Legendre my problem: Volume 2 PhL 11.1.09

Note: Part I contains various unmarked incorrect statements which are corrected in Part II. I decided to keep Part I in the doc as part of "the record".

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Overview of Part I: Original Contents of this Doc (2 pages 9.23.10)

In **Section 1** I quickly review how starting with $\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1$ for half sphere, we get to:

$$\Sigma_n P_n(y)\, I_n = 2 \qquad (*) \qquad \text{where} \qquad I_n \equiv \int_0^1 dx\, P_n(x)\, I(x)$$

$$\int_0^1 dx\, K(y,x)\, I(x) = 2 \qquad (**) \qquad \text{where} \qquad K(y,x) \equiv \Sigma_n P_n(y)\, P_n(x)$$

I later show in Part II Section 7 how to derive these from scratch. It is the $\Sigma_n P_n(y) I_n$ series whose existence will be coming into question below. [equation (**) is never further used in this doc.]

In **Section 2** I define $\psi_n(x) = (n+1/2)^{1/2} P_n(x)$ to be the orthonormal Legendres and I make sure the full charged sphere problem works out right with these. I am planning to ponder Hilbert Space once again, and I guess these little ψ_n are going to be the normalized basis vectors in a HS.

In **Section 2a** I rewrite everything in Section 2 using bra-ket notation where now $\psi_n(x) = \langle x|n\rangle$ and then these $|n\rangle$ are the normalized base states spanning the HS. I don't really know where I was going with this idea, but see next section maybe.

In **Section 3** I resume the flow of Section 1. I restate the two equations seen above, then I **review my own Stakgold notes on spanning sets** with the big example being the powers x^n. A spanning set "spans" a Hilbert Space H and L_2 on the interval (0,1) is such a space and the powers x^n form such a spanning set. The set of points in H which you can reach using a spanning set -- call this set S -- is "dense" in the HS, but does not equal the HS. In fact $H = \overline{S}$, the closure of S. Any spanning set that is *orthogonal* on the interval is a basis. The classic problem is that as you add more terms in a series to approach some desired point f in H (a function which is not a polynomial, and assume spanning set x^n) , the coefficients of the lower terms MAY "move around" and never stabilize. If this happens, the series with an infinite number of terms which is supposed to add up to f does not exist. I then applied these facts to the set $\{ P_n(x) \}$ on the interval (0,1) and concluded that the series $\Sigma_{n=0}^{\infty} P_n(x) I_n$ does not exist for x on (0,1) because the P_n are not orthogonal on this interval and therefore do not form a basis, just a spanning set. In Part II I will change this conclusion, but that lies ahead. Continuing on the (incorrect) *assumption* that the series does not exist, I concluded that my development of the half sphere solution was "wrong" going between these two steps:

$$2 \quad (1/4\pi) \int_\sigma d\Omega_\xi \{ \Sigma_n N_{0n} \Sigma_m (1/N_{mn})\, Y_{nm}(\Omega_\xi)\, Y^*_{nm}(\Omega_s) \}\, I(\Omega_\xi) = 1 \qquad \sigma = \text{half sphere}$$

$$3 \quad (1/4\pi) \Sigma_n N_{0n} \Sigma_m (1/N_{mn})\, Y^*_{nm}(\Omega_s) \int_\sigma d\Omega_\xi\, Y_{nm}(\Omega_\xi)\, I(\Omega_\xi) = 1 \qquad \sigma = \text{half sphere}$$

I felt line 2 was OK, but line 3 then creates this non-existent series. Things are a little clouded here by the presence of index m which Part II shows can be set to 0 so we are then talking P_n functions only. I then restate my conclusion this way

$$2 = \int_0^1 dx\, \Sigma_n P_n(y)\, P_n(x)\, I(x) \neq \Sigma_n P_n(y) \int_0^1 dx\, P_n(x)\, I(x) \equiv \Sigma_n P_n(y)\, I_n$$

where the ≠ shows where the non-existent series is "generated" by the sum/integral order interchange.

In **Section 4** I then make some vague comments (but dismiss them) about the fact that on an enclosing interval (-1,1), the P_n *do* form a basis. I will return to this with more clarity in Part II.

In **Section 5** I repeat the comments of Section 4. I can see now that I was doubting this conclusion that the series does not exist and was trying to get more traction on the idea.

In **Section** 6 I repeat the comments *yet again* in a more general framework,

$$\int_a^b dx \ \{\Sigma_n f_n(y)\, g_n(x)\}\, I(x) \ \neq \ \Sigma_n f_n(y) \ \int_a^b dx \ g_n(x)\, I(x) \ \equiv \Sigma_n f_n(y)\, I_n$$

where the ≠ applies if $f_n(y)$ is not a basis on (a,b). In retrospect, if $f_n(y)$ were powers y^n, then the series in the left term $\Sigma_n f_n(y)\, g_n(x)$ would not exist either! I will be countermanding these conclusions later if it is the case that the f_n form a basis on an interval which *encloses* (a,b).

In **Section 7** I restate the conclusion *still again*, this time as "Theorem 1". This theorem needs to be modified and I will do that in Part II. It is Theorem 1A and it appears in the Part II Overview!

In **Section 8** I "pretend" that the series in question *does* exist (despite all my work above), and I review some of my attempts to solve the half-sphere problem using this series. It all begins with $\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1/2$ which is the Stak "integral equation" we know and love. This is correctly converted to $\Sigma_{n=0}^{\infty} P_n(y)\, I_n = 1$ (assuming the series *does* exist) with the definition $I_n \equiv \int_0^1 dx\, P_n(x)\, I(x)$. I then (wrongly) assume that $I_n = \delta_{n,0}$ is the one and only possible solution of this little equation on (0,1), and is therefore the correct solution to it for the half sphere problem. This leads to the system of equations $\int_0^1 dx\, P_n(x)\, I(x) = \delta_{n,0}$. It then became the subject of my two "legendre my problem" docs to solve this equation for I(x). I then review my Plan C attempt (from my previous "leg my prob" doc) to solve for I(x). This numerical work suggested (correctly) that no solution I(x) exists! This Plan C is also reviewed in my previous doc's overview, so I won't review it again here. The review given below is also pretty good. Next, I review something I call Plan B, but which is really subsection 8 of my first "potential of half-spherical shell" doc. This Plan also led to the same conclusion that no I(x) exists. In retrospect, it is OK that no I(x) solution exists to a system of equations which are an incorrect representation of my half sphere problem of interest.

Overview of Part II: Slippery When Wet (+ Theorem 1A) (2 pages 9.24.10)

In **Section** 1 I state the expansion/projection transform associated with a set of functions φ_n complete on an interval (a,b). The expansion is a series which "exists" in the sense of stable coefficients. I don't care

really whether the high end of the series diverges or not, which is a distinct meaning of "existence". I am here only concerned with the first meaning.

In **Section 2** I comment that the expansion series certainly continues to "exist" if you decide to only look at it on an enclosed interval (A,B) within (a,b). The coefficients (projections) involve full (a,b) integrals still in this case. This is the "peephole" idea.

In **Section 3** I wander off briefly on something not relevant or useful.

In **Section 4** I observe that, if f(x) exists on (A,B), you can *extend* it to the full (a,b) by adding 0 pieces, call this $f_{ext}(x)$. Then you can expand $f_{ext}(x) = \Sigma_n\, c_n\, \varphi_n(x)$ on (a,b) [series exists]. The peephole view of this series on (A,B) is then $f(x) = \Sigma_n\, c_n\, \varphi_n(x)$ which series must then also exist. Moreover, the c_n involve only (A,B) integrals of f(x). By this process, I have proven the following theorem:

Theorem 1A. Let (a,b) enclose (A,B), let φ_n be complete on (a,b), and let f(x) be defined only on (A,B). Then the series $f(x) = \Sigma_n\, c_n\, \varphi_n(x)$ on (A,B) *exists*, and $c_n = \int_A^B dx\, f(x)\, \varphi_n(x)$.

Conversely, suppose $f(x) = \Sigma_n\, c_n\, \varphi_n(x)$ on (A,B) where there exists no enclosing interval (a,b) on which the φ_n are complete. In this case, the φ_n are only a spanning set on all (a,b) enclosing intervals. It is likely then that the series $\Sigma_n\, c_n\, \varphi_n(x)$ which attempts to equal f(x) will not exist, because the coefficients c_n will keep moving as the series is lengthened. An example would be $\varphi_n = x^n$. For any given finite last term N, you would determine the optimal c_n by a least squares fit. There is no formula for c_n as in the previous case. Of course if f(x) in this case happens to be a polynomial, the series *will* exist (and be finite).

