Legendre My Problem META.doc PhL 9.25.10

key phrases: half bowl, Stakgold integral equation method, Legendre polynomials
file: I will file this and related docs in my electrostatics "bowl" area, that seems the strongest key phrase

The "Legendre My Problem" series consists of two docs called Vol 1 and Vol 2 together with this META document. Each of those documents has a solid 3 page overview. I will not repeat that information here, but I do give a very *high level* summary of each doc below.

1. Statement of My Legendre Problem: Find the solution f(x) to the following set of equations:

$$\int_0^1 dx\, f(x)\, P_n(x) = \delta_{n0} \qquad n = 0,1,2..$$

It turns out there is no f(x) which solves all these equations.

2. How this problem arose: In *another* series of docs called

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I was in fact studying the "half spherical shell" problem, more generally the charged bowl problem. I was attempting a solution by the Stakgold "integral equation method" which I write as $\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$ in a symbolic form that serves to remind me of the Stakgold discussion. As shown in the first doc of the above series # (see Overview of Part C) this integral equation condition leads to the equation

$$\Sigma_n\, P_n(z_s)\, I'_n = 2 \qquad \text{half sphere} \qquad \text{valid } (0,1) \text{ for } z_s$$
$$\text{where} \quad I'_n = \int_0^1 dz_\xi\; P_n(z_\xi)\, I'(z_\xi) = 1 \ .$$

In Vol 2 Section 7 of "Leg my prob" I rederive the above from scratch, using this notation:

$$\Sigma_n\; P_n(z_s)\, \sigma_n(b) = V_0/(2\pi a) \qquad \text{valid } (\beta,1) \text{ for } z_s$$
$$\text{where} \quad \sigma_n(b) \equiv \int_b dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

Here things are for the general bowl, and we are in Jackson units instead of Stakgold units so we have a 4π difference. Also, I use radius a = 1 and $V_0 = 1$ so I can see things better. Having done all that, we can reselect V_0 such that $V_0/(2\pi a) = 1$, and then we have this sort of generic situation

$$\Sigma_n\; P_n(z_s)\, \sigma_n(b) = 1 \qquad (*) \qquad \text{valid } (\beta,1) \text{ for } z_s$$
$$\text{where} \quad \sigma_n(b) \equiv \int_b dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

where of course σ is the (unknown) charge density on the bowl and β is the bowl angle, and b refers both to the surface of the bowl and to this angle as a parameter. [You arrive at equation (*) very quickly if you just assume the usual spherical Smythian form for bowl potential and set r = a and $\sigma_n(b) = a_n$ and V_0=1.]

At this point, I observed that there is an obvious solution to equation (*), namely

$$\sigma_n(b) = \delta_{n0}$$

which is the same solution I used for the full sphere problem. I unthinkingly assumed this was the only possible solution, that is, that the solution was unique. This then led to this statement of the bowl problem:

$$\int_b dz_\xi \, P_n(z_\xi) \, \sigma(z_\xi) = \delta_{n0}$$

and in particular for the half-sphere bowl this says

$$\int_0^1 dz_\xi \, P_n(z_\xi) \, \sigma(z_\xi) = \delta_{n0}$$

and solving this for $\sigma(z_\xi)$ became "Legendre My Problem". But, as described below, this set of integral equations for n= 0,1,2.... in fact *has no solution* $\sigma(z_\xi)$!! So this was my great confusion.

In retrospect, you can think of equation (*) in this way:

$$\Sigma_n \; P_n(z_s) \, \sigma_n(b) = 1 \qquad \Rightarrow \qquad \mathbf{P}(z_s) \bullet \boldsymbol{\sigma}(b) = 1$$

Now knowledge of the dot product of two vectors and knowledge of one of those vectors **P** does not in fact uniquely determine the other vector **σ**. So this equation $\Sigma_n \, P_n(z_s) \, \sigma_n(b) = 1$ does *not* have a unique solution, and in fact $\boldsymbol{\sigma}(b) = \mathbf{e}_1$ [that is, $\sigma_n(b) = \delta_{n0}$] is not the correct solution for the half sphere.

So to summarize, the Legendre My Problem arose erroneously in my study of the half sphere.

