integral Lebedev *et. al.* give in Problem 501 (page 239) which uses $\alpha = \xi$ and $\beta_0 = u_0$:

Hint. To calculate the density, use the integral

$$\begin{aligned} \int_0^\infty \frac{\tau \sinh \pi\tau \cosh(\pi - \beta_0)\tau}{\cosh^2 \pi\tau} P_{-1/4+i\tau}(\cosh \alpha) d\tau &= \frac{2}{\pi} \frac{\sin \frac{1}{2}\beta_0}{2 \cosh \alpha - 2 \cos \beta_0} \\ &\times \left[1 + \frac{2 \cos \frac{1}{2}\beta_0}{\sqrt{2 \cosh \alpha - 2 \cos \beta_0}} \arctan \frac{2 \cos \frac{1}{2}\beta_0}{\sqrt{2 \cosh \alpha - 2 \cos \beta_0}} \right]. \end{aligned}$$

(c) Where to find Mehler integrals (some errata noted)

The largest source of such integrals known to the author is an extremely obscure 1961 Boeing Report #246 written by no less than Professor Fritz Oberhettinger and coworker Higgins. Integral (7.8) appears for example as page 20 #3. Integral (7.6) above as page 20 #5 for $y > \operatorname{cha}$, and the corresponding log form for $y < \operatorname{cha}$ has a typo in that the leading factor should be $1/2$ instead of $1/\sqrt{2}$. Similarly, result (7.7) appears as page 20 #6 expressed as $\tan^{-1}$ and in this form a certain exponent has a wrong sign. That erroneous exponent also appears in PBM mentioned next

$$6. \int_0^\infty \frac{\operatorname{ch} bx}{\operatorname{ch}^2 \pi x} P_{ix-1/2}(c) dx = \frac{1}{\sqrt{2(c - \cos b)}} - \frac{\sqrt{2(c - \cos b)}}{\pi} \operatorname{arctg} \sqrt{\frac{1 + \cos b}{c - \cos b}} \quad [c > 1].$$

// wrong

Using $\cot^{-1} = \pi/2 - \tan^{-1}$ for the principle branch of the arc trig functions, (7.7) above becomes

$$\begin{aligned} \int_0^{\infty} d\tau P_{i\tau-1/2}(y) \operatorname{ch}(b\tau) \operatorname{sech}^2(\pi\tau) \\ = (1/\sqrt{2}) (1/\sqrt{y-\cos b}) - (\sqrt{2}/\pi) (1/\sqrt{y-\cos b}) \tan^{-1}[\sqrt{1+\cos b}/\sqrt{y-\cos b}] \end{aligned} \quad (7.7A)$$

which shows the correct placement of the radicals. It can be shown that 7.7A is only valid for $|b| < \pi$, whereas 7.7 is valid for $|b| < 2\pi$.

The second largest source is volume 3 of the special functions series of PBM which has about 20 integrals of $P_{i\tau-1/2}(y)$ against elementary functions and many more against special functions.

Bateman ET 2 has a grand total of 6 Mehler integrals, and GR7 has somewhat more. Perhaps there is some recent collection of Mehler integrals unknown to the author.

8. Evaluating the toroidal bowl potential integral into elementary functions

Start with the form derived above

$$V(\xi, u) = V_0 \sqrt{2} \sqrt{\operatorname{ch}\xi - \cos u} \int_0^{\infty} d\tau P_{i\tau-1/2}(\operatorname{ch}\xi) \operatorname{ch}[(\pi-u_0)\tau] \operatorname{ch}[(\pi+u_0-u)\tau] / \operatorname{ch}^2(\pi\tau) \quad (2.8)$$

Expand the $\operatorname{ch} \operatorname{ch}$ product to get

$$V(\xi, u) = (V_0 / \sqrt{2}) \sqrt{\operatorname{ch}\xi - \cos u} \int_0^{\infty} d\tau P_{i\tau-1/2}(\operatorname{ch}\xi) \{ \cosh[(2\pi-u)\tau] + \cosh[(2u_0-u)\tau] \} / \operatorname{ch}^2(\pi\tau) \quad (8.1)$$

Use integral (7.7) above to find

$$\begin{aligned} V(\xi, u) = (V_0 / \sqrt{2}) (\sqrt{2}/\pi) \sqrt{\operatorname{ch}\xi - \cos u} * \\ \{ (1/\sqrt{\operatorname{ch}\xi - \cos(2\pi-u)}) \cot^{-1}[\sqrt{1+\cos(2\pi-u)}/\sqrt{\operatorname{ch}\xi - \cos(2\pi-u)}] \\ + (1/\sqrt{\operatorname{ch}\xi - \cos(2u_0-u)}) \cot^{-1}[\sqrt{1+\cos(2u_0-u)}/\sqrt{\operatorname{ch}\xi - \cos(2u_0-u)}] \} \end{aligned} \quad (8.2)$$

u_0 in $(0, \pi)$

u in $(u_0, 2\pi+u_0)$

$\operatorname{ch}\xi$ in $(1, \infty)$

and we can if we like replace $\cos(2\pi-u)$ by $\cos u$ in three places.

Before one can do any Maple plots, one needs to understand the meaning of the numerator radicals appearing in the above formula. Consider again the underlying integral (7.7)

$$\int_0^{\infty} d\tau P_{i\tau-1/2}(y) \operatorname{ch}(b\tau) \operatorname{sech}^2(\pi\tau) = (\sqrt{2}/\pi) (1/\sqrt{y-\cos b}) \cot^{-1}[\sqrt{1+\cos b}/\sqrt{y-\cos b}] \quad (7.7)$$

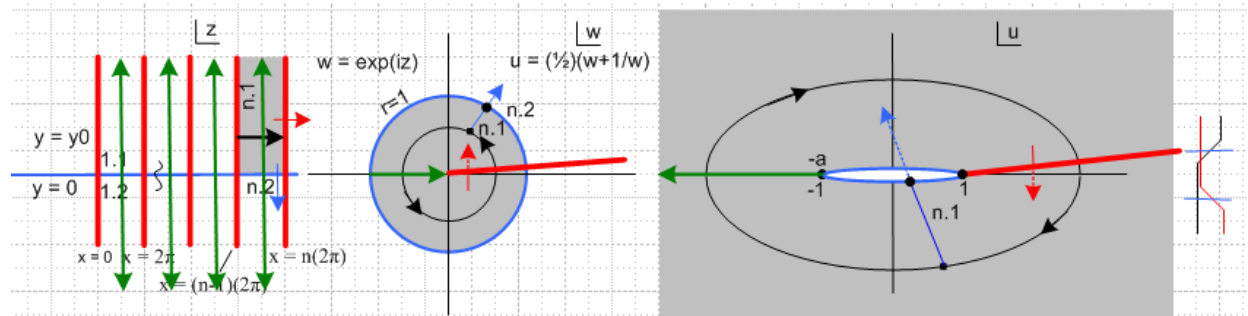
Written in this form, the LHS converges for $|b| < 2\pi$ and so does the RHS and we can regard the equation as matching two functions analytic in parameter b near $b = 0$. If the RHS is written in $\tan^{-1}$ form, the RHS is only analytic for $|b| < \pi$, which is why we use the $\cot^{-1}$ form. As parameter b makes its excursion from $b = 0$ to $b = 2\pi$, it has to pass by the value $b = \pi$, a place where $\sqrt{1+\cos b} = 0$. The correct analytic continuation is to swap branches of $\sqrt{1+\cos b}$ when passing through an odd multiple of π , but this fact is not very obvious when one simply writes " $\sqrt{1+\cos b}$ ". Maple, for example, always assumes the positive branch. This continuation issue arises in both $\cot^{-1}$ numerators $\sqrt{1+\cos u}$ and $\sqrt{1+\cos(2u_0-u)}$ appearing above. To clarify the meaning of our integral evaluation, we formalize this sign change as it occurs in both $\cot^{-1}$ numerators, by writing

$$V(\xi, u) = (V_0/\pi) \sqrt{\text{ch}\xi - \cos u} * \left\{ \begin{aligned} & (1/\sqrt{\text{ch}\xi - \cos u}) \cot^{-1}[(-1)^{\eta_1} \sqrt{1+\cos u} / \sqrt{\text{ch}\xi - \cos u}] \\ & + (1/\sqrt{\text{ch}\xi - \cos(2u_0-u)}) \cot^{-1}[(-1)^{\eta_2} \sqrt{1+\cos(2u_0-u)} / \sqrt{\text{ch}\xi - \cos(2u_0-u)}] \end{aligned} \right\} \quad (8.3)$$

$$\begin{aligned} \text{where} \quad \eta_1 &= \text{floor}[(3\pi-u)/2\pi] & u_0 &\text{ in } (0, \pi) \\ \eta_2 &= \text{floor}[(2u_0-u+\pi)/2\pi] & u &\text{ in } (u_0, u_0+2\pi) \end{aligned}$$

This form provides the correct analytic continuation of our RHS to any real value of u , even beyond the point where the LHS integral actually converges, though we only use it in the convergent region.

In order to rigorously verify this analytic continuation, one has to study the Riemann sheet structure of the function $f(z) = \sqrt{1 + \cos(z)}$. One first looks at $u(z) = \cos(z)$ which is plenty complicated by itself, and then one has to add a square root branch cut to get to $f(z)$. Here is the kind of picture one ends up having to ponder



The black arrow in the z plane crosses $\text{Re}(z) = \text{an odd multiple of } \pi$, and we can trace this in black to an elliptical "ant trail" in the u -plane. But the green cut is that from the radical in $\sqrt{1 + \cos(z)}$ and to analytically continue one must dive through the green cut to arrive at a "the other sheet" of this function, where it has the opposite sign. See Ahlfors's Chapter 3 on analytic mapping.

9. Using Maple to plot the bowl potential

We use Maple because we have it; any computer algebra system will do, though the syntax changes slightly. The code text is given in Appendix A below, here we use screen clips. The first order of business is just to enter the evaluated bowl potential and set a few parameter values,

```

> restart;
Construct the charged bowl potential in toroidal coordinates
> A := sqrt(cosh(xi)-cos(u));
> B := sqrt(1 + cos(u));
> C := sqrt(cosh(xi)-cos(2*u0-u));
> E := sqrt(1+cos(2*u0-u));
> eta1 := floor((3*Pi-u)/(2*Pi));
> eta2 := floor((2*u0-u+Pi)/(2*Pi));
> T1 := (1/A)*arccot((-1)^eta1*(B/A));
> T2 := (1/C)*arccot((-1)^eta2*(E/C));
> V := (V0/Pi)*A*(T1+T2);

```

$$V := \frac{V0 \sqrt{\cosh(\xi) - \cos(u)} \left(\frac{\arccot\left(\frac{(-1)^{\left\lfloor \frac{1}{2} \frac{3\pi - u}{\pi} \right\rfloor} \sqrt{1 + \cos(u)}}{\sqrt{\cosh(\xi) - \cos(u)}}\right)}{\sqrt{\cosh(\xi) - \cos(u)}} + \frac{\arccot\left(\frac{(-1)^{\left\lfloor \frac{1}{2} \frac{2u0 - u + \pi}{\pi} \right\rfloor} \sqrt{1 + \cos(-2u0 + u)}}{\sqrt{\cosh(\xi) - \cos(-2u0 + u)}}\right)}{\sqrt{\cosh(\xi) - \cos(-2u0 + u)}} \right)}{\pi}$$

```

> V0 := 1;
> a := 1;
> u0 := evalf(2*Pi/8);

```

$u0 = .7853981635$

Next, we need a special routine which takes a point (x,y) on a circle and returns a $\tan^{-1}$ angle which lies in the range $(0, 2\pi)$, where the angle is measured CCW away from the x axis. Maple has an internal function that does something like this, but we want to see what is happening, thus

Routine arctan2Pi

Given (x,y) somewhere on a circle, return the angle in $(0, 2\pi)$ measured CCW from the x axis.

Warning: returned result may include unevaluated multiples of Pi

```

> arctan2Pi := proc(x,y)
    local q;
    if type(x,numeric) and type(y,numeric) then
        if x = 0 and y = 0 then print("arctan2Pi(0,0) error." ); RETURN(0) fi;
        if x = 0 and y > 0 then RETURN(Pi/2) fi;
        if x = 0 and y < 0 then RETURN(3*Pi/2) fi;
        if x > 0 and y = 0 then RETURN(0) fi;
        if x < 0 and y = 0 then RETURN(Pi) fi;
        if x > 0 and y > 0 then q := 0 fi;
        if x < 0 and y > 0 then q := Pi fi;
        if x < 0 and y < 0 then q := Pi fi;
        if x > 0 and y < 0 then q := 2*Pi fi;
        RETURN(arctan(y/x)+q);
    else
        'arctan2Pi(x,y)';
    fi;
end:

```

Lastly, but very importantly, we have to figure out how to compute toroidal parameter u from Cartesian coordinates ρ and z (really cylindrical coordinates, but just think $\rho = x$). In the toroidal equations section above we wrote in (5.3)

$$\tan u = [-2az/(a^2 - \rho^2 - z^2)] \quad \Rightarrow \quad u = \tan^{-1}[-2az/(a^2 - \rho^2 - z^2)]$$

but the letters " $\tan^{-1}$ " do not indicate which branch of $\tan^{-1}()$ we need to be on for a given value of ρ and z . This information is not present in the equation above. For that reason, we compute $\cos u$ and $\sin u$ separately from these equations

$$\xi = \tanh^{-1}[2a\rho/(a^2 + \rho^2 + z^2)] \quad (5.3)$$

$$\begin{aligned} \rho &= a \sinh \xi / (\cosh \xi - \cos u) \\ z &= a \sin u / (\cosh \xi - \cos u) \end{aligned} \quad (5.2)$$

and then use the $(0, 2\pi)$ `arctan2Pi` routine shown above to get the proper u . Finally, we have to adjust this value of u so it fits into the Lebedev-adjusted range $(u_0, u_0 + 2\pi)$. And we have to worry about all the special cases that tend to blow things up when ignored. So here then is a routine to compute u :

Routine `getu`

Given a, u_0, ρ, z , compute toroidal bowl-label parameter u in range $(u_0, u_0 + 2\pi)$