In **Section 5** I apply a full $4\pi \int d\Omega_s\, Y_{nm}(\Omega_s)$ to both sides of the $V(s) = \Sigma\, q/R$ equation and this takes me down a wrong alley and all I conclude is the following, which does not contain my "series of interest",

$$\int dz_s\, P_n(z_s)\, V(z_s) = 4\pi a\, (2n+1)^{-1} \int_b\, dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

This equation is pretty useless since it just relates integrals of two unknown functions.

In **Section 6** I apply just a *bowl* integral $\int_b\, d\Omega_s\, Y_{nm}(\Omega_s)$ to both sides of the $V(s) = \Sigma\, q/R$ equation and this takes me down another wrong alley which is nevertheless mildly interesting. In place of the result quoted above, I get

$$V_0 \int_b\, dz_s\, P_n(z_s) \;=\; 2\pi a\, \Sigma_{n"}\; \{ \int_b dz_s\, P_n(z_s)\, P_{n"}(z_s) \}\, [\int_b\, dz_\xi\, P_{n"}(z_\xi)\, \sigma(z_\xi)]$$

I then give the names $k_n(b)$, $h_{nn"}(b)$ and $\sigma_{n"}(b)$ to the three integrals shown to get

$$k_n(b)\, V_0 \;=\; 2\pi a\, \Sigma_{n"}\, h_{nn"}(b)\, \sigma_{n"}(b)$$

I then rescale the vector k by defining $K(b)_n = k_n(b)\, V_0/(2\pi a)$ and write the above in matrix form:

$$\mathbf{K}(b) = H(b)\,\boldsymbol{\sigma}(b) \qquad \Rightarrow \qquad \boldsymbol{\sigma}(b) = H(b)^{-1}\,\mathbf{K}(b)$$

This seems to be a new formal method of solving the half sphere problem. You pick some N and compute the H and K objects since they are just known integrals of P_n functions, and get the $\sigma_n(b)$ from which you can get $\sigma(z)$ which is what you want -- the charge density on the half sphere. As you increase N, I think the solution would converge. I could try this in Maple some rainy day. However, this is not really relevant to the discussion at hand.

In **Section 7** I finally get around to deriving the equation which has the series of interest in it. In fact I obtain two related "charged bowl" equations each having a series:

$$V(z_s) = 2\pi a \,\Sigma_n\, P_n(z_s)\,\sigma_n(b) \qquad \text{valid for all } z_s \text{ in } (-1,1)$$

$$V_0 = 2\pi a \,\Sigma_n\, P_n(z_s)\,\sigma_n(b) \qquad \text{valid for } z_s \text{ in } (\beta,1) \qquad // \quad \sigma_n(b) \equiv \int_b dz_\xi\, P_n(z_\xi)\,\sigma(z_\xi)$$

The second equation is of course just the first with z_s located "on the bowl", and it contains our "series of interest". We ask: does this series exist or not? My answer is yes, by the following argument: there is no issue with the first equation's series existing since the P_n are complete on (-1,1). The second equation's series is just a peephole view of the first equation's series, so it also exists. Tempest in a teapot.

I also consider another series expansion $\sigma(z) = \Sigma_n\, \sigma'_n\, P_n(z)$ on $(\beta,1)$ and show this exists as well since the P_n are complete on an enclosing interval (-1,1).

In **Section 8** I review the conclusions above, admit that my "series does not exist" claim of Part I was incorrect, and convert a few equations from green-Jackson units to Stakgold units.

In **Section 9** I give a simple proof that $\int_0^1 dx\, f(x)\, P_n(x) = \delta_{n0}$ has no solution f(x). Had I been aware of this simple proof at the start, I never would have written these two docs!

Part I: Original Contents of this Doc

0. Sometimes taking a personal voyage to a place helps in your understanding of that place, compared to just reading about that place. I think the place involved here is the subject of "infinite dimensional Hilbert Spaces", but I am not yet quite sure. This was just another boring chapter of Stakgold Volume I that one has to slog through. But now I think I have been drawn into that place "in my face".

1. Review of Past Efforts on the half sphere problem. Change scene now to our problem of charge self-distribution on a hemispherical shell, or on any spherical cap smaller than a whole sphere. We start with the self-consistency integral equation

$$\int_{\sigma} E(s|\xi) I(\xi) dS_{\xi} = 1 \qquad \sigma = \text{half sphere}$$

We insert an expansion for 1/R in E, assume I has no azimuthal dependence, and we arrive here, where we define $y = z_s = \cos(\theta_s)$ and $x = z_\xi = \cos(\theta_\xi)$ where $dS_\xi = 1^2 d\Omega_\xi$: [will show wrong below!] [but then I later show it is right !!]

$$\Sigma_n P_n(y) \int_0^1 dx\, P_n(x)\, I(x) = 2 \qquad \text{half sphere} \qquad \text{valid (0,1) for } x_s = \cos(\theta_s) \equiv y$$

where the 2 comes from $E = 1/(4\pi R)$ and azimuthal integral $\rightarrow 2\pi$. This can be rewritten in two ways:

$$\Sigma_n P_n(y)\, I_n = 2 \qquad (*) \qquad \text{where} \qquad I_n \equiv \int_0^1 dx\, P_n(x)\, I(x)$$

$$\int_0^1 dx\, K(y,x)\, I(x) = 2 \qquad (**) \qquad \text{where} \qquad K(y,x) \equiv \Sigma_n P_n(y)\, P_n(x)$$

In either form, we see the stringency of the condition: the RHS is magically independent of y, whereas the LHS depends on y. The second form appears in my "canonical structures" PDF book by VSV that I luckily downloaded yesterday. For our purposes here, this simply confirms that I have not made a "math error" in developing the two lines shown above.

2. Let's look for a moment at the whole sphere problem. We can use normalized basis functions defined as follows:

$$\psi_n(x) = (n+1/2)^{1/2}\, P_n(x)$$

and then we can write orthogonality and completeness in this way

$$\int_{-1}^1 dy\, \psi_m(y)\psi_n(y) = \delta_{m,n} \qquad \delta(x-y) = \Sigma_n \psi_n(x)\psi_n(y)$$

Since I am extremely prone to errors, I will verify this left equation:

$$\int_{-1}^1 dy\, \psi_m(y)\psi_n(y) = \int_{-1}^1 dy\, [(m+1/2)^{1/2}\, P_m(x)]\, [(n+1/2)^{1/2}\, P_n(x)] =$$

$$= (m+1/2)^{1/2}(n+1/2)^{1/2} \int_{-1}^1 dy\, P_m(x)\, P_n(x)$$

$$= (m+1/2)^{1/2}(n+1/2)^{1/2}\, \delta_{nm}\, (n+1/2)^{-1} = (n+1/2)^{1}\, \delta_{nm}\, (n+1/2)^{-1} = \delta_{nm} \qquad \text{OK}$$

and from this we can write the following expansion theorem:

$I(x) = \Sigma_n I_n \psi_n(x)$ $\qquad I_n = \int_{-1}^{1} dx\, \psi_n(x) I(x)$

Now we can rewrite (*) for the whole sphere [which is $\Sigma_n P_n(y) I_n = 2$] in this way

$\Sigma_n \psi_n(y) J_n = 2$ $\qquad$ where $\qquad J_n = (n+1/2)^{-1/2} I_n \qquad I_n = (n+1/2)^{1/2} J_n$

and now let's make sure we did this right:

$\Sigma_n \psi_n(y) J_n = \Sigma_n [(n+1/2)^{1/2} P_n(x)][(n+1/2)^{-1/2} I_n] = \Sigma_n P_n(x) I_n$

We can then apply $\int_{-1}^{1} dy\, \psi_m(y)$ to both sides of our equation on the last line, we get

$$\int_{-1}^{1} dy\, \psi_m(y) \Sigma_n \psi_n(y) J_n = \int_{-1}^{1} dy\, \psi_m(y)\, 2$$

$$\Sigma_n J_n \int_{-1}^{1} dy\, \psi_m(y) \psi_n(y) = 2 \int_{-1}^{1} dy\, \psi_m(y)$$

$$\Sigma_n J_n \delta_{m,n} = 2\, \delta_{m,0}$$

$$J_m = 2\, \delta_{m,0}$$

$$I_n = (n+1/2)^{+1/2} J_n = (n+1/2)^{+1/2}\, 2\, \delta_{n,0} = (1/2)^{1/2}\, 2\, \delta_{n,0} = 2^{-1/2}\, 2\, \delta_{n,0} = \sqrt{2}\, \delta_{n,0}$$

Then we use our expansion to find

$$I(x) = \Sigma\, I_n \psi_n(x) = \sqrt{2}\, \psi_0(x) = \sqrt{2} [(0+1/2)^{1/2} P_0(x)] = \sqrt{2}\, 2^{-1/2}\, 1 = 1$$

which agrees with our solution elsewhere of this full sphere problem.