Note added 4.24.11. In my toroidal bowl doc Appendix G, I show that in fact the above equation is the first of this dual pair, where we identify $a_n = \sigma_n$ and $b = \theta_0$ as the bowl size label :

$$\begin{array}{llllll} \Sigma_{n=0}^{\infty} & P_n(\cos\theta) \, a_n = 1 & \theta \text{ in } (0,\theta_0) & A\Psi = f & // \; f_1(\theta) = 1 & \\ \Sigma_{n=0}^{\infty} \, (2n+1) & P_n(\cos\theta) \, a_n = 0 & \theta \text{ in } (\theta_0,\pi) & B\Psi = g & // \; g_2(\theta) = 0 & (E.12) \end{array}$$

I eventually show that the solution for general θ_0 is this famous result

$$\sigma_n = a_n = (1/\pi) \; \{ \; \sin(n\theta_0)/n + \sin[(n+1)\theta_0] \, /(n+1) \; \} \qquad (G.23)$$

and this then is the "other solution" which differs from $\sigma_n(b) = \delta_{n0}$. For the half sphere bowl we have $\theta_0 = \pi/2$ and this says,

$$\begin{array}{lll} \sigma_n = a_n = (1/\pi)(-1)^{(n-1)/2} \, 1/(n) & & n \text{ odd} \\ \qquad\quad = (1/\pi)(-1)^{(n)/2} \, 1/(n+1) & & n \text{ even} \end{array}$$

Inserting these into the first sum in E.12 above gives

$$\Sigma_{n=0,2,4..}^{\infty} [P_n(\cos\theta) + P_{n+1}(\cos\theta)] \{ (1/\pi) (-1)^{n/2} 1/(n+1) \} = 1$$

3. Is there a solution to Legendre My Problem? No. In Vol 2 section 9 I give a simple proof showing conclusively that the problem has no solution, but of course I was unaware of this proof when I wrote the two docs. Things were swirling around in their usual fuzzy way. I had $\sigma = 0$ on the non-bowl part of the sphere, I had $V = 1$ on the bowl part, so I had the "dual integral equations" (here series) in mind, but I let this information confuse the picture of the simple "transform math" I should have understood clearly.

4. What is Vol 1 about? After reviewing "general Stak theory" of SL transforms in Section 1, I state the Legendre transform in Section 2. In Section 3 I was worried about whether $\theta(x)f(x)$ was a "valid function" for use in a SL transform. Since SL transforms are OK for piecewise continuous functions, the answer here was yes. Then in Section 4 (Plan A) I stumbled at once onto the above proof that no solution exists, but I did not *recognize* it as a proof and ignored it. In Section 5 (Plan B) I flailed writing f as even + odd function, a totally wrong thing to do. In Section 6 (Plan C) I expanded both $P_n(x)$ and $f(x)$ onto a complete set of φ_n on (0,1) and ended up (with Maple work) convincing myself that the series you then get for f(x) diverges, suggesting (for the first time) that f(x) does not exist. In Section 7 I reached the same conclusion by studying the equations on a lattice (and more Maple).

5. What is Vol 2 about? This volume has two large parts written about a year apart!

5.I What is Vol 2 Part I about? In the earlier Part I, I state what is stated above as $\Sigma_n \; P_n(z_s) \; \sigma_n(b) = 1$ but appearing as $\Sigma_n \; P_n(y) \; I_n = 2$. I got concerned that this sum was only valid on the interval (0,1) and I was concerned that on this limited interval, the set of functions $\varphi_n = P_n$ was merely a "spanning set" like the powers $\varphi_n = x^n$. I knew that with just a spanning set, it is possible that a series like $f(y) \equiv \Sigma_n \; P_n(y) \; I_n$ *might* not exist because the coefficients I_n keep moving. This is the case for example with the powers x^n in a region containing a slope discontinuity of f(y) and in fact any kind of discontinuity. In terms of powers x^n we know that at such a location, you cannot expand in a power series. If you could, then the function and all derivatives would be the same as you approach the discontinuous point from the two sides, so it would not be a discontinuous point. I knew that our "series of interest" did in fact have a discontinuity at the edge of the bowl (I had been thinking that it was σ that was the problem, but I guess in retrospect it is the discontinuity of V that matters.) I convinced myself, for whatever reason, that the series $f(y) \equiv \Sigma_n \; P_n(y) \; I_n$ did not exist for the half sphere problem in the sense that the coefficients "keep moving". I was sold on this fact! As I looked back on my processing of equations, I had then to conclude that I was creating this non-existent series in a step where I interchanged the order between a summation and an integration. I then formalized my mathematical conclusion in an erroneous Theorem 1 which said that a series on φ_n does not (might not) exist if you are working on an interval that is less than the full interval of the complete set φ_n, and that in this case the φ_n are "merely a spanning set like x^n " and that therefore my series of interest $\Sigma_n \; P_n(y) \; I_n = 2$ does not exist. I let this wrong conclusion sit valid for the entire year as I did other things.