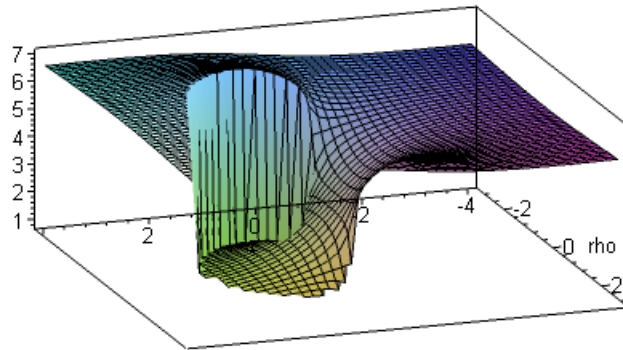
```
> getu := proc(rho, z)
  global a, u0;
  local xi, cosu, sinu, t1;
  if type(rho, numeric) and type(z, numeric) then
    xi := arctanh(2*a*rho/(a^2+rho^2+z^2));
    if rho = 0 then
      cosu := (z^2 - a^2)/(z^2+a^2);
      sinu := 2*a*z/(z^2+a^2);
    else
      cosu := cosh(xi) - (a/rho)*sinh(xi);
      sinu := (z/rho)*sinh(xi);
    fi;
    t1 := evalf(arctan2Pi(cosu, sinu));
    if t1 < u0 then t1 := t1 + evalf(2*Pi) fi; # get into proper range
    RETURN(t1);
  else
    'getu(rho, z)';
  fi;
end;
```

where the type functions relate to an obscure evaluation quirk of Maple and should be ignored. We used this `getu` function to plot the surfaces of u shown earlier: (you may have to stare at this picture for awhile, it is a camera shot from below the plane)

```

u := getu(rho,z):
plot3d(u, rho = -3..3, z = -4..4, axes=BOXED, numpoints=2000);

```



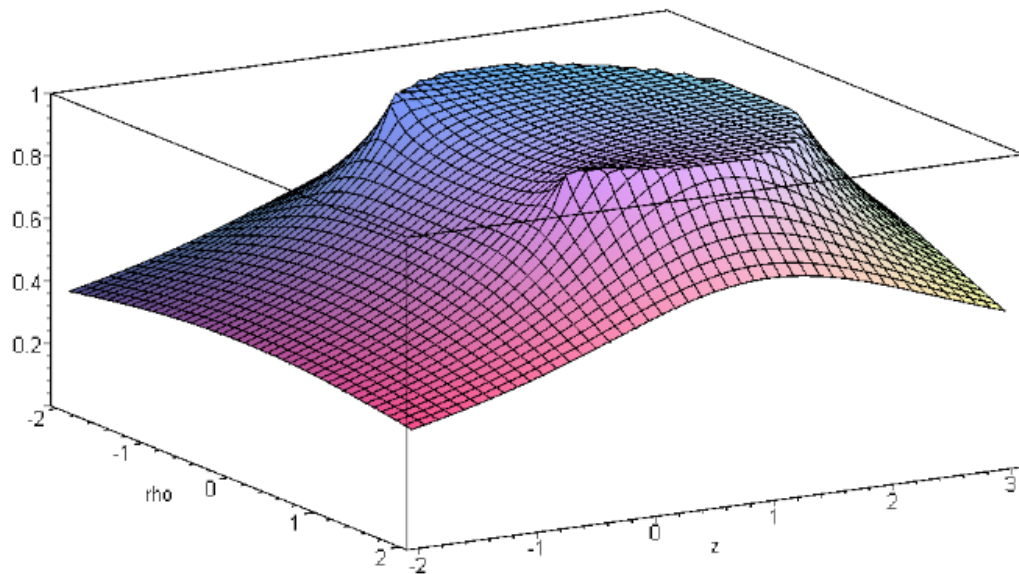
Finally, here is the code to plot the potential surface:

Plot the potential of the charged bowl (an azimuthal slice)

```

> xi := arctanh(2*a*abs(rho)/(a^2+rho^2+z^2)):
> u := getu(rho,z):
> plot3d(V, rho = -2..2, z = -2..3, numpoints=2000, axes=BOXED, view=0..1);

```



10. Potential and capacitance of a charged toroid.

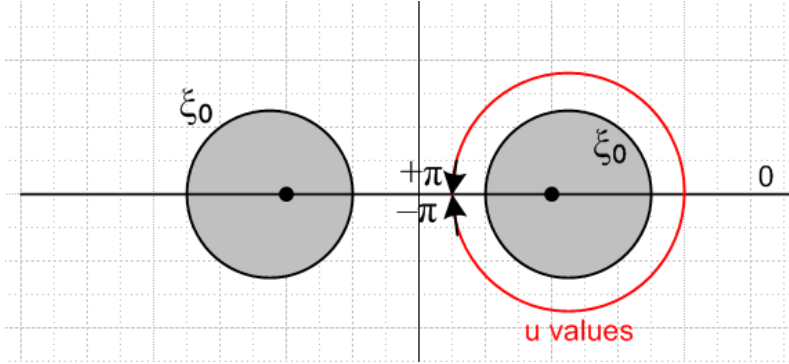
The solution to the charged toroid problem provides an interesting comparison to the bowl problem.

(a) The potential of a charged toroid

Looking back at the two toroidal atomic forms, we can see that for problems involving a toroid as boundary ($\xi = \xi_0$) we must be oscillatory in the other two coordinates u and ϕ . Thus we select

$$(1) \quad \sqrt{\text{expo}} \quad \text{osc} \quad \text{osc} \\ \sqrt{\text{ch}\xi - \cos u} \quad [P_{n-1/2}^m(\text{ch}\xi), Q_{n-1/2}^m(\text{ch}\xi)] \quad [\sin(nu), \cos(nu)] \quad [\sin(m\phi), \cos(m\phi)]$$

It is convenient for this problem to put bowl label u in the range $(-\pi, \pi)$ as suggested by this picture



Unlike the situation with the bowl problem, the red path of u values around the grey toroid core is unobstructed so the potential must be periodic in u with period 2π , and this fact causes quantization to integers of the parameter n appearing in the atomic form. Furthermore, we can see that with this range of u , the potential must be symmetric $V(\xi, u) = V(\xi, -u)$, so only the $\cos(nu)$ term survives. Thus we quickly arrive at the following Smythian form for the potential of a charged toroid,

$$V(\xi, u) = \sqrt{\text{ch}\xi - \cos u} \sum_{n=0}^{\infty} P_{n-1/2}(\text{ch}\xi) A_n \cos(nu) \quad (10.1)$$

where A_n are coefficients to be determined. As in the bowl case, the Q function is ruled out since it diverges at $\xi = 0$, whereas $\xi \rightarrow 0$ is part of the condition for being "far away" where the potential must vanish. (The other part of that condition is $u \rightarrow 0$.)

The boundary condition of constant potential on the toroid of label ξ_0 is this

$$V_0 / \sqrt{\text{ch}\xi_0 - \cos u} = \sum_{n=0}^{\infty} P_{n-1/2}(\text{ch}\xi_0) A_n \cos(nu) \quad (10.2)$$

Armed with knowledge of the Fourier series expansion and projection of $1/\sqrt{a - b \cos(x)}$

$$\begin{aligned} 1/\sqrt{a - b \cos(x)} &= (1/\pi) \sqrt{2/b} \sum_{n=0}^{\infty} \epsilon_n Q_{n-1/2}(a/b) \cos(nx) && // \text{ expansion} \\ \int_0^\pi dx \cos(nx) / \sqrt{a - b \cos(x)} &= \sqrt{2/b} Q_{n-1/2}(a/b) && // \text{ projection} \end{aligned} \quad (10.3)$$

(see GR7 p961 8.713.1 with $\mu = 0$ and $\nu = n-1/2$) and using

$$\int_0^\pi du \cos(nu) \cos(n'u) = (\pi/\epsilon_n) \delta_{nn}, \quad \text{where } \epsilon_n = 2 - \delta_{n,0} = \text{"Neumann's Factor"} \text{ (Watson)}$$

one can easily invert (10.2) to find coefficient A_n

$$A_n = V_0 (\sqrt{2} \epsilon_n / \pi) [Q_{n-1/2}(\text{ch} \xi_0) / P_{n-1/2}(\text{ch} \xi_0)] \quad (10.4)$$

and thus one arrives at the potential for a conducting charged toroid,

$$V(\xi, u) = V_0 (\sqrt{2} / \pi) \sqrt{\text{ch} \xi - \cos u} \sum_{n=0}^{\infty} \epsilon_n P_{n-1/2}(\text{ch} \xi) [Q_{n-1/2}(\text{ch} \xi_0) / P_{n-1/2}(\text{ch} \xi_0)] \cos(nu) \quad (10.5)$$

which can be compared with Morse and Feshbach p 1304 ($\xi = \mu$ and $u = \eta$)

$$\psi = \frac{V_0}{\pi} \sqrt{2(\cosh \mu - \cos \eta)} \sum_{n=0}^{\infty} \left[\frac{Q_{n-1/2}(\cosh \mu_0)}{P_{n-1/2}(\cosh \mu_0)} \right] P_{n-1/2}(\cosh \mu) \cos(n\eta) \quad (10.3.80)$$

// wrong

where the ϵ_n factor has been erroneously omitted. I asked Mark Feshbach if he knew of an errata collection for the massive 2000 page masterwork coauthored by his father, but he did not.

(b) Capacitance of a toroid

Since large r means $\xi \rightarrow 0$ and $u \rightarrow 0$, we can set $P_{n-1/2}(\text{ch} \xi) = 1$ and $\cos(nu) = 1$ in (10.5) to get

$$V(\xi, u) \approx V_0 (\sqrt{2} / \pi) \sqrt{\text{ch} \xi - \cos u} \sum_{n=0}^{\infty} \epsilon_n [Q_{n-1/2}(\text{ch} \xi_0) / P_{n-1/2}(\text{ch} \xi_0)] \quad (10.6)$$

As in the bowl problem, we know from (4.11) that $\sqrt{\text{ch} \xi - \cos u} \approx \sqrt{2} a / r$ in the "far limit", so

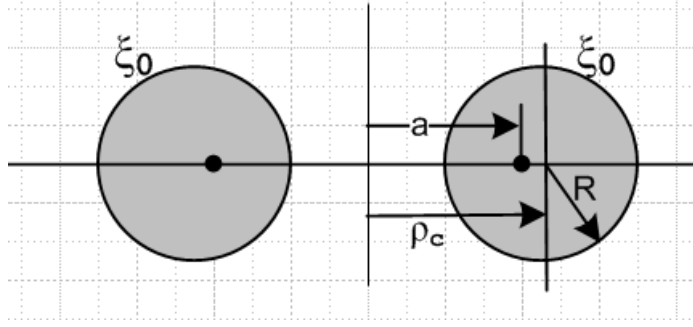
$$V(\xi, u) \approx V_0 (\sqrt{2} / \pi) \sqrt{2} a / r * \sum_{n=0}^{\infty} \epsilon_n [Q_{n-1/2}(\text{ch} \xi_0) / P_{n-1/2}(\text{ch} \xi_0)] \quad (10.7)$$

Thinking of this as $V = Q/r$ and using $Q = CV_0$ we find

$$Q = (2V_0 a / \pi) \sum_{n=0}^{\infty} \epsilon_n [Q_{n-1/2}(\text{ch} \xi_0) / P_{n-1/2}(\text{ch} \xi_0)]$$

$$C = (2a / \pi) \sum_{n=0}^{\infty} \epsilon_n [Q_{n-1/2}(\text{ch} \xi_0) / P_{n-1/2}(\text{ch} \xi_0)] \quad (10.8)$$

Sometimes this result is expressed in terms of R and ρ_c shown here (R is the tube radius, ρ_c is the toroid radius to the center line of the tube)



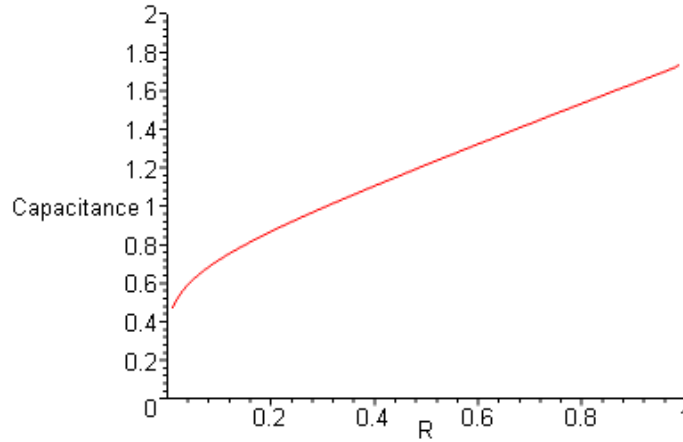
Toroidal equations (5.5) show that

$$\rho_c = a \coth \xi_0 \quad R = a / \sinh \xi_0 \quad \Rightarrow \quad \sqrt{\rho_c^2 - R^2} = a, \quad \rho_c/R = \cosh \xi_0 \quad (10.9)$$

so that the C formula can be written

$$C = (2/\pi) \sqrt{\rho_c^2 - R^2} \sum_{n=0}^{\infty} \epsilon_n [Q_{n-1/2}(\rho_c/R) / P_{n-1/2}(\rho_c/R)] \quad (10.10)$$

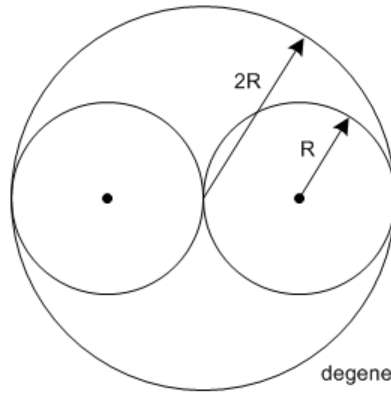
Setting $\rho_c = 1$, we can have Maple plot C as a function of R:



$R/\rho_c = 1$ describes a degenerate toroid where the hole just disappears (upper right of plot), and $R \rightarrow 0$ gives a limiting thin wire ring (lower left of plot). Both these limits are quite fascinating as discussed in some detail in Appendix B below.

The degenerate toroid limit $R=1$ gives the mysterious value $C = 1.7413$. The author would love to know if this number is a simple function of π , small integers and simple roots (see Appendix B).

A comparison of this degenerate toroid to a sphere which just encloses it ($r=2R$)



degenerate torus enclosed by a sphere, cross section

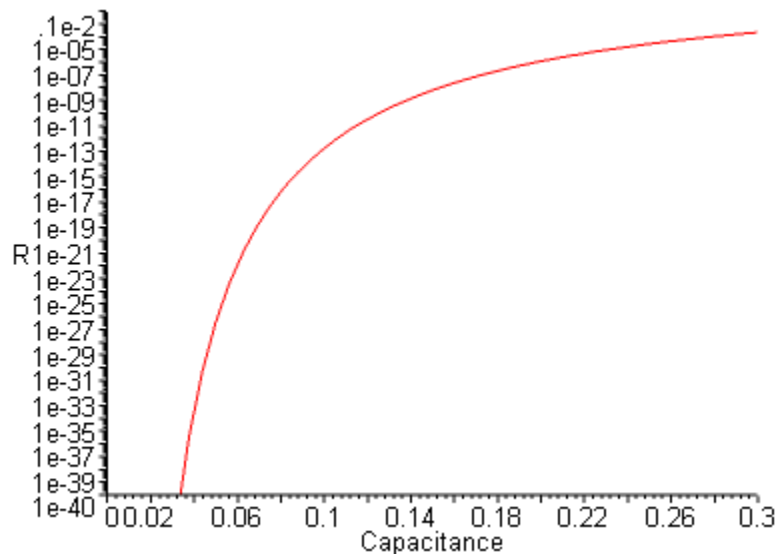
reveals these facts

$$\begin{array}{llll} C_{\text{toroid}} & = 1.7413 & & // \text{mystery number} \\ C_{\text{sphere}} & = 2.0000 & \text{ratio sphere/toroid} = 2.0000/1.741 & = 1.149 // C = (2R) \end{array}$$

$$\begin{array}{llll} \text{AREA}_{\text{toroid}} & = 4\pi^2 R^2 & & // \text{area} = 4\pi^2 R \rho_c \\ \text{AREA}_{\text{sphere}} & = 16\pi R^2 & \text{ratio sphere/toroid} = 4/\pi = 1.273 & // \text{area} = 4\pi(2R)^2 \end{array}$$

So the sphere has 27% more area and 15% more capacitance than the enclosed degenerate toroid. We expect the sphere to have more capacitance since it is the optimal shape for keeping the charges apart.