2a. Same as 2 in bra ket notation. Now let's rewrite some of our whole sphere equations in Hilbert Space notation:

$\int_{-1}^{1} dy\, \psi_m(y) \psi_n(y) = \delta_{m,n} \qquad \delta(x-y) = \Sigma_n \psi_n(x) \psi_n(y)$

$\langle m|n\rangle = \delta_{m,n} \qquad 1 = \Sigma_n |n\rangle\langle n| \qquad \delta(x-y) = \langle x|y\rangle$

$I(x) = \Sigma_n I_n \psi_n(x)$ $\qquad I_n = \int_{-1}^{1} dx\, \psi_n(x) I(x)$

$\langle x|I\rangle = \Sigma_n \langle x|n\rangle\langle n|I\rangle$ $\qquad$ $\langle n|I\rangle = \int_{-1}^{1} dx \langle n|x\rangle\langle x|I\rangle$

$\Sigma_n \psi_n(y) J_n = 2$ $\qquad$ where $\qquad$ $J_n = (n+1/2)^{-1/2} I_n$ $\qquad$ $I_n = (n+1/2)^{1/2} J_n$

$\Sigma_n \langle y|n\rangle\langle n|J\rangle = 2 = \langle y|J\rangle = J(y)$ $\qquad$ so J(y) = 2, but we want to know I(y) :

$\int_{-1}^{1} dy\, \psi_m(y) \Sigma_n \psi_n(y) J_n = \int_{-1}^{1} dy\, \psi_m(y)\, 2$

$\int_{-1}^{1} dy \langle m|y\rangle \Sigma_n \langle y|n\rangle J_n = 2 \int_{-1}^{1} dy \langle m|y\rangle\langle y|0\rangle$

$\Sigma_n J_n \langle m \left[\int_{-1}^{1} dy\, |y\rangle \langle y| \right] n\rangle = 2 \langle m|0\rangle$

$\Sigma_n J_n \langle m| n\rangle = 2 \langle m|0\rangle$

$J_m = 2 \delta_{m,0}$

$I_m = (m+1/2)^{1/2} J_m = 2/\sqrt{2}\ \delta_{m,0} = \sqrt{2}\ \delta_{m,0}$

$I(x) = \Sigma_n I_n \psi_n(x)$ $\qquad$ ie, $\qquad$ $\langle x|I\rangle = \Sigma_n \langle x|n\rangle\langle n|I\rangle$

$I(x) = \Sigma_n \sqrt{2}\ \delta_{n,0} \psi_n(x) = \sqrt{2}\ \psi_0(x) = \sqrt{2}\ (0+1/2)^{1/2} P_0(x) = 1$

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3. Resume Half Sphere. Now having done all of the above for the whole sphere, we return to our half sphere situation:

$\Sigma_n P_n(y) I_n = 2$ $\qquad$ (*) $\qquad$ where $\qquad$ $I_n \equiv \int_{0}^{1} dx\, P_n(x) I(x)$

$\int_{0}^{1} dx\, K(y,x) I(x) = 2$ $\qquad$ (**) $\qquad$ where $\qquad$ $K(y,x) \equiv \Sigma_n P_n(y) P_n(x)$

The functions $P_n(x)$ do not form a complete set of basis functions on (0,1), so we cannot "construct" our infinite dimensional Hilbert Space method used in 2 and 2a above. The functions $P_n(x)$ could just as well be x^n , another set of functions on (0,1) which do not form a basis on that interval. I want now to look back at Stakgold Chapter 2 on this subject and recall what was taught there. Here are some quotes from my raw notes.

David Hilbert (1862-1943, German) was a very big gun, see wiki. Did physics math support among other things, for QM and gen rel, stated the 23 problems, some still unsolved. Godel friend, etc.

definition: **"a spanning set"** . Let H = some Hilbert space. Suppose we can find a *countable* set of vectors f_1, f_2 ... (such as x^n) whose linear combinations form a space which is *dense* in H. If such a set can be found, then those f's are set to form **a spanning set** (which is not necessarily a basis, see later).

An immediate example of such a countable set would be the powers t^n or some polynomials of degree n in a Hilbert space consisting of functions of a real variable going to same. Here n = 1,2,3.... is where the countable comes in. We have already seen how such polynomials are dense in L_2.

So our Hilbert Space is L_2 on (0,1). The functions $P_n(x)$ or x^n on this interval are **countable** and no doubt are **dense** in L_2 so the set of $P_n(x)$ or x^n do form a **spanning set** for our problem of the half sphere.

definition: **"separable Hilbert Space"** : one which has at least one spanning set.

So without doubt L_2 on (0,1) is a **separable** Hilbert Space, I just listed two different spanning sets.

Dependence and Independence of Vectors: notion of a Basis (118)

Detail on the **spanning set**: a spanning set f_1, f_2 ... means that you can get arbitrarily close to an element x in H by doing a *finite* sum $x \approx \Sigma^n a_k f_k$. As you require a lower ε, you in general need more terms, that is, you have to increase n. ***If***, when you do this all the way to the dimensionality of the vector space N, the a_k *don't change* for earlier terms in the series, then the f_k also form a **basis** because then you have written $x = \Sigma^{nmax} a_k f_k$. For N = finite, a spanning set is always a basis because once you reach N terms in the sum, you have your a_k and you are done and the f_k must be a basis. For N = infinite, it is easy to find cases where the a_k don't stay put as you crunch down on your ε closeness requirement. I think, for example, that in L_2 , the set of powers $t^k = f_k$ forms a spanning set, but not a basis. However, any *orthogonal* set of spanning set functions f_k (perhaps made by G-S) do always form a basis, and this is formally stated as Theorem 1 a few pages down in these notes.

Notice that, even if f_k are not a basis, a spanning set spans a space that is **dense** in H since you can get arbitrarily close. Call this space S. If you add all the points that you cannot quite reach, that is the **closure** $\overline{S}$, and then you have $\overline{S} = H$. I think the closure of any space S that is dense in H is H.

So consider our situation above

$$\Sigma_n P_n(y) I_n = 2 \qquad (*) \qquad \text{where} \qquad I_n \equiv \int_0^1 dx\, P_n(x) I(x)$$

Here $P_n(x)$ form a spanning set but not a basis for the interval (0,1). We are thinking about

$$f(y) = 2 = \Sigma_{n=0}^{\infty} P_n(y) a_n \qquad\qquad f(y) = 2 \approx \Sigma_{n=0}^{N} P_n(y) a_n$$

For any given $N < \infty$, we can find a set of a_n that make this "fit" as good as possible. But as we increase N, the a_k **keep moving**, no matter how large we make N. The conclusion is this. Suppose you write

$$f(y) = \Sigma_{n=0}^{\infty} P_n(y) a_n \qquad \text{on} \qquad (0,1)$$

You can write these symbols on paper, but **such a series DOES NOT EXIST** ! There is no set of a_n that makes this series true! For finite N you can find a set of a_k that make a best fit.

Notice that we are not saying the series does not converge, we are saying it does not exist! If something does not exist, you cannot even talk about whether it converges or not.

Now let's go back and examine our developmental steps from the integral equation for I.

1 $\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1$ σ = half sphere

2 $(1/4\pi) \int_\sigma d\Omega_\xi \{ \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y_{nm}(\Omega_\xi) Y^*_{nm}(\Omega_s) \} I(\Omega_\xi) = 1$ σ = half sphere

3 $(1/4\pi) \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y^*_{nm}(\Omega_s) \int_\sigma d\Omega_\xi Y_{nm}(\Omega_\xi) I(\Omega_\xi) = 1$ σ = half

4. $(1/4\pi) \Sigma_n N_{0n}(1/N_{0n}) Y^*_{n0}(\Omega_s) \int_\sigma d\Omega_\xi Y_{n0}(\Omega_\xi) I(\Omega_\xi) = 1$ σ = half

5. $(1/2) \Sigma_n P_n(z_s) \int_0^1 dz_\xi P_n(z_\xi) I(z_\xi) = 1$ half sphere valid (0,1) for z_s

6. $(1/2) \Sigma_n P_n(z_s) I_n = 1$ half sphere valid (0,1) for z_s

We know step 1 is valid, and we know that step 6 has a series which does not exist! There is no set of coefficients I_n which make equation 6 true (sums are all n=0,∞) . Therefore equation 5 is also not valid, because the integral there is I_n. The problem presumably *has* some solution I(x), so these integrals which we call I_n would then be a set of actual numbers we could compute for all n. But that would imply that equation 6 is valid, which we know it is not. Again, equation 5 is not valid.