So in this Part I, I am not so much trying to solve Legendre My Problem $\int_b dz_\xi \; P_n(z_\xi) \; \sigma(z_\xi) = \delta_{n0}$. Instead, I am questioning the very existence of the equation $\Sigma_n \; P_n(z_s) \; \sigma_n(b) = 1$ which *leads* to the

Legendre My Problem. If this series does not exist, then you cannot assume that $\sigma_n = \delta_{n0}$ means anything for this series, and then we don't worry about Legendre My Problem.

5.II What is Vol 2 Part II about? As noted, this was written a year after Part I. I call this part "slippery when wet" because I could not get comfortable about the series not existing. I knew that the Canonical doc probably gave an explicit form for the series, so how could it not exist? So in Part II I thought more about the notion of an enclosing interval (a,b) and an enclosed interval (A,B) and a set of φ_n which are complete on the enclosing interval. I got the idea of taking a function f(x) defined only on (A,B) and extending it to be defined on all of (a,b) by "adding zero pieces", this new function being $f_{ext}(x)$. This idea, trivial as it is, assists in the proof that Leg My Prob has no solution. I also defined the peephole concept which trivially says if you only look at an existent series in a peephole, it still exists in that limited view. I then provided a correction to Theorem 1 of Part I as Theorem 1A. I quote that here:

Theorem 1A. Let (a,b) enclose (A,B), let φ_n be complete on (a,b), and let f(x) be defined only on (A,B). Then the series $f(x) = \Sigma_n c_n \varphi_n(x)$ on (A,B) *exists*, and $c_n = \int_A^B dx\, f(x)\, \varphi_n(x)$.

So this completely "countermanded" my main conclusion of Part I of a year earlier that the series did not exist. By that time I knew that Leg My Prob was not a correct statement of the half sphere problem. Finally in Part II Section 9 I formalized the trivial proof that Leg My Problem has no solution.

6. What is the historic logic flow here?

(1) trying to solve the half sphere in manner similar to the full sphere
(2) reduce half sphere problem to $\Sigma_n P_n(z_s) \sigma_n = 1$ problem to find σ_n
(3) think the only solution to this equation is $\sigma_n = \delta_{n0}$
(4) this leads to Legendre My Problem which is to solve $\int_0^1 dx\, \sigma(x)\, P_n(x) = \delta_{n0}$

(5) "experimental" (Maple) work suggests that solution σ(x) does not exist (vol 1)
(6) of course I knew a solution σ(x) did exist, how could that be?
(7) conclusion is that sum/integral order interchange was invalid, so step (2) above invalid and the series shown there does not exist, this explains the mystery. Based on Theorem 1 and spanning set idea. Vol 2.I.
(8) change mind: decide that series in (2) really does in fact exist.
(9) realize that $\sigma_n = \delta_{n0}$ is not the unique solution to $\Sigma_n P_n(z_s) \sigma_n = 1$ and in fact the half sphere must have some different solution. Later see this as a dot product and non-uniqueness obvious.
(10) Once $\sigma_n = \delta_{n0}$ no longer need be valid, Leg My Problem no longer needs to be solved.
(11) Just for the record, came up with a simple proof that Leg My Prob in fact has no solution!

7. Was this all a big waste of time? Well, I don't think so. I was forced to review many things I thought I had learned, such as the Stak SL theory, complete φ_n sets and such. I got more experience making Maple do interesting things. I was forced in the end to carefully do a complete logic threading of what was earlier just a fuzzy mess. And I have left things in a completely clean state. As I learned in my stint teaching physics at the U of U, you don't really learn things until you teach them many times, and here I have added a bit to the number of times I have encountered certain concepts. I am the teacher and the student in my current activities. Also, the mysteries I ran into provided motivation to review otherwise

cold and boring things. How many "things" of the math world were involved here? At first you might list of three or four items, but in reality the number is probably in the range of 50. Math consists of a huge number of interlocking and related elements, you can hardly ever deal with individual items in isolation. Here is a partial list:

Hilbert Space of functions
linear algebra, finite and infinite
power series expansions, when valid
spanning sets, coefficients keep moving
order interchange
integral equations
dual integral and series equations representing mixed boundary conditions
theory of complete set of functions
bra-ket notation
convergence and existence of series