As for the thin-wire-ring limit (left edge of the above plot), here is more detail (notice that the axes are now switched)



If we go from $R/\rho_c = 10^{-3}$ to $R/\rho_c = 10^{-40}$, capacitance drops only by a factor of 10, so one might say that a wire ring "holds its capacitance quite well" as the wire gets thinner and thinner. Ultimately the capacitance goes to 0 as the wire vanishes out of existence. This can be compared to the capacitance of a metal sphere which is shrunk to a point, in which case $C = R \rightarrow 0$ in a more reasonable fashion.

If a metal sphere carrying fixed charge Q is gradually shrunk to a point, work must be done to get the charges closer together which raises the sphere's potential V relative to infinity ($E = CV^2/2 = QV/2$). The charges are all piled on top of each other in the limit, so $C = Q/V$ goes to 0 quickly. In the toroid case, the charges can stay away from each other to some extent by being spread out around the wire ring, so V rises more slowly as the ring is made thinner.

[The slowness of the thin ring limit approach is due to the fact that the first term in the series for C dominates, and $Q_{-1/2}(z)/P_{-1/2}(z) \rightarrow (\pi^2/2) (1/\ln[8z])$ as $z \rightarrow \infty$, so $C(R; \rho_c=1) \approx \pi / \ln(8/R)$. Inverting one gets $R = 8 e^{-\pi/C}$ so that $\ln R = \ln 8 - \pi/C$ which is the plot shown above.]

(c) Charge density on a toroid

Using this formula (plus sign because ξ decreases moving outward from the toroid surface)

$$\sigma = + (1/4\pi) (1/h_\xi) \partial_\xi V(\xi, u) |^{\xi=\xi_0} \quad 1/h_\xi = (ch\xi - \cos u)/a \quad (10.11)$$

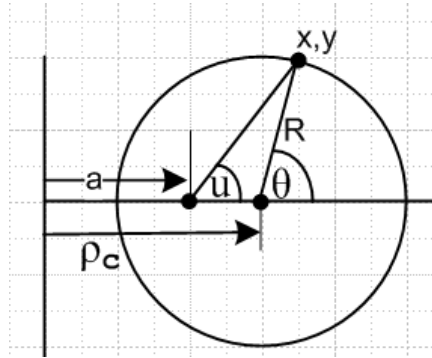
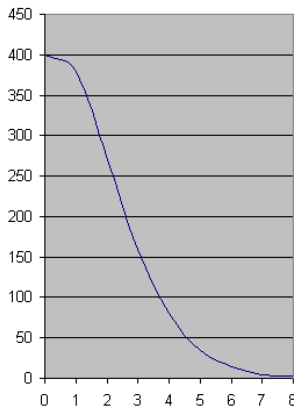
and the potential stated above

$$V(\xi, u) = V_0 (\sqrt{2}/\pi) \sqrt{ch\xi - \cos u} \sum_{n=0}^{\infty} \epsilon_n P_{n-1/2}(ch\xi) [Q_{n-1/2}(ch\xi_0)/P_{n-1/2}(ch\xi_0)] \cos(nu) \quad (10.5)$$

we get the following expression for the charge density σ on a charged toroid with label ξ_0 , ($R = a/sh\xi_0$)

$$\sigma(u) = (V_0/4\pi R) * \{ (\sqrt{2}/\pi) (\sqrt{ch\xi_0 - \cos u})^3 \sum_{n=0}^{\infty} \epsilon_n P_{n-1/2}'(ch\xi_0) [Q_{n-1/2}(ch\xi_0)/P_{n-1/2}(ch\xi_0)] \cos(nu) + (1/2) \} \quad (10.12)$$

where $P_{n-1/2}'(z)$ means $\partial_z P_{n-1/2}(z) = (z^2-1)^{-1/2} [P_{n+1/2}(z) - z P_{n-1/2}(z)]$. In the large z limit, it can be shown that $\sigma = (V_0/4\pi R) z_0 / \ln(8z_0)$, so for a thin wire the linear charge density is $\lambda = 2\pi R\sigma$ and then the total charge is $Q = 2\pi\rho_c \lambda = \pi V_0 / R \ln(8/R)$, consistent with the capacitance limit found above when $\rho_c=1$. Here is a plot of $\sigma(u)$ for $\xi_0 = \pi/4$, a toroid marked in our bipolar coordinates picture above, where the horizontal axis shows u in units of $\pi/8$ and the height is relative units (left picture)



One can convert from angle u to angle θ as shown using the following equations

$$\sin\theta = \text{sh}\xi_0 \sin u / (\text{ch}\xi_0 - \cos u) \quad \cos\theta = (\text{ch}\xi_0 \cos u - 1) / (\text{ch}\xi_0 - \cos u) \quad (10.13)$$

The main idea in the $\sigma(u)$ sum is that near $u = 0$ the terms are additive since $\cos(nu) \sim 1$, whereas in the "backward direction" especially near $u = \pi$ there is term interference from $\cos(nu) \sim (-1)^n$, causing the charge to concentrate on the outer toroidal surface just as one would expect. This situation is akin to the forward peak in a scattering amplitude in partial wave analysis. The plot was made from data computed in Maple using 20 terms and confirming term stability as discussed in Appendix B.

Appendix A. Maple code text for plotting the Bowl Potential

The following code can be cut from this document and pasted into the Maple V R5 white worksheet window. It then runs when you hit the enter key and plots should appear. Hopefully this also works in more current versions of Maple. See author's Maple User's Guide for how to break code into separate execution groups.

```
restart;
# Construct the charged bowl potential in toroidal coordinates
A := sqrt(cosh(xi) - cos(u)):
B := sqrt(1 + cos(u)):
C := sqrt(cosh(xi) - cos(2*u0-u)):
E := sqrt(1+cos(2*u0-u)):
eta1 := floor((3*Pi-u)/(2*Pi)):
eta2 := floor((2*u0-u+Pi)/(2*Pi)):
T1 := (1/A)*arccot((( -1)^eta1)*(B/A)):
T2 := (1/C)*arccot((( -1)^eta2)*(E/C)):
V := (V0/Pi)*A*(T1+T2);
V0 := 1:
a := 1:
u0 := evalf(2*Pi/8);

# Routine arctan2Pi
# Given (x,y) somewhere on a circle, return the angle in (0,2Pi) measured CCW from the
# axis.
# Warning: returned result may include unevaluated multiples of Pi
arctan2Pi := proc(x,y)
    local q;
    if type(x,numeric) and type(y,numeric) then
        if x = 0 and y = 0 then print("arctan2Pi(0,0) error." ); RETURN(0) fi;
        if x = 0 and y > 0 then RETURN(Pi/2) fi;
        if x = 0 and y < 0 then RETURN(3*Pi/2) fi;
        if x > 0 and y = 0 then RETURN(0) fi;
        if x < 0 and y = 0 then RETURN(Pi) fi;
        if x > 0 and y > 0 then q := 0 fi;
        if x < 0 and y > 0 then q := Pi fi;
        if x < 0 and y < 0 then q := Pi fi;
        if x > 0 and y < 0 then q := 2*Pi fi;
        RETURN(arctan(y/x)+q);
    else
        'arctan2Pi(x,y)';
    fi;
end:

# Routine getu
# Given a,u0, rho,z, compute toroidal bowl-label parameter u in range (u0,u0+2Pi)
```

```

getu := proc(rho,z)
  global a,u0;
  local xi,cosu,sinu,t1;
  if type(rho,numeric) and type(z,numeric) then
    xi := arctanh(2*a*rho/(a^2+rho^2+z^2));
    if rho = 0 then
      cosu := (z^2- a^2)/ (z^2+a^2);
      sinu := 2*a*z/(z^2+a^2);
    else
      cosu := cosh(xi) - (a/rho)*sinh(xi);
      sinu := (z/rho)*sinh(xi);
    fi;
    t1 := evalf(arctan2Pi(cosu,sinu));
    if t1 < u0 then t1 := t1 + evalf(2*Pi) fi; # get into proper range
    RETURN(t1);
  else
    'getu(rho,z)';
  fi;
end:

# Plot the potential of the charged bowl (an azimuthal slice)
xi := arctanh(2*a*abs(rho)/(a^2+rho^2+z^2));
u := getu(rho,z);
plot3d(V, rho = -2..2, z = -2..3,numpoints=2000,axes=BOXED, view=0..1);

u := getu(rho,z);
plot3d(u, rho = -3..3, z = -4..4,axes=BOXED,numpoints=2000);

```

Appendix B: Comments on computing the capacitance of a toroid and its degenerate limit.

The capacitance formula (10.10) given above is this,

$$C = (2R/\pi) \sqrt{z^2 - 1} \sum_{n=0}^{\infty} \epsilon_n [Q_{n-1/2}(z) / P_{n-1/2}(z)] \quad z = \text{ch}\xi = \rho_c/R \quad (\text{B.1})$$

It is convenient to remove the leading factor and talk about this function

$$T(z) \equiv \sqrt{z^2 - 1} \sum_{n=0}^{\infty} \epsilon_n [Q_{n-1/2}(z) / P_{n-1/2}(z)] \quad (\text{B.2})$$

Basically this is a "special function" that has no name. We can call it the "toroidal capacitance function". It is defined for our purposes on the range z in $(1, \infty)$. One can see from this fact

$$Q_\nu(z) / P_\nu(z) \approx \pi \{ \Gamma(1+\nu)^2 / [\Gamma(\nu+3/2) \Gamma(\nu+1/2)] \} (2z)^{-2\nu-1} \quad (\text{B.3})$$

that the series converges for $z > 1$. It is a simple matter to have Maple compute $T(z)$ for a variable number of terms to make sure enough terms are included to give a stabilized result. As $z \rightarrow 1$, the number of required terms increases ($Q_\nu(z)$ has a log singularity at $z=1$). The magic question is to find the limit $T(z=1)$, if it exists. This is the limit which determines the capacitance of a toroid for which the hole has just gone out of existence. This physical interpretation suggests that the limit must exist.

We use appropriate definitions of the P and Q functions as follows, and do some cursory checks:

P and Q functions for $z \geq 1$

For P, we select Bateman (24) which says this

$$P_\nu(z) = z^\nu F(-\nu/2, 1/2-\nu/2; 1; 1-1/z^2)$$

$$P(\nu, \xi) \equiv P_\nu(\cosh \xi) = (\cosh \xi)^\nu F(-\nu/2, 1/2-\nu/2; 1; \tanh^2 \xi)$$

> restart;

> P := (nu, xi) -> evalf(cosh(xi)^nu*hypergeom([-nu/2, 1/2-nu/2], [1], tanh(xi)^2));

$$P = (\nu, \xi) \rightarrow \text{evalf}\left(\cosh(\xi)^\nu \text{hypergeom}\left(\left[-\frac{1}{2}\nu, \frac{1}{2}-\frac{1}{2}\nu\right], [1], \tanh(\xi)^2\right)\right)$$

Now show that these two functions agree at least somewhere!

> evalf(P(3, 0.5));

1.893115746

> evalf(LegendreP(3, cosh(0.5)));

1.893115745

For Q, we select Bateman (36) which says this

$$Q_\nu(z) = 2^\nu [\Gamma(1+\nu)]^2 (z+1)^{-\nu-1} / \Gamma(2+2\nu) * F(1+\nu, 1+\nu, 2+2\nu, 2/(1+z))$$

$$Q(\nu, \xi) = Q_\nu(\cosh \xi) = 2^\nu [\Gamma(1+\nu)]^2 (\cosh \xi + 1)^{-\nu-1} / \Gamma(2+2\nu) * F(1+\nu, 1+\nu, 2+2\nu, 2/(1+\cosh \xi))$$

> Q := (nu, xi) ->

evalf(2^(nu)*GAMMA(1+nu)^2*(cosh(xi)+1)^(-nu-1)*(1/GAMMA(2+2*nu))*hypergeom([1+nu, 1+nu], [2+2*nu], 2/(1+cosh(xi))));

>

$$Q = (\nu, \xi) \rightarrow \text{evalf}\left(\frac{2^\nu \Gamma(1+\nu)^2 (\cosh(\xi) + 1)^{(-\nu-1)} \text{hypergeom}\left([1+\nu, 1+\nu], [2+2\nu], 2 \frac{1}{\cosh(\xi) + 1}\right)}{\Gamma(2+2\nu)}\right)$$

Now show that these two functions agree at least somewhere!