Equation 2 is valid because we have just inserted a classical expansion for 1/R. If that expansion did not exist, we would not be seeing it in the literature! This expansion is a double series which exists and which converges for any points ξ and s on the sphere.

If equation 3 were valid, then equation 4 would follow from it, undeniably because we know the azimuthal integral is 0 unless m = 0,

Conclusion: The error is made going from equation 2 to equation 3. This is a place I always knew was a potential problem, but thought is was OK here. This is an interchange of order of summation and integration. Doing this interchange creates a series that does not exist!

4. Integration Range. Look again at our claim:

$$2 = \int_0^1 dx \, \Sigma_n P_n(y) P_n(x) I(x) \neq \Sigma_n P_n(y) \int_0^1 dx \, P_n(x) I(x) \equiv \Sigma_n P_n(y) I_n$$

Why does the integration range have to relate to our interval of interest (a,b)? Suppose (0,1) is a subset of (a,b), for example, (a,b) = (-1,1). Then on this full interval, the series on the right DOES exist, and there ARE some I_n which satisfy $2 = \Sigma_{n=0}^{\infty} P_n(y)\, I_n$. That is OK, but then we have to regard the equation

$$2 = \int_0^1 dx\, \Sigma_n P_n(y)\, P_n(x)\, I(x) = \Sigma_n P_n(y)\, I_n$$

as being valid on y in *all* of (-1,1). But that is like saying $\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$ where σ is the half sphere, but we allow s to go onto the lower half sphere. This is then the WRONG integral equation for charge self consistency.

5. Force I(x) = 0 on lower half sphere. Suppose we insist that I(ξ) = 0 on the lower half sphere and consider $\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$ where now σ is the entire sphere. This equation then applies for ξ (within the integration) anywhere on the whole sphere, but s only on the upper half sphere. We then write our integral equation and posited sum in either of these manners,

$$2 = \int_{-1}^1 dx\, \Sigma_n P_n(y)\, P_n(x)\, I(x) = \Sigma_n P_n(y)\, I_n \qquad I_n = \int_{-1}^1 dx\, P_n(x)\, I(x)$$

or

$$2 = \int_0^1 dx\, \Sigma_n P_n(y)\, P_n(x)\, I(x) = \Sigma_n P_n(y)\, I_n \qquad I_n = \int_0^1 dx\, P_n(x)\, I(x)$$

BUT either of these equations is only valid on y ε (0,1). The equation is known to be *invalid* outside of this y range, in fact, because the potential is not going to be "1" on the lower half sphere. So we have to restrict the equation to being valid on L_2 over just (0,1). But in this case, the series shown does not exist.

We could consider a *different* integral equation which is true on y ε (-1,1), and in that case the series DOES exist and $I_n = 2\, \delta_{n,0}$ and I(x) = 1.But this equation is the whole sphere problem, and our assumption that I(x) = 0 on the lower half sphere is no good.

6. More General Situation. Assume then that we have a Fredholm equation of this form:

$$\int_a^b dx\ \{\Sigma_n f_n(y)\, g_n(x)\}\, I(x) = h(y)$$

where we have written the kernel as an infinite diagonal sum which presumably exists and the f and g are just some functions. The whole theory of Fredholm equations is based on this equation applying for y on the range (a,b) which appear as endpoints (the Volterra equation has an upper endpoint y instead of b). So we assume that the equation is meant to be true on *all* of (a,b) and we have L_2 on (a,b) as our HS. Then we try to process the LHS as follows

$$\int_a^b dx\ \{\Sigma_n f_n(y)\, g_n(x)\}\, I(x) \stackrel{?}{=} \Sigma_n f_n(y) \int_a^b dx\ g_n(x)\, I(x) \equiv \Sigma_n f_n(y)\, I_n$$

If the functions $f_n(y)$ do not form a basis on (a,b), then this series does not exist, and we have to replace the symbol =?= with ≠ !!! So in this situation, interchanging the order of integration and summation is not allowed !!! . There exists no set of numbers I_n such that $h(y) = \Sigma_{n=0}^{\infty} f_n(y) I_n$ on (a,b).

This was exactly our situation where (a,b) = (0,1) and $f_n = g_n = P_n$.

7. Theorem 1: Consider the expansion $h(y) = \Sigma_{n=0}^{\infty} f_n(y) I_n$. Whether this series exists or not depends on the range over which you mean the expansion to be true. Suppose $f_n(y)$ form a basis on (a,b). Then this series exists only on the range (a,b). For any other range, the series does NOT exist. If we know that h(y) is a periodic function on the range (a,b+N(b-a))for N=integer, then in that special case the series would also exist, but not for general h(y). So the default situation really is that the above series does NOT exist. For it to exist, the functions have to form a basis on the interval in question.

So you really need to think about this any time you do a change of integration order. The interchange is not valid any time you create a series that does not exist!

8. Review of attempts to solve: $\int_0^1 dx\, P_n(x) I(x) = \delta_{n,0}$

I erroneously converted the half-sphere integral equation into the following series form, with the intended region of validity y in (0,1). [Here I rescale the RHS to make things simpler, which would correspond to integral equation $\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1/2$, except of course it is invalid]

$\Sigma_{n=0}^{\infty} P_n(y) I_n = 1$ (*) where $I_n \equiv \int_0^1 dx\, P_n(x) I(x)$

I now know that the series does not exist on (0,1), so there is no point in continuing. But what happens if you assume it *does* exist? Then you are led down mysterious alleyways that go nowhere, to wit:

Plan C Reviewed

The first thing you do, assuming the expansion (*) of 1 exists, is to claim that you have an immediate solution for the coefficients I_n (of the equation $\Sigma_{n=0}^{\infty} P_n(y) I_n = 1$).

$I_n = \delta_{n,0}$

[In retrospect, it is true that this *is* a solution of the equation $\Sigma_{n=0}^{\infty} P_n(y) I_n = 1$ for y on (0,1), but it is not *the* solution which is correct for the half sphere. But I did not know that when writing this doc.]

This then leads you to try to find a function I(x) on (0,1) which has these properties

$\int_0^1 dx\, P_n(x) I(x) = \delta_{n,0}$ (*)

[This equation comes from our definition that $I_n \equiv \int_{-1}^{1} dz_\xi \; P_n(z_\xi)\, I(z_\xi)$ to which we add our knowledge that $I(z_\xi) = 0$ for the lower sphere, which tells is we can also write $I_n = \int_{0}^{1} dz_\xi \; P_n(z_\xi)\, I(z_\xi)$. So we are then combining the definition of I(n), knowledge that I=0 on the bottom, and our incorrect solution that $I_n = \delta_{n,0}$ to get the equation (*) shown above. We then try to solve this incorrectly derived equation.]

In Section 6 (Plan C) of my document "legendre my problem.doc", I tried to solve these equations in the following manner. I found a set of complete orthonormal basis functions

$$\varphi_n(x) = (2n+1)^{1/2}\, P_n(-1+2x)$$

on the interval (0,1), and I tried expanding both I(x) and $P_n(x)$ onto these functions, certainly allowable. My expansions were [I call the coefficients of I(x) f_n to not confuse them with I_n shown above]

$$I(x) = \Sigma_m\, f_m\, \varphi_m(x) \qquad f_m = \int_0^1 dx\, \varphi_n(x) f(x) = (2n+1)^{1/2} \int_0^1 P_n(-1+2x)\, f(x)$$

$$P_n(x) = \Sigma_k\, G_{nk}\, \varphi_k(x) \qquad G_{nk} = \int_0^1 dx\, P_n(x)\, \varphi_k(x) = \text{known numbers}$$

Insertion of these expansions into (*) and use of orthonormality then yielded

$$\Sigma_m\, G_{nm}\, f_m = \delta_{n,0} \qquad \text{or} \qquad G\, f = 1$$

I considered this a square matrix equation, and figured maybe I could invert it and find a solution for the vector f whose components are f_m θ

$$f = G^{-1}\, 1 \qquad \Rightarrow$$

$$f_n = \Sigma_m\, [\, G^{-1}]_{nm}\, \delta_{m,0} = [\, G^{-1}]_{n0} = \text{(first column of matrix } G^{-1})$$

I had Maple compute the matrix G and G^{-1} for various finite summation end values N. This was an exact calculation, since G_{nk} are well-defined numbers. What I found was that the first column of G^{-1} has numbers which "don't move" -- you just get a longer list of numbers as you increase N. For N = 6 for example we got this for f_n

$$[1, -\sqrt{3}, 3\sqrt{5}, -11\sqrt{7}, 45\sqrt{9}, -197\sqrt{11}\,]$$

I then used these resulting values for f_n and installed them into my valid expansion above to get I(x), and then I plotted I(x). I even then recomputed $\int_0^1 dx\, P_n(x) I(x)$ to make sure I was getting $\delta_{n,0}$, and this all worked out fine. BUT, what I seemed to find was that I(x) so-computed was not approaching any stable limit. For any finite N, I was able to obtain an exact function I(x) which satisfied my requirement

$$\int_0^1 dx\, P_n(x)I(x) = \delta_{n,0} \qquad \text{where solution is:} \qquad I(x) = \Sigma_m f_m \varphi_m(x)$$

Now this series for I(x) on (0,1) is "legal series", but I was finding that it diverges! The f_m coefficients just keep getting larger and larger as m increases. I had a sequence of functions, call it this

$$I_N(x) = \Sigma_{m=0}^{N} f_m \varphi_m(x)$$

How do you test to see whether a sequence converges or not? You really need to know something about what the f_m do in size, and my method here makes that pretty hard to see. But all my numerical evidence was that the f_m were getting large without limit, and it certainly seemed that $I_N(x)$ was not approaching a limit. Each N made a new function that was different from the previous value of N. I thought maybe this sequence was approaching some kind of distribution, if not a function. So this is where this little effort stopped. The conclusion seemed to be this: no function I(x) exists such that $\int_0^1 dx\, P_n(x)I(x) = \delta_{n,0}$ on the interval (0,1). As you increase N, you force I(x) to bend more ways to maintain all previous orthogonalities *and* have one new one.

I was very disturbed by this apparent fact, since it seemed to say that my half sphere shell had no solution for its charge density I(x). But now I know that the set of conditions $\int_0^1 dx\, P_n(x)I(x) = \delta_{n,0}$ is *not valid* for this half sphere problem. Thus the fact that these conditions are satisfied by no function is less bothersome. And later I learned that the half sphere solution is $I(x) = (\pi/2)\ 1/\sqrt{x}$, and this very obviously does not satisfy $\int_0^1 dx\, P_n(x)I(x) = \delta_{n,0}$ as Maple happily shows for various n.

Plan B Reviewed

I came back for more of the same I think, a slight variation of the above. I start with

$$I_n = \int_0^1 dx\, P_n(x)I(x) = \delta_{n,0}$$

and then activate those same $\varphi_n(x)$ on (0,1) as above. This time we try this

$$\varphi_n(x) = \Sigma_{m=0}^{\infty} k_{nm} P_m(x) \qquad (*)$$

which we now know is a non-existent sum on (0,1) [say for some particular value of n] . The reason is that the P_m are not a complete set on (0,1) so the k_{nm} for given n should "keep moving" as we increase N and do best bit to $\varphi_n(x)$. We can "extend" the range of both sides to (-1,1). This gets us involved with values like $P_n(-3)$ on the LHS, but never mind. After extension, the sum does exist at least, and we can solve for k_{nm} ,

$$k_{nm} = (1/2)\,(2m+1)\,(2n+1)^{1/2} \int_{-1}^1 dx\, P_m(x)\, P_n(-1+2x) = (1/2)\,(2m+1) \int_{-1}^1 dx\, P_m(x)\, \varphi_n(x)$$

$$k_{n0} = (1/2)\,(2n+1)^{1/2} \int_{-1}^{1} dx\, P_n(-1+2x) \qquad \text{// special case for use below}$$

which are a well defined set of numbers for finite n and m.

Now we go back to our (0,1) world where we have the expansion theorem

$$f_n = \int_0^1 dx\, \varphi_n(x) I(x) \qquad I(x) = \Sigma_{n=0}^{\infty} f_n \varphi_n(x) \qquad x = \cos\theta$$

If we insert (*) into the projection here we get

$$f_n = \int_0^1 dx\, \{ \Sigma_{m=0}^{\infty} k_{nm} P_m(x) I(x) \} = \Sigma_{m=0}^{\infty} k_{nm} \int_0^1 dx\, P_m(x) I(x) = \Sigma_{m=0}^{\infty} k_{nm}\, \delta_{m0} = k_{n0}$$

Notice that these expansion coefficients f_n are the same as those used in Plan A above. There we only knew that $f_n = [\, G^{-1}]_{\,n0,}$ but now we have a nice expression for f_n as the integral k_{n0}. So the rest of the story comes out exactly as in Plan A above. That is to say, we have our candidate series for I(x) on (0,1)

$$I(x) = \Sigma_{n=0}^{\infty} f_n \varphi_n(x) = \Sigma_{n=0}^{\infty} k_{n0}\, \varphi_n(x)$$

Maple gives us these values for the first 11 k_{n0} coefficients

```
> with(orthopoly);
> phi := (2*n+1)^(1/2)*P(n,-1+2*x):
> kn0 := (1/2)*(2*n+1)^(1/2) * int(P(n,-1+2*x), x=-1..1):
> for n from 0 to 10 do kn0 od;
```

$$1$$
$$-\sqrt{3}$$
$$3\sqrt{5}$$
$$-11\sqrt{7}$$
$$45\sqrt{9}$$
$$-197\sqrt{11}$$
$$903\sqrt{13}$$
$$-4279\sqrt{15}$$
$$20793\sqrt{17}$$
$$-103049\sqrt{19}$$
$$518859\sqrt{21}$$

which you see agree with the smaller f_n list shown in Plan A above. Our series diverges!

Summary: we attempt to solve $\int_0^1 dx\, P_n(x) I(x) = \delta_{n,0}$ for I(x). This is an equation which we earlier found had no solution, but we try anyway. Our candidate solution is

$I(x) = \Sigma_{n=0}^{\infty} k_{n0}\ \varphi_n(x)$ where $k_{n0} = (1/2)(2n+1)^{1/2} \int_{-1}^{1} dx\ P_n(-1+2x)$

Although this series at least "exists" on (0,1) since $\varphi_n(x)$ form a complete basis on (0,1), in order to make the series solve our condition $\int_0^1 dx\ P_n(x)I(x) = \delta_{n,0}$ we find that the series severely diverges. So again our conclusion is that there is no solution I(x) that meets all these conditions. For any finite N we can find I(x) that meets N conditions. Each time we increment N by 1, the solution I(x) changes violently everywhere in the range. The sequence $I_N(x)$ defined as partial sums does not converge.

And I must hasten to add: this is no longer bothersome because $\int_0^1 dx\ P_n(x)I(x) = \delta_{n,0}$ is now understood to be an invalid encapsulation of our surface charge integral equation for the half sphere.

Task Still Remaining: Solve $\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1$ for $I(\xi)$! [I much later solved this integral equation using a Polyanin lookup.]

Part II: Slippery When Wet 9.22.10.

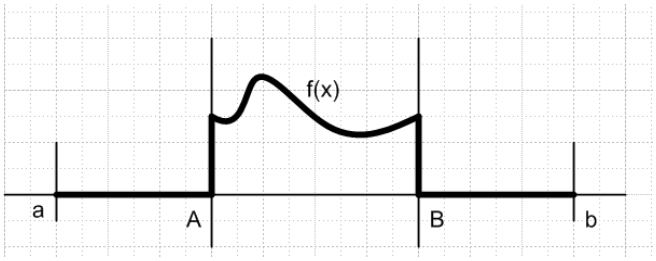
I still have not nailed down this series existence issue, it keeps squirming around. Let's try it once again. In what follows, Σ_n always means $\Sigma_{n=0}^{\infty}$.

1. Full (a,b) interval series. Suppose you have $f(x) = \Sigma_n f_n\varphi_n(x)$ on interval (a,b) where φ_n are complete and orthonormal on that interval. This series exists, all is well, and we know that $f_n = \int_a^b f(x)\ \varphi_n(x)$ are the coefficients. We assume f(x) is some reasonable function so f_n exists and the series converges.

2. Enclosed (A,B) view of series. The series of course also is fine if we examine it just on some subinterval (A,B) which is contained in (a,b). On (A,B) the series exists and converges. The coefficients are the same f_n shown above. They are NOT $F_n \equiv \int_A^B f(x)\ \varphi_n(x)$, although the F_n can surely be computed.

3. Aside: Since we can compute these F_n , we can at least *ask* about the possible series $\Sigma_n F_n\varphi_n(x)$. Since the F_n differ from the f_n, we know that $\Sigma_n F_n\varphi_n(x) \neq f(x)$ on all of (a,b) because the f_n are unique for a given f(x). On the interval (a,b), since $\varphi_n(x)$ are complete, it is possible that $\Sigma_n F_n\varphi_n(x) = g(x)$, ie, the series converges to some well defined function. Since φ_n are complete on (a,b), the only malfunction here would be if this series failed to converge. If it converges, then the series is good. In this case, we would conclude that we can write $F_n = \int_a^b g(x)\ \varphi_n(x)$. Then we have $F_n = \int_a^b g(x)\ \varphi_n(x) = \int_A^B f(x)\ \varphi_n(x)$.