> evalf(Q(1, 1));

.1911607780

> evalf(LegendreQ(1, cosh(1)));

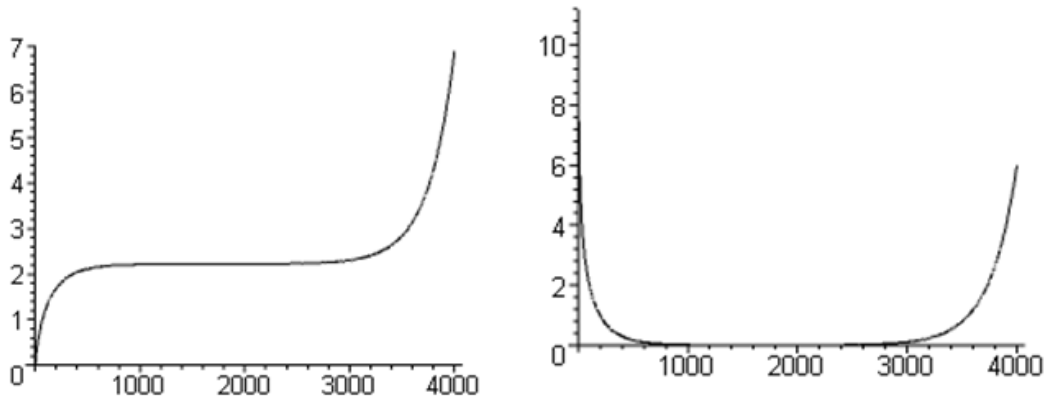
.191160779

>

In order to study the behavior of sums involving P and Q functions, we first considered this "sum rule" (which is the 10.3 expansion above with $x = 0$, $b = 1$ and $a = z$)

$$R(z) \equiv \sqrt{z-1} \sum_{n=0}^{\infty} \varepsilon_n Q_{n-1/2}(z) = (\pi/\sqrt{2}) = 2.221441469 \quad (\text{B.4})$$

Our Maple program was able to obtain the correct sum to three decimal places down to $z = 1.00001$ where it had to add 2000 terms. Because this series is in a sense only half as convergent as our intended Q/P series, the partial sums (left) and terms (right) exhibited characteristics of an asymptotic series, for example [the horizontal axis shows the number of terms added]



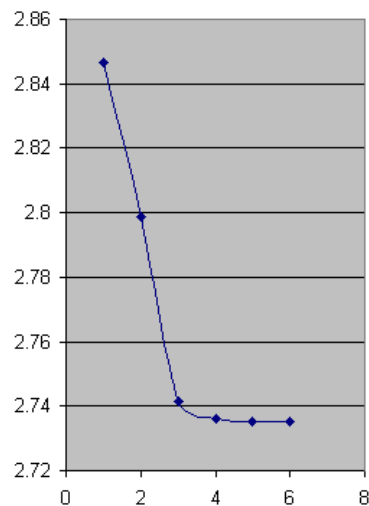
The series $R(z)/\sqrt{z-1}$ really is truly convergent, and this Maple behavior arises (we think) from the way Maple computes the hypergeometric function for very large parameters and z near 1, an area known to be problematic. The Q/P series partial sums (below) did not exhibit this asymptotic series behavior.

After this warmup exercise, we turned to the $T(z)$ series (B.2) and computed it the same way, always with a stable number of terms. Our results were as follows:

z	$T(z)$	terms used	$-\log_{10}(z-1)$
1.1	2.8465	15	1
1.01	2.7987	140	2
1.001	2.7414	300	3
1.0001	2.7359	600	4
1.00001	2.7353	1200	5
1.000001	2.7353	4000	6

(B.5)

suggesting that we are approaching a limit close to 2.7353 for the value of $T(1)$. We can plot the above points as follows:



where the x axis is $-\log_{10}(z-1)$. The point is that we are coming into the limit $T(1)$ with zero slope.

Using the Wronskian of P and Q, one can show that this slope is given by

$$T'(z) = \{ T(z) - \sqrt{z^2 - 1} \sum_{n=0}^{\infty} \epsilon_n / [P_{n-1/2}(z)]^2 \} / (z^2 - 1) \quad (B.6)$$

If we conjecture that $T'(1)$ is not infinite, which the picture certainly suggests, we conclude that

$$T(1) = \lim_{z \rightarrow 1} \{ \sqrt{z^2 - 1} \sum_{n=0}^{\infty} \epsilon_n / [P_{n-1/2}(z)]^2 \} \quad (B.7)$$

and we have then a second way to compute $T(1)$. This series is more stable because it avoids the $\ln(z-1)$ singularity of the Q functions. Using this series Maple yielded the following results

$z = 1.1$	above = 2.898517346	
$z = 1.01$	above = 2.751570937	
$z = 1.001$	above = 2.736973437	
$z = 1.0001$	above = 2.735058209	
$z = 1.00001$	above = 2.735362078	
$z = 1.000001$	above = 2.735354499	10,000 terms
$z = 1.0000001$	above = 2.735353885	40,000 terms

(B.8)

We verified most of these results by jamming strings like the following into the Wolfram Alpha website computation window (the argument 3 indicates off-the-cut Legendre functions in Mathematica)

$$\text{sqrt}(1.0000001^2 - 1) * (2 * \text{sum}(1 / \text{LegendreP}(n - 1/2, 0, 3, 1.0000001)^2, n = 1 \text{ to } 39999) + 1 / \text{LegendreP}(-1/2, 0, 3, 1.0000001)^2)$$

Though it timed out on this case, it did all the other cases listed, confirming Maple's results.

Our conclusion is the following strange fact

$$T(1) \approx 2.735353 \quad (B.9)$$

and therefore we find that the capacitance of a degenerate toroid is given by

$$C(\rho_c = R = 1) = (2/\pi)T(1) \approx 1.741380 \quad (B.10)$$

It is interesting that $T(1)$ is slightly larger than $e \approx 2.718282$, but for sure $T(1) \neq e$. We do not know how to obtain this limit in any other way. We know of no integral representations of inverse Legendre functions, nor of any published series involving them, though PBM has very many series involving Legendre functions and products of same combined with other functions. Perhaps a variation for $z \geq 1$ of Bateman HT I p 149 (11) could be used.

We sometimes read about the interchange of limits giving different results in situations where uniform convergence is lacking in both limits, or where one of the limits does not exist. We have here exactly a case of this second situation:

$$\lim_{z \rightarrow 1} \lim_{N \rightarrow \infty} \{ \sqrt{z^2 - 1} \sum_{n=0}^N \epsilon_n / [P_{n-1/2}(z)]^2 \} = T(1) = 2.735353$$

whereas

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \lim_{z \rightarrow 1} \left\{ \sqrt{z^2 - 1} \sum_{n=0}^N \epsilon_n / [P_{n-1/2}(z)]^2 \right\} \\
&= \sum_{n=0}^{\infty} \epsilon_n \lim_{z \rightarrow 1} \left\{ \sqrt{z^2 - 1} \right\} \quad \text{since } \lim_{z \rightarrow 1} P_v(z) = 1 \\
&= \infty * 0
\end{aligned}$$

The situation is even worse with the P/Q series since $\lim_{z \rightarrow 1} Q_v(z)/P_v(z) = \infty/1 = \infty$.

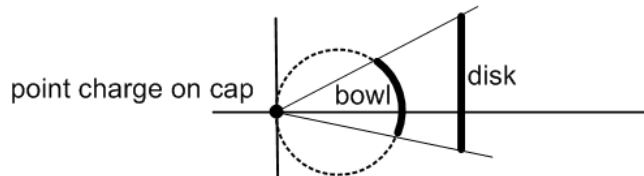
Appendix C: Kelvin's approach to the Charged Bowl Problem.

Lord Kelvin (William Thomson) published his own Collected Works in 1872 and again in 1884 and the latter is the form available on the web as a Microsoft digitized document. The Contents gives section numbers, not page numbers. There are 673 sections filling some 600 pages, and the section numbers just increment through all the many Collected papers.

In the first collected paper (page 1, the ellipsoid paper, "On the Uniform Motion of Heat...", 14p, 1842) Kelvin derives the potential of and charge distribution on a charged conducting ellipsoid and gets the correct result (2.1) above. He bases things on a certain geometric distance p which turns out to be proportional to σ on the ellipsoid: $p = 1/\sqrt{x^2/a^4 + y^2/b^4 + z^2/c^4}$ and $\sigma = Qp/(4\pi abc)$ where a, b, c are the semimajor axes and Q is the total charge on the ellipsoid (see McDonald's paper). His method is very unusual, there is no formal ellipsoidal coordinate system here, but at age 18 he did the whole thing by brute force. He wanders around between the heat, gravitational and electrostatic manifestations of potential theory, but is mostly concerned with heat.

This brings us then to his bowl paper (paper XV, page 178, sections 231-248, "Determination of the distribution...", 14p, 1869). He opens by quoting the above σ formula (he refers to σ by the symbol ρ) and then specializes it first to an elliptical plate then to a circular disk, noting the disk capacitance $C = 2a/\pi$. He finds that $\sigma_{disk}(\rho) = [Q/(4\pi a)] (1/\sqrt{a^2 - \rho^2})$ on each side, where ρ is our modern-day cylindrical coordinate and a the disk radius. This is the key result upon which his bowl theory is built. Kelvin notes retrospectively that George Green obtained this result in 1832. (See green Jackson 3.179 for σ both sides.)

Deviating slightly from Kelvin's order of presentation, we tack sideways and ponder the **inversion** of his charged disk into a grounded spherical bowl with a point charge located on the bowl's cap. The point charge appears at the inversion origin in the bowl space when one adds a constant potential in the disk space to bring the disk potential down to 0 so it will map into a grounded bowl in the bowl space. By shifting the charged disk off the symmetry axis, the point charge can be made to appear at any desired point on the bowl's cap:



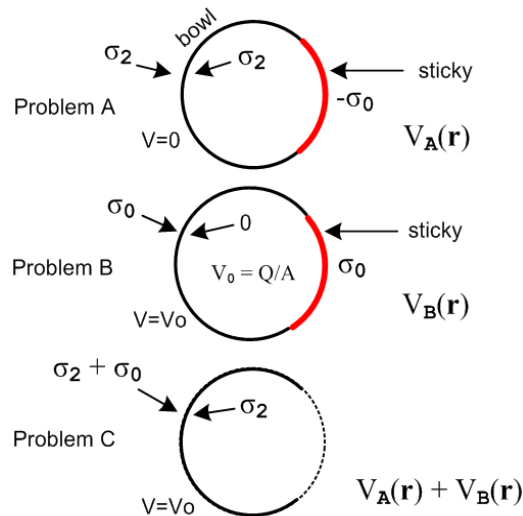
Since Kelvin knows the disk's charge density $\sigma_{\text{disk}}(\rho)$ just stated above, he knows by inversion σ on the bowl. He does not mention the disk's potential, but that maps into the bowl potential by inversion as well. So in the bowl space, we have the grounded bowl with a point charge on the cap, and we know σ on the bowl. Since σ_{disk} is the same on both sides of the disk, σ is the same on both sides of the bowl. Basically, the bowl potential is the Green's Function for the bowl with a restricted placement of the Green's point charge.

Kelvin then does two superpositions.

He **first superposes** an infinite number these Green's Function situations to obtain a target uniform charge density $-\sigma_0$ on the bowl's cap. In doing this the point charge has to be properly scaled for each point in the integration as it is made to wander over the cap region. The corresponding bowl σ 's are also being superposed in this process. One then ends up with the final σ on the bowl (let's call it σ_2 , same on both surfaces) as a superposition integral which can be evaluated into trig and inverse trig functions. One can interpret this σ_2 as the charge induced on each surface of a grounded bowl due to the presence of uniform $-\sigma_0$ on the cap. This induced charge on the bowl is brought in from infinity on the traditional "infinitesimally thin wire" grounding the bowl to The Great Metal Sphere at Infinity.

This cap of charge density $-\sigma_0$ is what we call "sticky charge". It is an infinitely thin layer of charge that is magically glued in place so it cannot move, just as the point charge in a Green's function problem is glued to its location and cannot move.

In a **second superposition**, Kelvin superposes Problem A and Problem B to get Problem C:



Problem A is the grounded conducting bowl of radius R in the presence of a cap of uniform sticky charge $-\sigma_0$ as just described above. The bowl is at $V=0$ and has σ_2 on both surfaces.

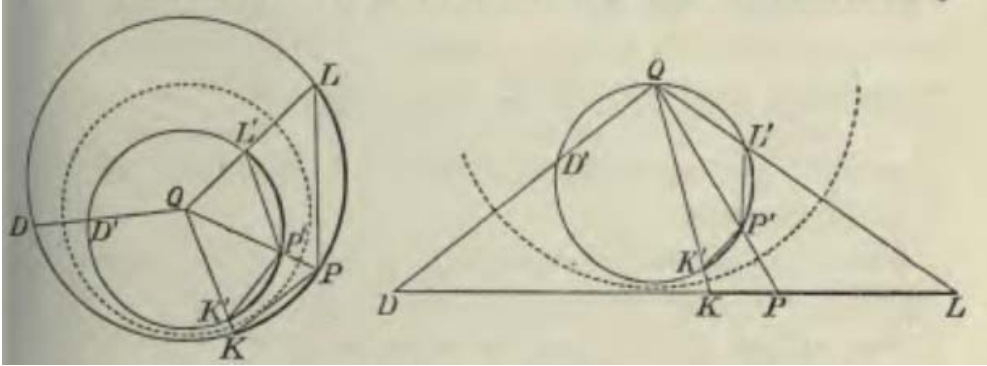
Problem B is the same conducting bowl carefully prepared to have a uniform free charge density $+\sigma_0$ on its outer surface along with a cap of sticky charge also of $+\sigma_0$. Since we have then a full sphere of uniform σ_0 , the potential *on and inside* the entire bowl sphere is $V_0 = Q/R = (\sigma_0 4\pi R^2)/R = 4\pi R \sigma_0 = \text{constant}$. By the same argument used for a full conducting sphere, this bowl must have $\sigma = 0$ on its inner surface. One might prepare this Problem B by first shrink-wrapping a neutral conducting bowl's sphere with a full shell of sticky σ_0 . One then unglues the sticky charge covering just the bowl part of the sphere. The released charges don't move because there is no tangential E field to make them move.

Each of these two problems represents a valid solution to the Laplace equation in the presence of the same conductor. Kelvin forms a third solution by superposing these two problems as Problem C. In this superposed Problem C one has: (1) $V = 0 + V_0 = 4\pi R\sigma_0$ on the bowl ; (2) $\sigma_{in} \equiv \sigma_2 + 0 = \sigma_2$ on the bowl's inner surface; (3) $\sigma_{out} \equiv \sigma_2 + \sigma_0$ on the bowl's outer surface; (4) exact cancellation of the sticky charge on the cap. Notice that $\sigma_{out} - \sigma_{in} = \sigma_0 = V_0/(4\pi R) = \text{a constant}$.

But Problem 3 is recognized as exactly our "charged bowl problem" for a bowl with potential V_0 , and we have just shown that $\sigma_{out} - \sigma_{in} = \sigma_0 = V/(4\pi R) = \text{constant}$, which is the famous result. And of course we know $\sigma_{in} = \sigma_2$ which came with Problem A. The underlying fact is that a Laplace solution is unique, so if you find something that works, that is the answer. Kelvin's result for $\sigma_{in} = \sigma_2$ is stated in (4.9) or (4.11) above.

Kelvin uses his hard-won formulas to compute σ_{in} and σ_{out} at several locations on spherical bowls of various shapes. He does this to 5 decimal places and shows the results in a full page graphic. Had he stopped here, he would have had a great paper, but he had more to say.

Kelvin now knows all about "the charged bowl" problem. Just as he started with a "charged disk" and obtained by inversion the Green's function for a bowl, he now starts with the "charged bowl" and inverts it into either another bowl or a disk, and this time the "inversion-generated point charge" can be made to appear anywhere relative to that bowl or disk. Here are his drawings for these two cases



The inversion sphere is shown dashed, and the point charge is at the inversion origin Q. Thus Kelvin has obtained (in theory) the *fully general Green's Function* for a bowl or a disk. His paper only discusses the charge densities, but the method applies as well to the potential. One can start with the known disk potential (green Jackson p 92 (3.178) with typo corrected)

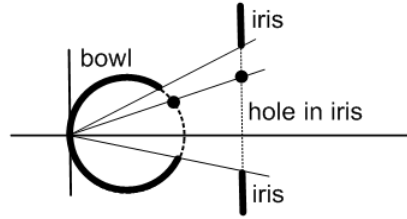
$$V_{disk}(\rho, z) = (q/a) \sin^{-1} [2a / [\sqrt{(\rho-a)^2+z^2} + \sqrt{(\rho+a)^2+z^2}]] \quad (C.1)$$

and process it through all of Kelvin's steps above.

Appendix D: Smythe's approach to the Charged Bowl Problem.

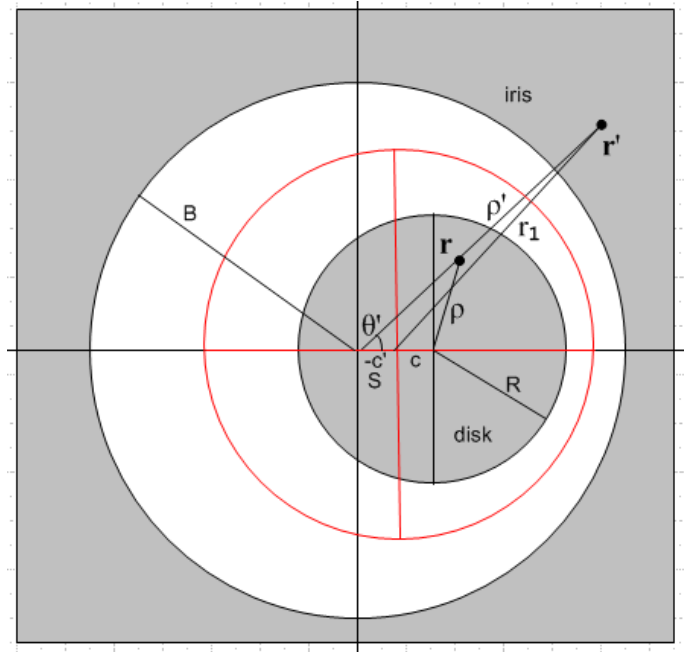
In the Kelvin discussion above we saw an inversion of the on-cap bowl Green's Function problem to an off-axis charged disk problem. If the bowl and its on-cap point charge are rotated together so the bowl surface touches the inversion origin, the same bowl Green's Function problem (with the same inversion

origin and inversion sphere) can be inverted into an iris Green's Function problem with the point charge in the hole of the iris ("iris" being an infinite grounded conducting plane containing a circular hole). With the bowl in this position, the bowl's cap maps into the hole in the iris and a point charge on the cap maps into a point charge in the hole of the iris:



If one could somehow solve this iris Green's Function problem, one could thereby gain full knowledge of the on-cap bowl Green's Function problem, and one could then carry out Kelvin's two superpositions described in the previous Appendix and thereby obtain the potential and charge densities for the charged bowl problem.

This is the approach taken by Smythe in an intriguing set of Problems (38 through 42 starting on page 203 of his 2nd Edition book). The starting point is to figure out the Green's Function for the iris with point charge in the hole, and that can be done by inverting the charged disk using the following planar inversion arrangement (the point charge appears in the iris space at the inversion origin = red crosshair, a distance S from the center of the iris hole)



If the iris space has primed coordinates, the solution to this inversion problem, starting with the Jackson charged-disk potential quoted above, is as follows:

$$V'(\mathbf{r}') = 2q_1/(\pi r_1) \cos^{-1} [\sqrt{2} B r_1 / \{ \sqrt{m + (B^2 - S^2)} \sqrt{[(\rho' - B)^2 + z'^2]} [(\rho' + B)^2 + z'^2] \}] \quad (\text{D.1})$$

where

$$m = (\rho'^2 + z'^2)(S^2 - B^2) + B^2(B^2 - S^2 + 2r_1^2)$$

$$r_1^2 = \rho^2 + S^2 - 2S\rho'\cos\theta' + z'^2 \quad \theta' = 0 \text{ in direction of the point charge}$$

$$\sigma'(\mathbf{r}') = - [q_1 / (2\pi^2 r_1^2)] \sqrt{B^2 - S^2} / \sqrt{\rho'^2 - B^2} \quad // \text{ either side of the iris, } z'=0 \text{ in } r_1 \quad (\text{D.2})$$

where q_1 is the size of the point charge in the hole and B is the radius of the hole. The expression for σ' here is remarkably simple and only appears after considerable brute-force Maple algebra which secretly implements all those Kelvin geometry theorems.

In **Problem 38** Smythe asks his reader to come up with σ' as shown above.

In **Problem 39** he asks the reader to integrate the point charge problem around a circle, to obtain the charge density on the iris for a circular ring of charge centered in the hole. That result is still amazingly simple:

$$\sigma_2(\rho) = - (q/2\pi^2) (\sqrt{B^2 - S^2} / \sqrt{\rho^2 - B^2}) / (\rho^2 - S^2) \quad // \text{ either side} \quad (\text{D.3})$$

where now we remove the primes, and q is the total charge on the ring in the hole.

In **Problem 40** we are instructed to invert this grounded iris + ring charge in hole into a grounded bowl + ring charge on the cap (see first picture above).

Then in **Problem 42** we do a weighted integral of this bowl-cum-ring situation to obtain a uniform charge density on the cap set to the target amount $-\sigma_0$. This corresponds to Kelvin's first superposition described above. We then do Kelvin's second superposition and out pop all the charged bowl results.

Going back now, in **Problem 40** Smythe tells his reader to use "Green's Reciprocation Theorem" to find the potential anywhere on the cap of a charged bowl of potential V_0 . This theorem concerns the charges and potentials on a set of conductors in two "situations", one primed and one unprimed, $\sum_i V_i Q_i' = \sum_i V_i' Q_i$. In our application there are only two metal objects, one is the bowl, the other is an imagined fine wire in the location of the ring of charge on the bowl's cap. One situation is a charged bowl, the other is the bowl + ring-on-cap Green's function. One quickly finds that

$$V(\theta) = V_0(2/\pi) \sin^{-1}[\cos(\alpha/2)/\cos(\theta/2)], \quad (\text{D.4})$$

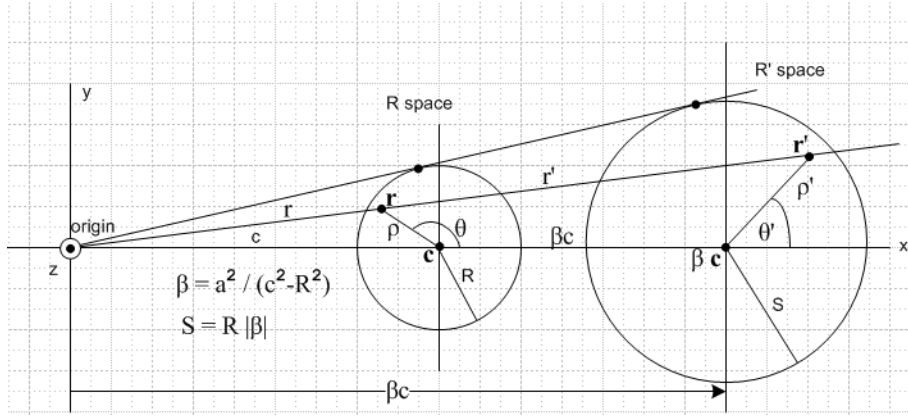
where θ and α are shown in our drawing above equation (4.10).

This result has great significance which is brought out in **Problem 41**. One now knows that $V = V_0$ on the bowl, and $V = V(\theta)$ on the cap, so we have a fully prescribed Dirichlet problem and we can then obtain the potential everywhere by assuming an appropriate Smythian form and inverting to find the coefficients. For the exterior problem that atomic superposition is $V(r, \theta) = \sum_{n=0}^{\infty} A_n r^{-n-1} P_n(\cos\theta)$ and the usual Legendre inversion gives

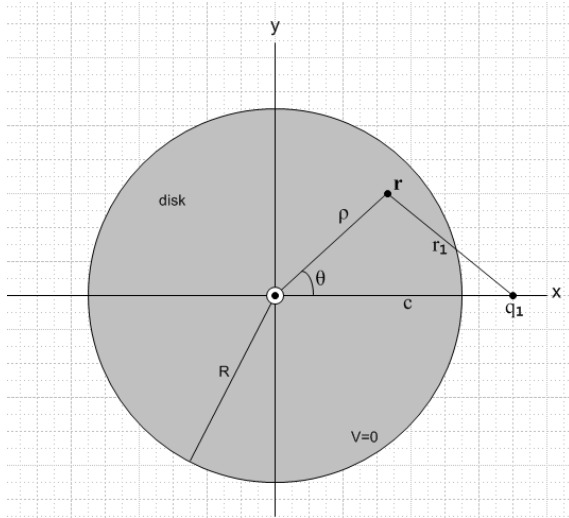
$$A_n = R^{n+1} (2n+1)(1/2) \int_0^\pi P_n(\cos\theta) \sin\theta V(R, \theta, \varphi) d\theta = V_0 R^{n+1} (2n+1)(1/2) \{ (2/\pi) \int_0^\alpha d\theta \sin\theta P_n(\cos\theta) \sin^{-1}[\cos(\alpha/2)/\cos(\theta/2)] + \int_\alpha^\pi d\theta \sin\theta P_n(\cos\theta) \} \quad (\text{D.5})$$

where R is the radius of the bowl's sphere. We find that $A_0 = V_0 (R/\pi) \{ \pi - \alpha + \sin\alpha \}$. For large r , we have $V \rightarrow A_0/r$ so A_0 is in fact the total charge on the bowl as seen from far away, and we find that capacitance is $C = R (1/\pi) \{ \pi - \alpha + \sin\alpha \}$, in agreement with (4.15) above.

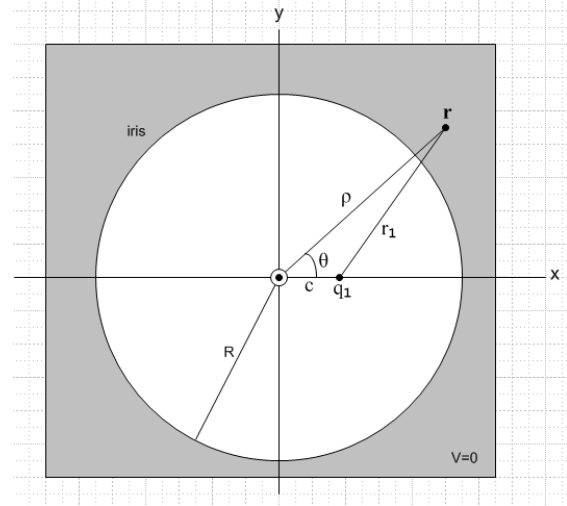
We close this appendix with a remark on a related problem which is the Green's Function problem for a disk with the point charge in the plane of (and outside) the disk. One can solve this disk Green's function problem using an inversion picture like this



where R' space contains a charged disk, and R space holds our desired Green's function problem for the disk with point charge at the marked origin. Note that the point charge is distance c from the center of the grounded disk. When this problem is solved, one finds a striking resemblance between its solution and that of the grounded iris with point charge in the hole. We now compare these two problem solutions



the disk problem



the iris problem

$$V_{\text{disk}}(\rho, \theta, z) = q_1 (2/\pi r_1) \cos^{-1} \left\{ \sqrt{2} R r_1 / \sqrt{m + (c^2 - R^2) \sqrt{[(R-\rho)^2 + z^2][(R+\rho)^2 + z^2]}} \right\}$$

$$m = (\rho^2 + z^2)(c^2 - R^2) + R^2(R^2 - c^2 + 2r_1^2)$$

$$r_1^2 = (\rho^2 + c^2 + z^2 - 2c\rho \cos \theta)$$
(D.6)

$$\sigma_{\text{disk}}(\mathbf{r}) = - q_1 (1/(2\pi^2 r_1^2)) \sqrt{c^2 - R^2} / \sqrt{R^2 - \rho^2}$$

$$r_1^2 = (\rho^2 + c^2 - 2c\rho \cos \theta)$$

// each side

(D.7)

total charge induced on the disk = $-q_1 (2/\pi) \tan^{-1}(R/\sqrt{c^2-R^2})$

$$V_{\text{iris}}(\rho, \theta, z) = q_1(2/\pi r_1) \cos^{-1} \left[\frac{(\sqrt{2} R r_1 / \sqrt{m + (R^2 - c^2) \sqrt{[(\rho-R)^2 + z^2] [(\rho+R)^2 + z^2]}})}{m} \right]$$

$$m = (\rho^2 + z^2)(c^2 - R^2) + R^2(R^2 - c^2 + 2r_1^2)$$

$$r_1^2 = (\rho^2 + c^2 + z^2 - 2c\rho \cos\theta) \quad (\text{D.8})$$

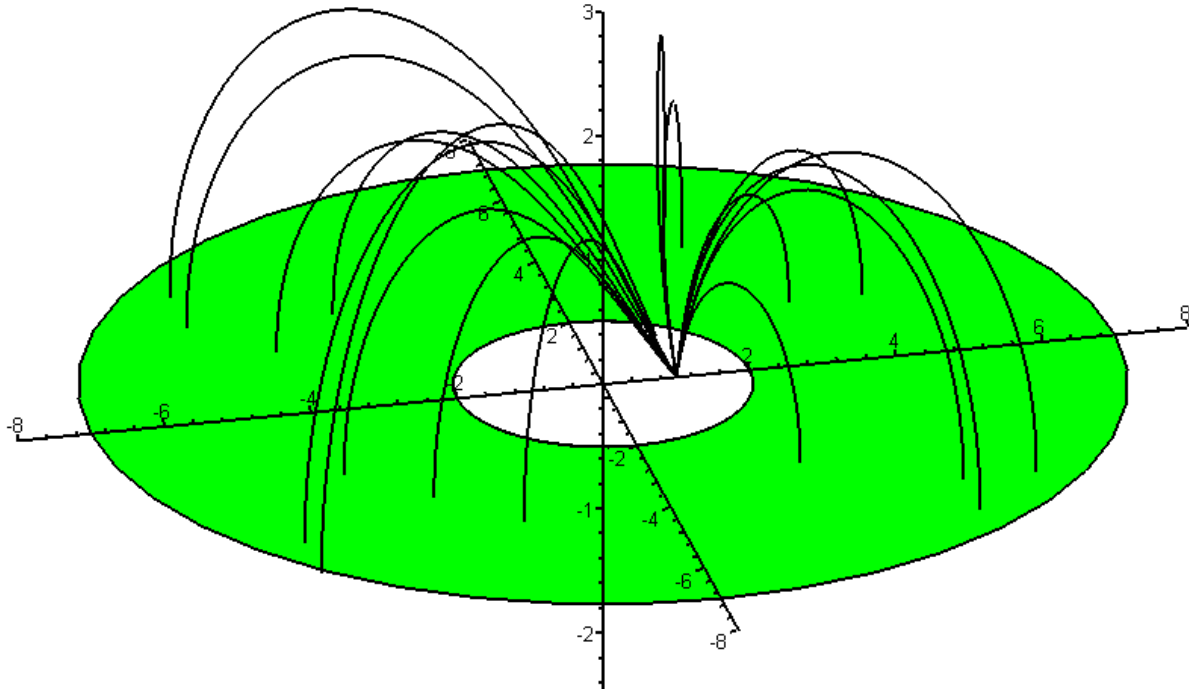
$$\sigma_{\text{iris}}(\rho, \theta) = -q_1 [1/(2\pi^2 r_1^2)] \sqrt{R^2 - c^2} / \sqrt{\rho^2 - R^2} \quad // \text{ each side}$$

$$r_1^2 = (\rho^2 + c^2 - 2c\rho \cos\theta) \quad (\text{D.9})$$

total charge induced on the iris = $-q_1$

It turns out that the two problems are related by analytic continuation of the variable ρ from $\rho < R$ for the disk problem to $\rho > R$ for the iris problem. The path going directly through $\rho = R$ is blocked by a branch cut of $f(\rho) = \sqrt{(\rho-R)^2 + z^2} = \sqrt{(\rho - a_+)(\rho - a_-)} = \sqrt{\rho - a_+} \sqrt{\rho - a_-}$ joining the points $a_{\pm} = R \pm iz$, so one must continue around either of the branch points with the result that $\sqrt{(\rho-R)^2 + z^2} \rightarrow -\sqrt{(R-\rho)^2 + z^2}$, and this is the only difference in the disk and iris solution sets shown above.