4. Extension by 0 of the enclosed interval function. Here is one possibly useful concept. We are handed f(x) on the internal interval (A,B) on which the φ_n are not a complete set. We can "extend" f(x) to the larger interval (a,b) [on which φ_n are a complete set] by making it be 0 outside (A,B):



In this case we can expand the extended function on the complete set φ_n so that $f(x) = \Sigma_n F_n\varphi_n(x)$ where $F_n = \int_A^B f(x)\ \varphi_n(x)$ [$= \int_a^b f(x)\ \varphi_n(x) = f_n$] as above. In this situation, the series $f(x) = \Sigma_n F_n\varphi_n(x)$ certainly "exists" despite the fact that a counter theorem wants to make it not exist. The counter theorem is that on (A,B), the φ_n do not form a complete set, they are just a spanning set, and therefore $\Sigma_n F_n\varphi_n(x)$ on (A,B) does not exist as would be the case if the φ_n were just powers. But we see that the counter theorem is "overridden" if (1) the φ_n are a complete set on some enclosing interval, and (2) on the enclosing interval we extend f(x) by making it be 0 outside the enclosed interval.

5. Bowl problem applying $\int d\Omega_s\ Y_{nm}(\Omega_s)$ to both sides of V(s) equation.

Let's now try to apply this idea to our half sphere situation. Start with this general statement, where b means an integral over the bowl, and where we use the Stak version of the 1/R expansion, but let's change units so E = 1/R so we are in green Jackson units. Then

$$V(s) = \int_b dS_\xi\ E(s|\xi)\sigma(\xi) = \int_b dS_\xi\ 1/R\ \sigma(\xi) = a^2 \int_b d\Omega_\xi\ 1/R\ \sigma(\xi)$$

$$= a^2 \int_b d\Omega_\xi\ \{ (1/a) \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"})\ Y_{n"m"}(\Omega_\xi)\ Y^*_{n"m"}(\Omega_s)\}\ \sigma(\xi)$$

$$= a \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"})\ Y^*_{n"m"}(\Omega_s) [\int_b d\Omega_\xi\ Y_{n"m"}(\Omega_\xi)\ \sigma(\xi)]$$

The object [...] is analogous to $\int_A^B f(x)\ \varphi_n(x)$ of our last section, an integral over an enclosed interval so to speak. Now apply $\int d\Omega_s\ Y_{nm}(\Omega_s)$ to both sides where we are doing a full 4π integration. We get

$$\int d\Omega_s\ Y_{nm}(\Omega_s)\ V(s)$$

$$= a \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) \{ \int d\Omega_s\ Y_{nm}(\Omega_s)\ Y^*_{n"m"}(\Omega_s) \} [\int_b d\Omega_\xi\ Y_{n"m"}(\Omega_\xi)\ \sigma(\xi)]$$

Using the Stak p 395 result we have $\int d\Omega_s\, Y_{nm}(\Omega_s)\, Y^*{}_{n''m''}(\Omega_s) = \delta_{nn''}\delta_{mm''}\, N_{nm}$ so we get

$$\int d\Omega_s\, Y_{nm}(\Omega_s)\, V(s)$$

$$= a\, \Sigma_{n''}\, N_{0n''}\, \Sigma_{m''}\, (1/N_{m''n''})\, \{\, \delta_{nn''}\delta_{mm''}\, N_{nm}\, \}\, [\ \int_b d\Omega_\xi\, Y_{n''m''}(\Omega_\xi)\, \sigma(\xi)\ ]$$

$$= a\ N_{0n}\, (1/N_{mn})\, \{\, N_{nm}\, \}\, [\ \int_b d\Omega_\xi\, Y_{nm}(\Omega_\xi)\, \sigma(\xi)\ ]$$

$$= a\ N_{0n}\ [\ \int_b d\Omega_\xi\, Y_{nm}(\Omega_\xi)\, \sigma(\xi)\ ]$$

$$= a\ \{4\pi/(2n+1)\}\ [\ \int_b d\Omega_\xi\, Y_{nm}(\Omega_\xi)\, \sigma(\xi)\ ]$$

$$= 4\pi a\, (2n+1)^{-1}\, [\ \int_b d\Omega_\xi\, Y_{nm}(\Omega_\xi)\, \sigma(\xi)\ ]$$

Meanwhile, we know that V(s) is really $V(\theta_s)$ only. So the LHS becomes

$$\int d\Omega_s\, Y_{nm}(\Omega_s)\, V(s) = \delta_{m0} \int d\Omega_s\, Y_{n0}(\Omega_s)\, V(s) = 2\pi\delta_{m0} \int dz_s\, P_n(z_s)\, V(z_s)$$

We can make the same argument on the [...] integral so say that

$$\int_b d\Omega_\xi\, Y_{nm}(\Omega_\xi)\, \sigma(\xi) = 2\pi\delta_{m0} \int_b dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

Our overall equation then becomes:

$$2\pi\delta_{m0} \int dz_s\, P_n(z_s)\, V(z_s) = 4\pi a\, (2n+1)^{-1}\, 2\pi\delta_{m0} \int_b dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

If $m \neq 0$, this says $0 = 0$ which is fine. So only real content is when $m = 0$ and we have

$$\int dz_s\, P_n(z_s)\, V(z_s) = 4\pi a\, (2n+1)^{-1} \int_b dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

where on the LHS he have the enclosing region integral (-1,1), and on the RHS an enclosed region (0,1) for the half shell bowl. *If* we had a full sphere at this point, then $V(z_s) = V_0$ for all z_s in (-1,1) and we then get

$$\int dz_s\, P_n(z_s)\, V(z_s) = 4\pi a\, (2n+1)^{-1} \int_b dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

$$2\, \delta_{n0}\, V_0 = 4\pi\, a(2n+1)^{-1} \int_b dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

$$2\,V_0 = 4\pi\, a \int_b dz_\xi\, \sigma(z_\xi) \;=\; a\, 2\,\sigma_0 \qquad\qquad \Rightarrow \qquad V_0 = 4\pi\, a\, \sigma_o$$

Q would then be $Q = 4\pi a^2\sigma_0 = 4\pi a^2(V_0/4\pi a) = V_0 a$, so $C = Q/V_0 = a$.

So where has this left us? For the bowl situation, we end up with this general equation:

$$\int dz_s\, P_n(z_s)\, V(z_s) = 4\pi a\, (2n+1)^{-1} \int_b dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

where we know $V(z_s) = V_0$ on the bowl, but we don't know it beyond the bowl edge. And we don't know $\sigma(z_s)$ on the bowl. So both sides of this equation contain unknown functions. Unfortunately, I don't see how this equation can be connected with our discussion of section 4 above. I have managed to get rid of all azimuthal stuff and just have the Legendre functions appear.

6. Bowl problem applying $\int_b d\Omega_s\, Y_{nm}(\Omega_s)$ to both sides of V(s) equation.

Suppose we were to apply $\int_b d\Omega_s\, Y_{nm}(\Omega_s)$ to both sides instead of a full integral. Then looking at the above, we would have:

$$\int_b d\Omega_s\, Y_{nm}(\Omega_s)\, V(s)$$

$$= a\, \Sigma_{n"}\, N_{0n"}\, \Sigma_{m"}\, (1/N_{m"n"})\, \{\int_b d\Omega_s\, Y_{nm}(\Omega_s)\, Y^*_{n"m"}(\Omega_s)\}\, [\;\; \int_b d\Omega_\xi\, Y_{n"m"}(\Omega_\xi)\, \sigma(\xi)\;\;]$$

We could right at this point make our two replacements:

$$\int_b d\Omega_s\, Y_{nm}(\Omega_s)\, V(s) \;=\; \delta_{m0} \int_b d\Omega_s\, Y_{n0}(\Omega_s)\, V(s) \;=\; 2\pi\delta_{m0} \int_b dz_s\, P_n(z_s)\, V(z_s)$$

$$\int_b d\Omega_\xi\, Y_{n"m"}(\Omega_\xi)\, \sigma(\xi) \;=\; 2\pi\delta_{m"0} \int_b dz_\xi\, P_{n"}(z_\xi)\, \sigma(z_\xi)$$

and rewrite the above as

$$2\pi\delta_{m0} \int_b dz_s\, P_n(z_s)\, V(z_s)$$

$$= a\, \Sigma_{n"}\, N_{0n"}\, \Sigma_{m"}\, (1/N_{m"n"})\, \{\int_b d\Omega_s\, Y_{nm}(\Omega_s)\, Y^*_{n"m"}(\Omega_s)\}\, [\;\; 2\pi\delta_{m"0} \int_b dz_\xi\, P_{n"}(z_\xi)\, \sigma(z_\xi)\;]$$

$$= 2\pi a\, \Sigma_{n"}\, N_{0n"}\, (1/N_{0n"})\, \{\int_b d\Omega_s\, Y_{nm}(\Omega_s)\, Y^*_{n"0}(\Omega_s)\}\, [\; \int_b dz_\xi\, P_{n"}(z_\xi)\, \sigma(z_\xi)\;]$$

$$= 2\pi a\, \Sigma_{n"}\;\; \{\int_b d\Omega_s\, Y_{nm}(\Omega_s)\, Y^*_{n"0}(\Omega_s)\}\, [\; \int_b dz_\xi\, P_{n"}(z_\xi)\, \sigma(z_\xi)\;]$$

But of course we know that

$$\int_b d\Omega_s\, Y_{nm}(\Omega_s)\, Y^*_{n''0}(\Omega_s) = \delta_{m0} \int_b d\Omega_s\, Y_{n0}(\Omega_s)\, Y^*_{n''0}(\Omega_s)$$

$$= 2\pi\, \delta_{m0} \int_b dz_s\, P_n(z_s)\, P_{n''}(z_s)$$

so we then get

$$2\pi\delta_{m0} \int_b dz_s\, P_n(z_s)\, V(z_s)$$

$$= 2\pi a\, \Sigma_{n''} \{2\pi\, \delta_{m0} \int_b dz_s\, P_n(z_s)\, P_{n''}(z_s)\} \left[\int_b dz_\xi\, P_{n''}(z_\xi)\, \sigma(z_\xi) \right]$$

Again if $m \neq 0$ we get $0 = 0$, so we take $m = 0$ to get

$$\int_b dz_s\, P_n(z_s)\, V(z_s)$$

$$= 2\pi a\, \Sigma_{n''} \{ \int_b dz_s\, P_n(z_s)\, P_{n''}(z_s) \} \left[\int_b dz_\xi\, P_{n''}(z_\xi)\, \sigma(z_\xi) \right]$$

But we know $V(z_s) = V_0$ for the entire LHS integration, so we can rewrite as

$$V_0 \int_b dz_s\, P_n(z_s)$$

$$= 2\pi a\, \Sigma_{n''} \{ \int_b dz_s\, P_n(z_s)\, P_{n''}(z_s) \} \left[\int_b dz_\xi\, P_{n''}(z_\xi)\, \sigma(z_\xi) \right]$$

We might then define some known integrals

$$\int_b dz_s\, P_n(z_s) \equiv k_n(b)$$

$$\int_b dz_\xi\, P_n(z_\xi)\, P_{n''}(z_\xi) = h_{nn''}(b)$$

Then our equation above becomes

$$k_n(b)\, V_0 = 2\pi\, a\, \Sigma_{n''}\, h_{nn''}(b) \left[\int_b dz_\xi\, P_{n''}(z_\xi)\, \sigma(z_\xi) \right]$$

Finally, let's define our "enclosed integral" projection this way

$$\sigma_{n''}(b) \equiv \int_b dz_\xi \, P_{n''}(z_\xi) \, \sigma(z_\xi)$$

Then we have

$$k_n(b) \, V_0 = 2\pi a \, \Sigma_{n''} \, h_{nn''}(b) \, \sigma_{n''}(b)$$

So this is what happened to our early statement that $\int_\sigma E(s|\xi) I(\xi) dS_\xi = V_0$. It is a matrix equation

$$K(b) = H(b) \, \sigma(b) \qquad\qquad K(b)_n = k_n(b) \, V_0/(2\pi a)$$

We can solve this if H can be inverted, so get

$$\sigma(b) = H(b)^{-1} \, K(b)$$

So this is all "well and good", but again, I see no connection to section 4 above. It does suggest a new method of solving the bowl problem. You might use this last line to find that values of $\sigma_{n''}(b)$. We do know that

$$\sigma_{n''}(b) \equiv \int_b dz_\xi \, P_{n''}(z_\xi) \, \sigma(z_\xi) = \int dz_\xi \, P_{n''}(z_\xi) \, \sigma(z_\xi)$$

because we know $\sigma(z_\xi) = 0$ beyond the bowl. Therefore we can invert to get $\sigma(z_\xi)$ in terms of $\sigma_{n''}(b)$. Here then we are "extending" the function $\sigma(z_\xi)$ by having it be 0 outside our enclosed interval. In this last step we are then in effect adding the second dual integral equation boundary condition.

7. The V(s) equation without applying any $\int d\Omega_s \, Y_{nm}(\Omega_s)$.

I was hoping in doing the above work to arrive at an equation *of this form*:

$$(1/2) \, \Sigma_n \, P_n(z_s) \int_0^1 dz_\xi \, P_n(z_\xi) \, \sigma(z_\xi) = 1 \qquad \text{half sphere} \qquad \text{valid (0,1) for } z_s$$

and then I could talk about whether this series exists or not. But neither of the two threads I just traced out in sections 5 or 6 led to such an equation! So now I have to go study Part C of "potential of half-spherical shell.doc" to see how this arises. In both my threads above, I applied some sort of $\int d\Omega_s \, Y_{nm}(\Omega_s)$ to both sides. In case 5 it was the full integral, in case 6 the bowl only integral. But in Part C we don't apply any operator like this.

So let's back up to the start of section 5 above where we had

$$V(s) = \int_b dS_\xi \, E(s|\xi)\sigma(\xi) = \int_b dS_\xi \, 1/R \, \sigma(\xi) = a^2 \int_b d\Omega_\xi \, 1/R \, \sigma(\xi)$$

$$= a^2 \int_b d\Omega_\xi \{ (1/a) \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) Y_{n"m"}(\Omega_\xi) Y^*_{n"m"}(\Omega_s)\} \sigma(\xi)$$

$$= a \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) Y^*_{n"m"}(\Omega_s) [\int_b d\Omega_\xi Y_{n"m"}(\Omega_\xi) \sigma(\xi)]$$

[This is in Jackson units, and I can get back to Part C Stak units by multiplying V by 4π.]

Right at this point, let us use this fact:

$$\int_b d\Omega_\xi Y_{n"m"}(\Omega_\xi) \sigma(\xi) = 2\pi\delta_{m"0} \int_b dz_\xi P_{n"}(z_\xi) \sigma(z_\xi)$$

to get

$$V(s) = a \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) Y^*_{n"m"}(\Omega_s) [\int_b d\Omega_\xi Y_{n"m"}(\Omega_\xi) \sigma(\xi)]$$

$$= a \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) Y^*_{n"m"}(\Omega_s) [2\pi\delta_{m"0} \int_b dz_\xi P_{n"}(z_\xi) \sigma(z_\xi)]$$

$$= a \Sigma_{n"} N_{0n"} (1/N_{0n"}) Y^*_{n"0}(\Omega_s) [2\pi \int_b dz_\xi P_{n"}(z_\xi) \sigma(z_\xi)]$$

$$= 2\pi a \Sigma_n P_n(z_s) [\int_b dz_\xi P_n(z_\xi) \sigma(z_\xi)]$$

[Aside: we can at this point, if we like, write this as

$$V(s) = 2\pi a \int_b dz_\xi \sigma(z_\xi) \{ \Sigma_n P_n(z_s) P_n(z_\xi) \}]$$

As at the end of section 6 we define then

$$\sigma_n(b) \equiv \int_b dz_\xi P_n(z_\xi) \sigma(z_\xi)$$

and we end up with

$$V(z_s) = 2\pi a \Sigma_n P_n(z_s) \sigma_n(b)$$

This equation (assuming series exists) is valid for all z_s in (-1,1). We know that when z_s lies on the bowl, this equation becomes:

$$V_0 = 2\pi a \Sigma_n P_n(z_s) \sigma_n(b)$$ half sphere valid (0,1) for z_s

and THIS then is the equation I have been looking for.

So, does this series exist or not?

series $f(z_s) \equiv \Sigma_n \; P_n(z_s)\, \sigma_n(b)$ valid for z_s in $(\beta,1)$ $\beta = 0$ for half sphere

where $\sigma_n(b) \equiv \int_b dz_\xi \, P_n(z_\xi)\, \sigma(z_\xi)$ // $= F_n$

Well, I think the answer is quite simple. The series

$V_0 = 2\pi a \, \Sigma_n \; P_n(z_s)\, \sigma_n(b)$

is just a "peephole view" looking at the enclosed interval $(\beta,1)$ of the following series

$V(z_s) = 2\pi a \, \Sigma_n \; P_n(z_s)\, \sigma_n(b)$

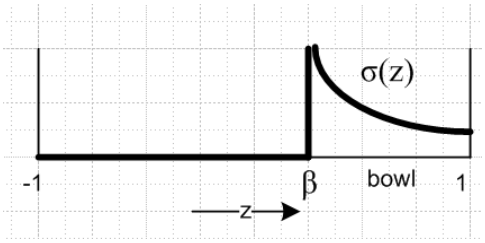
which is defined on the full enclosing interval (-1,1). On this interval the P_n are a complete set of functions, and the series exists and all is well. If the enclosing series exists, then of course if you examine it just on the enclosed interval, it still exists.