Using the iris Green's Function potential (D.8), one can compute an electric field line by computing $\hat{n} = \nabla V$ and tracking it in space. In this Maple plot, starting points on the iris were selected by a random number generator, and all roads lead to the Green's point charge in the hole:



As expected, each field line launches itself at right angles to the iris surface.

Appendix E: Dual Equations and their connection to the Charged Bowl Problem

Sneddon uses the term "dual relations" to cover both dual integral equations and dual series equations, but we shall just call them "dual equations". We can represent a pair of dual equations this way:

$$\begin{array}{llll} A\Psi = f & f = (f_1, f_2) & \text{presented as} & [A\Psi]_1 = f_1 \text{ on } I_1 \\ B\Psi = g & g = (g_1, g_2) & \text{presented as} & [B\Psi]_2 = g_2 \text{ on } I_2 \end{array} \quad (\text{E.1})$$

Here A and B are invertible linear operators (perhaps Hankel Transform integral operators) and the other symbols stand for functions of a real variable x. The range I of the variable x is partitioned into two regions I_1 and I_2 so the full interval of interest is $I = I_1 \cup I_2$. The notation $f = (f_1, f_2)$ means that $f(x) = f_1(x)$ on I_1 and $f(x) = f_2(x)$ on I_2 . The dual equation problem is easily stated: given the prescribed functions f_1 and g_2 , find the partner functions f_2 and g_1 (so that you then know f and g on all of I), and also find the solution function $\Psi(x)$ on all of I.

Although A and B are invertible operators, neither dual equation can be inverted because information is missing. For example, one could write $\Psi = A^{-1}f$, but we don't know f, we only know f_1 . The same is true for the second equation: we can write $\Psi = B^{-1}g$, but we only know g_2 . Only by considering both equations of the dual pair can a solution be found. In electrostatics problems, one usually has a Dirichlet boundary condition on interval I_1 and a Neumann boundary condition on I_2 , so a dual equation really represents a mixed boundary value problem.

Here is the formal trick used to solve the problem. Although A and B are operators, it is useful to think of them as matrices. And it is useful further to think of these matrices as consisting of submatrices so that the row and column spaces are partitioned in the sense of $I = I_1 \cup I_2$. Then for example we could say

$$A\Psi = f \quad \leftrightarrow \quad \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad (\text{E.2})$$

The game is to find invertible lower and upper triangular matrices (really operators) L and U such that $LA = UB$. It might seem at first that the existence of such L and U would be dubious, but an analysis of matrix decomposition theorems shows that in general such L and U do in fact exist. If we apply L to our first equation in (E.1) and U to the second equation, and if we define

$$S \equiv LA = UB \quad (\text{E.3})$$

(which S will also be invertible) then our dual equations become

$$\begin{array}{l} S\Psi = Lf \\ S\Psi = Ug \end{array} \quad (\text{E.4})$$

We can then examine the right hand sides of these two equations in our matrix language

$$Lf = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} L_{11}f_1 \\ L_{21}f_1 + L_{22}f_2 \end{pmatrix}$$

$$Ug = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} U_{11}g_1 + U_{12}g_2 \\ U_{22}g_2 \end{pmatrix}$$

Therefore, since both left hand sides are $S\Psi$, we find that

$$Lf = Ug = \begin{pmatrix} L_{11}f_1 \\ U_{22}g_2 \end{pmatrix} \quad (E.5)$$

where recall that we *know* L and U , and we *know* f_1 and g_2 since they are the prescribed functions (the driving terms on the RHS of the dual equations, often one of these is 0). The solution to the problem is then obtained as follows. First take either $S\Psi$ equation in (E.4) (say the first one) and invert to get

$$\Psi = S^{-1}(Lf) \leftrightarrow \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} S^{-1}_{11} & S^{-1}_{12} \\ S^{-1}_{21} & S^{-1}_{22} \end{pmatrix} \begin{pmatrix} L_{11}f_1 \\ U_{22}g_2 \end{pmatrix} \quad (E.6)$$

Since we know L, U, S, f_1, g_2 , we have solved the problem for Ψ . To obtain the unknown partner functions we use the original equations:

$$\begin{aligned} A\Psi = f & \leftrightarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} & \text{so} & f_2 = A_{21}\Psi_1 + A_{22}\Psi_2 \\ B\Psi = g & \leftrightarrow \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} & \text{so} & g_1 = B_{11}\Psi_1 + B_{12}\Psi_2 \end{aligned} \quad (E.7)$$

Hopefully we have clarified the "basic idea" with this little matrix viewpoint summary. In practice there is of course a *lot* of fine detail. An illustration of the nature of this detail appears in Appendix G below.

It might be noted that Sneddon also considers Triple Equations and the above analysis then involves 3×3 matrices since the variable range is partitioned into $I = I_1 \cup I_2 \cup I_3$. An example of such a problem is the "charged barrel problem" where barrel means a spherical shell with two polar caps removed.

As examples of dual equations, we can take a quick look at two famous problems.

The Beltrami unit disk problem. (Beltrami's 1881 work is reviewed by Sneddon.) An appropriate Smythian form for the potential can be constructed from cylindrical atoms:

$$V(\rho, z) = \int_0^\infty dk \, e^{-k|z|} J_0(k\rho) [k^{-1} a(k)] \quad (E.8)$$

The prescribed potential on the disk is $f_1(\rho)$, and the prescribed charge density is $\sigma \sim \partial_z V = 0$ outside the disk in the $z=0$ plane. Again, this is a "mixed boundary value problem". The dual equations are then

$$\begin{aligned} \int_0^\infty dk \, k^{-1} J_0(k\rho) a(k) &= f_1(\rho) & \rho < 1 & & A\Psi = f \\ \int_0^\infty dk \, J_0(k\rho) a(k) &= 0 & \rho > 1 & & B\Psi = g \quad // \, g_2 = 0 \end{aligned} \quad (E.9)$$

Here $\Psi(k) = a(k)$, the function to be solved for. A is an integral operator with kernel $K(\rho, k) = k^{-1} J_0(k\rho)$ while B has kernel $J_0(k\rho)$. The matrix sense of A is $A_{\rho, k} = K(\rho, k)$ where both indices of A are continuous real numbers, and similarly for B. Both A and B can be regarded as modified Hankel transforms. The formal matrix method above gives us $a(k)$ which we insert into the Smythian form to find solution $V(\rho, z)$. The partner functions are f_2 which is the potential outside the disk in the $z=0$ plane, and g_1 which is the charge density on the disk (sum of both sides).

The appropriate L and R operators for this problem turn out to be gussied-up Abel transform operators (called I and K by Sneddon) known as Erdélyi-Kober operators. These operators can be *interpreted* as fractional integral operators (fractional meaning α is continued off the integers; the disk problem uses $\alpha = 1/2$),

$$\begin{aligned} \mathcal{R}_\alpha \{ f(t); x \} &= (1/\Gamma(\alpha)) \int_0^x dt' f(t') (x-t')^{\alpha-1} && \text{Riemann-Liouville 1850} \\ \mathcal{W}_\alpha \{ f(t); x \} &= (1/\Gamma(\alpha)) \int_x^\infty dt' f(t') (t'-x)^{\alpha-1} && \text{Weyl 1917} \end{aligned} \quad (E.10)$$

but the fact is that they are really just generalized Abel transforms with fancy names. Like any respectable transform, the Abel transform is invertible (see Appendix F). The disk problem is then solved as outlined above using the Abel transforms L and U and the Hankel transforms A and B.

Setting $f_1(\rho) = V_0 = 1$ of course gives the "charged conducting disk problem".

The Corresponding Bowl Problem. An appropriate Smythian form for the unit-radius bowl potential, inside and outside the bowl, can be constructed from spherical atoms,

$$\begin{aligned} V_i(r, \theta) &= \sum_{n=0}^{\infty} a_n r^n P_n(\cos \theta) \\ V_o(r, \theta) &= \sum_{n=0}^{\infty} a_n r^{-n-1} P_n(\cos \theta) \end{aligned} \quad (E.11)$$

The prescribed potential is $f_1(\theta)$ on the bowl, and the prescribed charge density is $\sigma \sim \partial_r V = 0$ on the cap. The dual equations are these: (θ_0 is the polar angle which defines the circular lip of the bowl)

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(\cos \theta) a_n &= f_1(\theta) && \theta \text{ in } (0, \theta_0) && A\Psi = f \\ \sum_{n=0}^{\infty} (2n+1) P_n(\cos \theta) a_n &= 0 && \theta \text{ in } (\theta_0, \pi) && B\Psi = g \quad // \quad g_2 = 0 \end{aligned} \quad (E.12)$$

Now we have $\Psi_n = a_n$ and the matrix sense of A is $A_{\theta, n} = P_n(\cos \theta)$ where θ is a continuous angle index and n is a discrete index. As in the previous problem, A is an infinite x infinite matrix.

Vinogradov *et. al* show (Section 1.4.3, "Noble's method") that for our bowl problem the appropriate L, U operators are

$$\begin{aligned} L h(\theta) &= (1/\sqrt{2}) \partial_\theta \int_0^\theta d\varphi \sin \varphi h(\varphi) / \sqrt{\cos \varphi - \cos \theta} && // = K_1 h(\theta) \\ R h(\theta) &= (1/\sqrt{2}) \int_\theta^\pi d\varphi \sin \varphi h(\varphi) / \sqrt{\cos \varphi - \cos \theta} && // = K_2 h(\theta) \end{aligned} \quad (E.13)$$

where we note that the Volterra θ endpoints cause these operators to have triangular matrices as kernels. If we apply L to the first equation above and R to the second, we get (after some work)

$$\begin{aligned} \sum_{n=0}^{\infty} \cos[(n+1/2)\theta] a_n &= L f_1(\theta) & \theta < \theta_0 & \text{or} & S\Psi = Lf & \text{on } I_1 \\ \sum_{n=0}^{\infty} \cos[(n+1/2)\theta] a_n &= 0 & \theta > \theta_0 & \text{or} & S\Psi = Ug = 0 & \text{on } I_2 \end{aligned} \quad (E.14)$$

where both left hand sides are now of the form $S\Psi$ as discussed above in (E.4). In the case of the charged bowl, $f_1(\theta) = V_0 = 1$, we find that $L f_1(\theta) = L 1 = \cos(\theta/2)$. Then using (G.4),

$$\int_0^{\pi} d\theta \cos([n+1/2]\theta) \cos([m+1/2]\theta) = \delta_{n,m} (\pi/2) \quad (E.15)$$

one can trivially invert the equations (E.14) treated as a single equation on $(0, \pi)$ to get $\Psi = S^{-1} Lf$, or,

$$a_n = (2/\pi) \int_0^{\theta_0} d\theta \cos([n+1/2]\theta) \cos(\theta/2) = (1/\pi) \{ \sin(n\theta_0)/n + \sin[(n+1)\theta_0]/(n+1) \} \quad (E.16)$$

which is the famous result.

In Sneddon's treatment, the $P_n(\cos\theta)$ are replaced by Jacobi polynomials $P_n^{(\alpha, \beta)}(\cos\theta)$ so the bowl problem is seen to be a special case with $\alpha = \beta = 0$.

Appendix F: The Abel Transform in various forms

Here we just state the generalized Abel Transform in three different ways. The historical Abel transform problem (tautochrone) involves $\alpha = 1/2$, while "generalized" allows $0 < \alpha < 1$. The Abel transform can be expressed for α outside this range by doing one parts integration for each integer step of shift required (Sneddon). A derivation of the Abel Transform is given in Sneddon Section 2.3 and Vinogradov Section 1.5 in terms of a generic monotonic speed function $h(u)$ where denominators are $[h(x) - h(t)]^\alpha$.

In the S1 forms, the upper endpoint is the variable, while in the S2 forms it is the lower endpoint.

For the linear form we have explicitly done the ∂_t derivatives on the third line of each transform. The starting point is to consider $g(x) = \int_a^x dt k(x-t)f(t)$ and assume that the kernel can be written as $k(x-t) = \partial_x s(x-t) = -\partial_t s(x-t)$. This allows a parts integration which moves the derivative off the kernel and onto the function $f(t)$, giving these general results,

$$\begin{aligned} g(x) &= \int_a^x dt k(x-t)f(t) \Rightarrow \partial_x g(x) = f(a) k(x-a) + \int_a^x dt k(x-t) f'(t) \\ g(x) &= \int_x^b dt k(t-x)f(t) \Rightarrow \partial_x g(x) = -f(b) k(b-x) + \int_x^b dt k(t-x) f'(t) \end{aligned}$$

Linear Form of the Generalized Abel Transform:

$$S1: \quad \int_a^x dt f(t) / [x - t]^\alpha = g(x)$$

$$\begin{aligned}
\Rightarrow f(t) &= (1/\pi)\sin(\pi\alpha) \partial_t \left\{ \int_a^t du g(u) / [t-u]^{1-\alpha} \right\} \\
&= (1/\pi)\sin(\pi\alpha) \left\{ g(a) / [t-a]^{1-\alpha} + \int_a^t du g'(u) / [t-u]^{1-\alpha} \right\}
\end{aligned} \tag{F.1}$$

$$\begin{aligned}
\text{S2: } \int_x^b dt f(t) / [t-x]^\alpha &= g(x) \\
\Rightarrow f(t) &= -(1/\pi) \sin(\pi\alpha) \partial_t \left\{ \int_t^b du g(u) / [u-t]^{1-\alpha} \right\} \\
&= -(1/\pi) \sin(\pi\alpha) \left\{ -g(b) / [b-t]^{1-\alpha} + \int_t^b du g'(u) / [u-t]^{1-\alpha} \right\}
\end{aligned} \tag{F.2}$$

Quadratic Form of the Generalized Abel Transform:

$$\begin{aligned}
\text{S1: } \int_a^x dt f(t) / [x^2 - t^2]^\alpha &= g(x) \\
\Rightarrow f(t) &= (2/\pi)\sin(\pi\alpha) \partial_t \left\{ \int_a^t du u g(u) / [t^2 - u^2]^{1-\alpha} \right\}
\end{aligned} \tag{F.3}$$

$$\begin{aligned}
\text{S2: } \int_x^b dt f(t) / [t^2 - x^2]^\alpha &= g(x) \\
\Rightarrow f(t) &= -(2/\pi) \sin(\pi\alpha) \partial_t \left\{ \int_t^b du u g(u) / [u^2 - t^2]^{1-\alpha} \right\}
\end{aligned} \tag{F.4}$$

Trig Form of the Generalized Abel Transform:

$$\begin{aligned}
\text{S1: } \int_a^x dt f(t) / [\cos(t) - \cos(x)]^\alpha &= g(x) \\
\Rightarrow f(t) &= \pi^{-1}\sin(\pi\alpha) \partial_t \left\{ \int_a^t du \sin(u)g(u) / [\cos(u) - \cos(t)]^{1-\alpha} \right\}
\end{aligned} \tag{F.5}$$

$$\begin{aligned}
\text{S2: } \int_x^b dt f(t) / [\cos(x) - \cos(t)]^\alpha &= g(x) \\
\Rightarrow f(t) &= -\pi^{-1}\sin(\pi\alpha) \partial_t \left\{ \int_t^b du \sin(u)g(u) / [\cos(t) - \cos(u)]^{1-\alpha} \right\}
\end{aligned} \tag{F.6}$$

and this last case we write again with $\alpha = 1/2$

$$\begin{aligned}
\text{S1: } \int_a^x dt f(t) / \sqrt{\cos t - \cos x} &= g(x) \\
\Rightarrow f(t) &= \pi^{-1} \partial_t \left\{ \int_a^t du \sin u g(u) / \sqrt{\cos u - \cos t} \right\} \quad // \text{ Sneddon 2.3.5}
\end{aligned} \tag{F.7}$$

$$\text{S2: } \int_x^b dt f(t) / \sqrt{\cos x - \cos t} = g(x)$$

$$\Rightarrow f(t) = -\pi^{-1} \partial_t \left\{ \int_t^b du \sin u g(u) / \sqrt{\cos t - \cos u} \right\} \quad // \text{Sneddon 2.3.6} \quad (\text{F.8})$$

When $\alpha = 1/2$, the trig form always involves factors of the form $1/\sqrt{\cos a - \cos b}$. There is doubtless a deeper (perhaps group theoretic) explanation, but due to the following integral representations of the P function

$$P_n(\cos\theta) = (\sqrt{2}/\pi) \int_0^\theta d\varphi \cos[(n+1/2)\varphi] / \sqrt{\cos\varphi - \cos\theta} \quad // \text{GR7 8.823} \quad (\text{F.9})$$

$$P_n(\cos\theta) = (\sqrt{2}/\pi) \int_\theta^\pi d\varphi \sin[(n+1/2)\varphi] / \sqrt{\cos\theta - \cos\varphi} \quad // \theta \rightarrow \pi - \theta, \varphi \rightarrow \pi - \varphi$$

there is an close connection between Legendre functions and Abel transforms in the solution of dual series equations of the type discussed in Appendix E. The integrals (F.9) are called the Mehler-Dirichlet integrals for P_n and they are derived in Section 15.231 of Whittaker and Watson.

Corresponding Bessel J_0 representations also expose this Abel transform connection (though in the Abel quadratic form)

$$J_0(k\rho) = (1/\pi) \int_0^\rho dx \cos(kx) / \sqrt{\rho^2 - x^2} \quad // \text{Vinogradov 1.156}$$

$$J_0(k\rho) = (1/\pi) \int_\rho^\infty dx \cos(kx) / \sqrt{x^2 - \rho^2} \quad // \text{Vinogradov 1.157} \quad (\text{F.10})$$

Here are two interesting expansions which in certain circumstances allow for the solution of dual equation problems by the use of *two sequential* Abel transforms. The first operates in the Legendre world, the second in the Bessel world:

$$\sum_{n=0}^{\infty} P_n(\cos\theta) P_n(\cos\theta') = (1/\pi) \int_0^{\min(\theta, \theta')} d\varphi \left\{ 1/\sqrt{\cos\varphi - \cos\theta} - 1/\sqrt{\cos\varphi - \cos\theta'} \right\} \quad (\text{F.11})$$

$$\int_0^\infty J_n(rx) J_n(r'x) dx = (2/\pi) (rr')^{-n} \int_0^{\min(r, r')} ds s^{2n} / [\sqrt{r^2 - s^2} \sqrt{r'^2 - s^2}] \quad (\text{F.12})$$

The main idea is that on the RHS the variables of interest appear in *factorized* Abel-transform-ready form. The first appears as Vinogradov 1.107 and the second as 2.169. Vinogradov *et. al* have much to say about Abel transforms in the context of dual equations. We shall derive (F.11) in Appendix G, and use it.

Since this is supposed to be a document about the toroidal charged bowl problem, we note this alternate evaluation of the above double Bessel integral

$$\int_0^\infty J_n(rx) J_n(r'x) dx = (1/\pi) (rr')^{-1/2} Q_{n-1/2}[(r^2 + r'^2)/(2rr')] \quad (\text{F.13})$$

So in the midst of a cylindrical and spherical coordinates discussion, we are somewhat surprised to find ourselves staring at a toroidal Q function (see Section 2 (b) above).

Appendix G: Solving the Charged Bowl Problem using a double Abel transform

The steps given in this lengthy and tedious Appendix are typical of what one encounters in a dual equations problem. In particular, one often encounters painful integrals. Sometimes there are "easy ways" to do integrals, but when the easy ways are not known, one must resort to brute force, as done below. This Appendix closely follows Section 1.4.1 of Vinogradov *et. al.* They in turn are presenting 1964 work of W.E. Williams. Unlike Vinogradov, we try to show all of the waypoints of the calculation.

In Appendix E we outlined a matrix method for solving dual equations in this framework

$$\begin{array}{llll} A\Psi = f & f = (f_1, f_2) & \text{presented as} & [A\Psi]_1 = f_1 \text{ on } I_1 \\ B\Psi = g & g = (g_1, g_2) & \text{presented as} & [B\Psi]_2 = g_2 \text{ on } I_2 \end{array} \quad (E.1)$$

where the problem is to solve for the "potential" Ψ and the partner functions f_2 and g_1 if one is given the driving functions f_1 and g_2 . We placed the $V_0=1$ $R=1$ charged bowl problem into this framework according to

$$\begin{array}{llll} \sum_{n=0}^{\infty} P_n(\cos\theta) a_n = 1 & \theta \text{ in } (0, \theta_0) & A\Psi = f & // f_1(\theta) = 1 \\ \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta) a_n = 0 & \theta \text{ in } (\theta_0, \pi) & B\Psi = g & // g_2(\theta) = 0 \end{array} \quad (E.12)$$

In the matrix method, one formally finds Ψ first, and then from that gets the partner functions f_2 and g_1 . As one quickly learns from reading Sneddon's reviews, there are *many* variations of this general method. Sometimes the batting order is to first find the partner function g_1 and then from it obtain Ψ and f_2 . That is exactly what we are going to do here.

(a) Find g_1

The starting point is $B\Psi = g = (g_1, g_2)$ where $g_2 = 0$ since there is no charge density on the cap of a charged bowl. The unknown partner function g_1 is proportional to the total charge density on the bowl. We have

$$g_1(\theta) = \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta) a_n = \partial_{\mathbf{r}} V_{\mathbf{i}} - \partial_{\mathbf{r}} V_{\mathbf{o}} = 4\pi[\sigma_{\mathbf{i}}(\theta) + \sigma_{\mathbf{o}}(\theta)] \quad \theta \text{ in } (0, \theta_0)$$

from Gauss's Law where σ is expressed in cgs units. For a bowl of radius R ,

$$g_1 = 4\pi R(\sigma_{\mathbf{i}} + \sigma_{\mathbf{o}}). \quad (G.0)$$

The second equation of (E.12) above may be written as ($z = \cos\theta$ in all that follows)

$$\sum_{n=0}^{\infty} (2n+1) P_n(z) a_n = \begin{cases} g_1(\theta) & \theta \text{ in } (0, \theta_0) \\ 0 & \theta \text{ in } (\theta_0, \pi) \end{cases} \quad (G.1)$$

Applying $\int_{-1}^1 dz P_n(z)$ to both sides and using $\int_{-1}^1 dz P_n(z) P_m(z) = \delta_{n,m} 2/(2n+1)$ one finds that

$$a_n = (1/2) \int_{-1}^1 dz P_n(z) g(\theta) = (1/2) \int_0^{\theta_0} d\theta \sin\theta P_n(z) g_1(\theta) \quad (G.2)$$

If we install this a_n into our first dual equation of (E.12), $\sum_{n=0}^{\infty} P_n(z') a_n = 1$, we get

$$\int_{-1}^1 dz g(\theta) \left[(1/2) \sum_{n=0}^{\infty} P_n(z) P_n(z') \right] = 1$$

or

$$\int_{-1}^1 dz g(\theta) K(\theta, \theta') = 1 \quad \text{where} \quad K(\theta, \theta') \equiv (1/2) \sum_{n=0}^{\infty} P_n(z) P_n(z') \quad (G.3)$$

Thus, $K(\theta, \theta')$ is the kernel of an integral equation we want to solve for g .

Digressing momentarily, if one considers the "string problem with the left end offset "

$$u''(\theta) = -n^2 u(\theta) \quad u(0) = 1 \quad u(\pi) = 0$$

one finds that the eigenfunctions $\cos([n+1/2]\theta)$ form a complete set with completeness

$$\sum_{n=0}^{\infty} \cos([n+1/2]x) \cos([n+1/2]x') = (\pi/2) \delta(x-x') \quad (G.4)$$

and if the right end instead is offset, one gets the same result with $\cos \rightarrow \sin$.

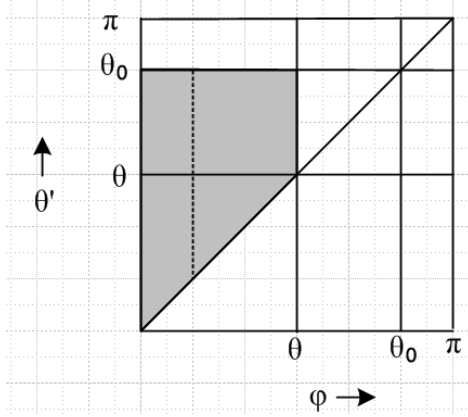
Now, using the integral representation (F.9) for each P function in (G.3) along with (G.4), we get

$$K(\theta, \theta') = (1/2\pi) \int_0^{\min(\theta, \theta')} d\phi \left\{ 1/\sqrt{\cos\phi - \cos\theta} - 1/\sqrt{\cos\phi - \cos\theta'} \right\} \quad (G.5)$$

We have thus derived (F.11) quoted in Appendix F. Equation (G.3) then says

$$1 = \int_0^{\theta_0} d\theta' [\sin\theta' g_1(\theta')] (1/2\pi) \int_0^{\min(\theta, \theta')} d\phi \left\{ 1/\sqrt{\cos\phi - \cos\theta} - 1/\sqrt{\cos\phi - \cos\theta'} \right\} \quad (G.6)$$

where we have used that fact that $g(\theta) = 0$ on (θ_0, π) . Pondering the region of the double integration



we can swap the integration order to obtain

$$2\pi = \int_0^{\theta} d\varphi \frac{1}{\sqrt{\cos\varphi - \cos\theta}} \int_{\varphi}^{\theta_0} d\theta' [\sin\theta' g_1(\theta')] \frac{1}{\sqrt{\cos\varphi - \cos\theta'}} \quad \theta \leq \theta_0 \quad (\text{G.7})$$

The game is now to solve this equation for g_1 using a double Abel transform. We can break (G.7) into two pieces, each an Abel transform, in this way

$$2\pi = \int_0^{\theta} d\varphi \frac{h(\varphi)}{\sqrt{\cos\varphi - \cos\theta}} \quad \varphi \leq \theta \leq \theta_0 \quad (\text{G.8})$$

$$h(\varphi) \equiv \int_{\varphi}^{\theta_0} d\theta' [\sin\theta' g_1(\theta')] \frac{1}{\sqrt{\cos\varphi - \cos\theta'}} \quad (\text{G.9})$$

Using Abel transform (F.7) applied to (G.8) we find that

$$h(\varphi) = 2 \sin\varphi / \sqrt{1 - \cos\varphi} \quad \varphi \text{ in } (0, \theta_0) \quad (\text{G.10})$$

Using this for the left side of (G.9) we apply Abel transform (F.8) to get

$$[\sin\theta g_1(\theta)] = - (2/\pi) \partial_{\theta} I \quad (\text{G.11})$$

where

$$I(\theta) = \int_{\theta}^{\theta_0} du \sin^2 u \left[\frac{1}{\sqrt{1 - \cos u}} \right] \left[\frac{1}{\sqrt{\cos\theta - \cos u}} \right] \quad (\text{G.12})$$

This integral can be transformed into a simpler form which can be evaluated,

$$\begin{aligned} I(\theta) &= \int_{\theta}^{\theta_0} du \sin^2 u \left[\frac{1}{\sqrt{1 - \cos u}} \right] \left[\frac{1}{\sqrt{\cos\theta - \cos u}} \right] \\ &= \int_a^b \left[\frac{dx}{\sqrt{1-x^2}} \right] (1-x^2) / \left[\sqrt{1-x} \sqrt{\cos\theta - x} \right] \quad // x = \cos u, a = \cos\theta_0, b = \cos\theta \\ &= \int_a^b dx \sqrt{1+x} / \sqrt{b-x} \end{aligned}$$

$$\begin{aligned}
&= \int_0^c dy \sqrt{d-y} / \sqrt{y} && // y = b-x, \quad c = b-a, \quad d = b+1, \quad d-c = a+1 \\
&= 2 \int_0^{\sqrt{c}} dx \sqrt{d-x^2} && // y = x^2 \\
&= \sqrt{c} \sqrt{d-c} + d \sin^{-1}(\sqrt{c} / \sqrt{d})
\end{aligned} \tag{G.13}$$

Then using $\partial_{\theta} = -\sin\theta (\partial_c + \partial_d)$ we can rewrite (G.11) as

$$[\sin\theta g_1(\theta)] = - (2/\pi) \partial_{\theta} I = (2/\pi) \sin\theta (\partial_c + \partial_d) I$$

so that

$$\begin{aligned}
g_1(\theta) &= (2/\pi) (\partial_c + \partial_d) I = (2/\pi) \{ \sqrt{d-c} / \sqrt{c} + \sin^{-1}(\sqrt{c} / \sqrt{d}) \} \\
&= (2/\pi) \{ \sqrt{a+1} / \sqrt{b-a} + \sin^{-1}(\sqrt{b-a} / \sqrt{b+1}) \}
\end{aligned}$$

Using the fact that $\sin^{-1}(\sqrt{b-a} / \sqrt{b+1}) = \pi/2 - \sin^{-1}[\sqrt{1+a} / \sqrt{1+b}]$ we find

$$\begin{aligned}
g_1(\theta) &= (2/\pi) \{ \sqrt{1+a} / \sqrt{b-a} + \pi/2 - \sin^{-1}[\sqrt{1+a} / \sqrt{1+b}] \} \\
&= (2/\pi) \{ \sqrt{2} \cos(\theta_0/2) / \sqrt{\cos\theta - \cos\theta_0} + \pi/2 - \sin^{-1}[\cos(\theta_0/2) / \cos(\theta/2)] \}
\end{aligned}$$

which is our corrected version of Vinogradov equation 1.111 (which has three typos, something very rare in their book). So the final result can be stated

$$g_1(\theta) = (2/\pi) \{ \sqrt{2} \cos(\theta_0/2) / \sqrt{\cos\theta - \cos\theta_0} + \cos^{-1}[\cos(\theta_0/2) / \cos(\theta/2)] \} \quad \theta \leq \theta_0 \tag{G.14}$$

To summarize: using a double Abel transform, we have solved our dual equation problem for the partner function g_1 .

(b) Run a check on g_1

As a check on (G.14), since $g_1 = 4\pi R(\sigma_1 + \sigma_o)$ from (G.0), we can examine the Kelvin bowl charge densities from (4.6) and (4.9)

$$\begin{aligned}
\sigma_o + \sigma_1 &= (V_o/2\pi^2 R) \{ \sqrt{1-\cos\alpha} / \sqrt{\cos\alpha - \cos\theta} - \tan^{-1} [\sqrt{1-\cos\alpha} / \sqrt{\cos\alpha - \cos\theta}] \} + V_o/(4\pi R) \\
&= V_o/(2\pi^2 R) \{ \sqrt{1-\cos\alpha} / \sqrt{\cos\alpha - \cos\theta} - \tan^{-1} [\sqrt{1-\cos\alpha} / \sqrt{\cos\alpha - \cos\theta}] + \pi/2 \}
\end{aligned}$$

But for comparison we have to flip the z axis which takes all $\cos \rightarrow -\cos$,

$$\sigma_o + \sigma_1 = V_o/(2\pi^2 R) \{ \sqrt{1+\cos\theta_0} / \sqrt{\cos\theta - \cos\theta_0} - \tan^{-1} [\sqrt{1+\cos\theta_0} / \sqrt{\cos\theta - \cos\theta_0}] + \pi/2 \}$$

and finally we use the fact that ("draw a small triangle")

$$\tan^{-1} [\sqrt{1+\cos\theta_0} / \sqrt{\cos\theta - \cos\theta_0}] = \sin^{-1} [\sqrt{1+\cos\theta_0} / \sqrt{1+\cos\theta}] = \sin^{-1} [\cos(\theta_0/2) / \cos(\theta/2)]$$

to get

$$\sigma_o + \sigma_i = V_o / (2\pi^2 R) \{ \sqrt{2} \cos(\theta_0/2) / \sqrt{\cos\theta - \cos\theta_0} + \pi/2 - \sin^{-1} [\cos(\theta_0/2) / \cos(\theta/2)] \}$$

Setting $V_o = 1$ and $R = 1$ we then have

$$g_1 = 4\pi(\sigma_i + \sigma_o) = (2/\pi) \{ \sqrt{2} \cos(\theta_0/2) / \sqrt{\cos\theta - \cos\theta_0} + \pi/2 - \sin^{-1} [\cos(\theta_0/2) / \cos(\theta/2)] \}$$

in agreement with (G.14).

(c) Find a_n

The next step is to determine coefficients a_n by inserting (G.14) for $g_1(\theta)$ into (G.2). This gives

$$a_n = (1/\pi) \int_0^{\theta_0} d\theta \sin\theta P_n(\cos\theta) \{ \sqrt{1+\cos\theta_0} / \sqrt{\cos\theta - \cos\theta_0} + \cos^{-1} [\sqrt{1+\cos\theta_0} / \sqrt{1+\cos\theta}] \} , \quad (G.15)$$

another unruly integral. The first term can be evaluated using GR7 p 790 7.225.2 (ignore the $P^{-1/2}$ typo)

$$\begin{aligned} \int_{\cos\theta_0}^1 dz P_n(z) / \sqrt{z - \cos\theta_0} &= (n+1/2)^{-1} (1 - \cos\theta_0)^{-1/2} [T_n(\cos\theta_0) - T_{n+1}(\cos\theta_0)] \\ &= (n+1/2)^{-1} (1 - \cos\theta_0)^{-1/2} [\cos(n\theta_0) - \cos[(n+1)\theta_0]] \end{aligned} \quad (G.16)$$

so that

$$\begin{aligned} a_n(\text{1st term}) &= (2/\pi) \cot(\theta_0/2) [\cos(n\theta_0) - \cos[(n+1)\theta_0]] / (2n+1) \\ &= (2/\pi) \cot(\theta_0/2) [2 \sin[(n+1/2)\theta_0] \sin(\theta_0/2)] / (2n+1) \\ &= (4/\pi) \cos(\theta_0/2) \sin[(n+1/2)\theta_0] / (2n+1) \\ &= (2/\pi) \{ \sin(n\theta_0) + \sin[(n+1)\theta_0] \} / (2n+1) \end{aligned} \quad (G.17)$$

The second term integral

$$a_n(\text{2nd term}) = (1/\pi) \int_0^{\theta_0} d\theta \sin\theta P_n(\cos\theta) \cos^{-1} [\sqrt{1+\cos\theta_0} / \sqrt{1+\cos\theta}] \quad (G.18)$$

is an incomplete range integral of P_n times an inverse trig function of an algebraic argument, making this integral a bit difficult to look up in a table. This is where brute force comes in. We put P_n into its integral representation (F.9), and then reverse the integration order using $\int_0^{\theta_0} d\theta \int_0^{\theta} d\phi = \int_0^{\theta_0} d\phi \int_{\phi}^{\theta_0} d\theta$ to get

$$a_n(\text{2nd term}) = (\sqrt{2}/\pi^2) \int_0^a d\varphi \cos[(n+1/2)\varphi] R$$

where

$$R \equiv \int_0^{\theta_0} d\theta \sin\theta / \sqrt{\cos\varphi - \cos\theta} * \cos^{-1}[\sqrt{1+\cos\theta_0} / \sqrt{1+\cos\theta}] \quad (\text{G.19})$$

R may be evaluated by setting $x = \cos(\theta/2)$, $a = \cos(\varphi/2)$ and $b = \cos(\theta_0/2)$ to get

$$\begin{aligned} R &= (2\sqrt{2}) \int_b^a dx x / \sqrt{a^2 - x^2} * \sec^{-1}(x/b) \\ &= -(2\sqrt{2}) \int_b^a dx \partial_x (a^2 - x^2)^{1/2} \sec^{-1}(x/b) \quad // \text{ set up for parts} \\ &= (2\sqrt{2}) \int_b^a dx (a^2 - x^2)^{1/2} \partial_x \sec^{-1}(x/b) \quad // \text{ the "parts" vanish} \\ &= (2\sqrt{2} b) \int_b^a dx x^{-1} (a^2 - x^2)^{1/2} (x^2 - b^2)^{-1/2} \quad // \text{ arc trig function is now gone} \\ &= (\sqrt{2} b) \int_\beta^\alpha dy/(y) (\alpha - y)^{1/2} (y - \beta)^{-1/2} \quad // x^2 = y, a^2 = \alpha, b^2 = \beta \\ &= (\sqrt{2} b) (c-1) \int_0^\infty dz \sqrt{z} / [(z+1)(z+c)] \quad // z = (\alpha - y)/(y - \beta) \quad c = \alpha/\beta = a^2/b^2 \\ &= (\sqrt{2} b) \pi(\sqrt{c} - 1) \quad // \text{ regulated partial fractions, } \infty \rightarrow \Lambda \rightarrow \infty \\ &= (\sqrt{2} b) \pi ((a/b) - 1) = (\sqrt{2} \pi) (a - b) \\ &= \sqrt{2} \pi (\cos(\varphi/2) - \cos(\theta_0/2)) \end{aligned} \quad (\text{G.20})$$

After this saga, we are left with

$$\begin{aligned} a_n(\text{2nd term}) &= (2/\pi) \int_0^{\theta_0} d\varphi \cos[(n+1/2)\varphi] [\cos(\varphi/2) - \cos(\theta_0/2)] \\ &= (1/\pi) [(n+1 - n \cos\theta_0) \sin(n\theta_0) - n \sin\theta_0 \cos(n\theta_0)] / [n(n+1)(2n+1)] \\ &= (1/\pi) \{ [\sin(n\theta_0)/n - \sin[(n+1)\theta_0]/(n+1)] / (2n+1) \} \end{aligned} \quad (\text{G.21})$$

At this point then we have shown that

$$\begin{aligned} a_n(\text{1st term}) &= (1/\pi) \{ 2\sin(n\theta_0) + 2\sin[(n+1)\theta_0] \} / (2n+1) \\ a_n(\text{2nd term}) &= (1/\pi) \{ \sin(n\theta_0)/n - \sin[(n+1)\theta_0]/(n+1) \} / (2n+1) \end{aligned} \quad (\text{G.22})$$

Adding we find

$$\begin{aligned} a_n &= (1/\pi) \{ [2+1/n] \sin(n\theta_0) + [2-1/(n+1)] \sin[(n+1)\theta_0] \} / (2n+1) \\ &= (1/\pi) \{ (2n+1)/n * \sin(n\theta_0) + (2n+1)/(n+1) * \sin[(n+1)\theta_0] \} / (2n+1) \\ &= (1/\pi) \{ \sin(n\theta_0)/n + \sin[(n+1)\theta_0]/(n+1) \} \end{aligned} \quad (\text{G.23})$$

which agrees with the result found in (E.16).

(d) Find Ψ and f_2

For the unit-radius bowl the potential is given by (E.11) with (G.23)

$$\begin{aligned} V_i(r, \theta) &= (1/\pi) \sum_{n=0}^{\infty} \{ \sin(n\theta_0)/n + \sin[(n+1)\theta_0]/(n+1) \} r^n P_n(\cos\theta) \\ V_o(r, \theta) &= (1/\pi) \sum_{n=0}^{\infty} \{ \sin(n\theta_0)/n + \sin[(n+1)\theta_0]/(n+1) \} r^{-n-1} P_n(\cos\theta) \end{aligned} \quad (G.24)$$

Finally, the other partner function f_2 is the potential on the cap, $\theta_0 < \theta < \pi$,

$$f_2(\theta) = (1/\pi) \sum_{n=0}^{\infty} \{ \sin(n\theta_0)/n + \sin[(n+1)\theta_0]/(n+1) \} P_n(\cos\theta) \quad (G.25)$$

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Kelvin (W. Thomson), *Reprint of Papers on Electrostatics and Magnetism, 2nd. Ed.*, (Macmillan, London, 1884). As are several other items in this Reference list, this out-of-print book is available on the web (this one "Digitized by Microsoft"). Converting PDF to DJVU makes things smaller and faster and maintains searchability.

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K.T. McDonald, "Conducting Spherical Shell with a Circular Orifice", PDF file downloadable from his Princeton website. This paper discusses the bowl problem in several ways, and reviews some of Kelvin's papers noted above. The current paper tries to fill in Section 2.5 of this document.

P. Moon and D.E. Spencer, *Field Theory Handbook, Including Coordinate Systems, Differential Equations and their Solutions* (Springer-Verlag, Berlin, 1961). This book is not about quantum field theory or anything like that, it is about curvilinear coordinate systems, how the Laplace and Helmholtz equations appear in each system, and what the solutions of these equations look like. This husband and wife team wrote several excellent books. Long ago they were strangely involved in an accident involving a test of general relativity.

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(PBM) A.P. Prudnikov, Y.A. Brychkov and O.I. Marichev, *Integrals and Series: More Special Functions, Vol 3* (Gordon and Breach, New York, 1996). A huge 5 volume set similar to the Bateman / Erdelyi set plus Gradshteyn & Ryzhik. The Russian editions are perhaps more available and can easily be used by non-Russian readers. Here are a few useful words:

оглавление = table of contents	ряды = series	сумма = sum
глава = chapter	преобразовать = transform	
интеграл = integral	функция = function	

<http://www.babylon.com/define/118/russian-english-dictionary.html>

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S.S Vinogradov, P.D. Smith, and E.D Vinogradova, *Canonical Problems in Scattering and Potential Theory, Part I: Canonical Structures in Potential Theory*, (Chapman & Hall/CRC, New York, 2001). The bowl problem is treated in Sections 1.4.1-3 of this excellent but very advanced monograph. (See (G.14) for a correction to their equation 1.111 for $g(\theta)$.) Chapter 2 incorporates in compact form nearly the entire huge panorama of Sneddon's book. The book is advanced in that it treats impressive variations of already-difficult problems in electrostatics, surfaces with holes and slots and missing pieces. For example,

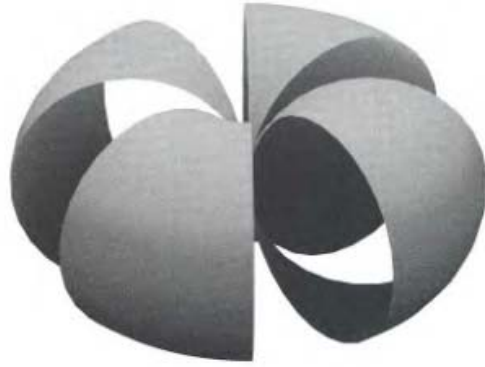


Figure 5.7
A degenerate toroidal shell with four azimuthal cuts.

E.T. Whittaker and G.N. Watson, *A Course in Modern Analysis*, 4th Ed. , (Cambridge University Press, Cambridge, 1927). The Mehler-Dirichlet integral is derived in Section 15.231 of this and at least two earlier editions. Beware that the 2nd edition is being sold by Merchant Press and does not include Chapter 23 on Ellipsoidal Harmonics and Lamé Functions. See Kelvin reference above.