Conclusion: The series on the RHS of this equation -- which equation is only true for limited z_s --

$V_0 = 2\pi a \, \Sigma_n \; P_n(z_s)\, \sigma_n(b)$ half sphere valid (0,1) for z_s

does in fact exist,

Now, what about the idea of "extending" the function $\sigma(z_\xi)$ to the full range by making it be 0 outside the bowl. Physically we know we can do this, because we know σ really is 0 outside the bowl. Using this method, we can show that a slightly different series also exists. Consider what the charge density looks like in our bowl problem, where we regard the full curve as the extended σ or $\sigma_{ext}(z)$,



$\sigma_{ext}(z)$ exists on the full (-1,1), and matches $\sigma(z)$ on $(\beta,1)$. We know we can expand $\sigma_{ext}(z)$ on the $P_n(z)$ in an "existent" series with some coefficients which we write this way

$\sigma_{ext}(z) = \Sigma_n \, \sigma'_n \, P_n(z)$ where $k_n \, \sigma'_n = \int_{-1}^{1} dz \, P_n(z)\, \sigma_{ext}(z) = \int_{\beta}^{1} dz \, P_n(z)\, \sigma(z).$

where $k_n = 1/(n+1/2)$ is a normalizing factor since the P_n are not orthonormal, recall

$$f_n = \int_{-1}^{1} dx\ f(x)\, P_n(x) \qquad\qquad f(x) = (1/2)\Sigma_n(2n+1) f_n\, P_n(x)$$

If we now look in the peep-hole bowl region, we can write this as

$$\sigma(z) = \Sigma_n\, \sigma'_n\, P_n(z) \qquad \text{where} \quad k_n\, \sigma'_n = \int_{-1}^{1} dz\, P_n(z)\, \sigma_{ext}(z) = \int_{\beta}^{1} dz\, P_n(z)\, \sigma(z).$$

So this is another "series" that we can consider being valid on (β,1) which is not the full (-1,1) but the series definitely exists because it is a peep-hole view of another series we know exists.

8. A quick review and statement of equations in Stakgold Units.

(a) We derived this equation, where I now suppress the parameter b ,

$$V(z_s) = 2\pi a\ \Sigma_n\, P_n(z_s)\, \sigma_n \qquad \text{where} \quad \sigma_n \equiv \int_b dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

and the $\int_b$ came from our original integral equation " $\int_b E(s|\xi) I(\xi) dS_\xi = 1$ ". This equation "knows about" the fact that σ = 0 outside the bowl. If σ were not zero outside the bowl, this equation as stated would not be true because Σ q/R would have to include this extra σ.

(b) The equation is valid on the entire (-1,1) for z_s , on which the P_n are complete, so the series which appears, namely $\Sigma_n\, P_n(z_s)\, \sigma_n$, does not have "existence issues" in the sense that all the coefficients keep moving because P_n is only a spanning set. If we take a peep-hole view of the above equation, we get

$$V_0 = 2\pi a\ \Sigma_n\, P_n(z_s)\, \sigma_n \qquad \text{for } z_s \text{ in } (\beta,1) \qquad \sigma_n \equiv \int_b dz_\xi\, P_n(z_\xi)\, \sigma(z_\xi)$$

so this series also exists since it is a peephole view of the previous series.

(c) As a separate subject, we can consider a series expansion of σ(z) and we find on all of (-1,1) that

$$\sigma_{ext}(z) = \Sigma_n\, \sigma'_n\, P_n(z) \qquad \text{where} \quad k_n\, \sigma'_n = \int_{-1}^{1} dz\, P_n(z)\, \sigma_{ext}(z) = \int_{\beta}^{1} dz\, P_n(z)\, \sigma(z).$$

In this case, the peephole view is

$$\sigma(z) = \Sigma_n\, \sigma'_n\, P_n(z) \qquad \text{where} \quad k_n\, \sigma'_n = \int_{-1}^{1} dz\, P_n(z)\, \sigma_{ext}(z) = \int_{\beta}^{1} dz\, P_n(z)\, \sigma(z).$$

The first series is defined on all of (-1,1) and thus has no existence issues. The second equation is a peephole view of the first, and therefore also has no existence issues.

(d) The coefficients are different for the two series considered in (b) and (c), but involve the exact same integrals. The difference is the normalizing factor k_n which involves (2n+1).

(e) So, my earlier conclusion that the change of order of integration and summation created a non-existent series was WRONG !!! I now have to backtrack and fix that up wherever that conclusion is used.

(f) If we want to convert our last several equations to Stak units, multiply V by 4π. Thus:

$2V(z_s) = a \Sigma_n \; P_n(z_s) \sigma_n(b)$

$2V_0 = a \Sigma_n \; P_n(z_s) \sigma_n(b)$ half sphere valid (0,1) for z_s

and of course I was using $V_0 = 1$ and sphere radius a = 1, so these then become

$2V(z_s) = \Sigma_n \; P_n(z_s) \sigma_n(b)$

$2 = \Sigma_n \; P_n(z_s) \sigma_n(b)$ half sphere valid (0,1) for z_s

Our "kernel form" equation above was $V(s) = 2\pi a \int_b dz_\xi \; \sigma(z_\xi) \; \{ \Sigma_n \; P_n(z_s) \; P_n(z_\xi) \} \;]$ in Jackson, so we can convert it as well to Stak to get

$2 \; V(s) = a \int_b dz_\xi \; \sigma(z_\xi) \; \{ \Sigma_n \; P_n(z_s) \; P_n(z_\xi) \}$

and again with a = 1 and s on the bowl and $V_0 = 1$ this becomes

$2 = \int_b dz_\xi \; \sigma(z_\xi) \; \{ \Sigma_n \; P_n(z_s) \; P_n(z_\xi) \} = \int_b dz_\xi \; \sigma(z_\xi) \; K(z_s, z_\xi)$

9. Proof that $\int_0^1 dx \; f(x) \; P_n(x) = \delta_{n0}$ has no solution f(x).

I now realize that I had a proof sitting there in my first doc [end of Section 4, "how do we resolve this mystery"] . Here I will formalize it, motivated a bit by what happened in this second doc.

We start with our equation to be solved

$\int_0^1 dx \; f(x) \; P_n(x) = \delta_{n0}$ $\qquad n = 0,1...\infty$

Assume that some solution f(x) exists. Extend that solution to the full (-1,1) interval by adding a zero piece to the left, call this $f_{ext}(x)$. Surely we can do that. Then we have $f_{ext}(x) = f(x) \; \theta(x)$ on the interval

(-1,1). We can certainly "work with" this function. Here is the Legendre expansion and projection for this function, on the interval (-1,1) for which the P_n are a complete set:

$$c_n = \int_{-1}^{1} dx\ f_{ext}(x)\, P_n(x) \qquad\qquad f_{ext}(x) = \Sigma_{n=0}^{\infty}(n+1/2)c_n\, P_n(x)$$

But of course we know then that

$$c_n = \int_{-1}^{1} dx\ f_{ext}(x)\, P_n(x) \;=\; \int_{-1}^{1} dx\ f(x)\,\theta(x)\, P_n(x) \;=\; \int_{0}^{1} dx\ f(x)\, P_n(x) = \delta_{n0}$$

The expansion then says:

$$f_{ext}(x) = \Sigma_{n=0}^{\infty}(n+1/2)c_n\, P_n(x) = \Sigma_{n=0}^{\infty}(n+1/2)\,\delta_{n0}\, P_n(x) = (1/2)$$

We then end up with the following statement on (-1,1):

$$f(x)\,\theta(x) = 1/2$$

For x = -1/2, say, this says

$$0 = 1/2$$

But that is a contradiction. Therefore our "assumption" that a solution f(x) exists must be false. QED.

In simple terms, we have a Legendre expansion on (-1,1) and we are given $c_n = \delta_{n0}$ right at the start. This *completely determines* $f_{ext}(x)$ to be $f_{ext}(x) = 1/2$ on the interval (-1,1). We cannot on the one hand accept the data $c_n = \delta_{n0}$ and then on the other hand try to insist that the solution have the form $f_{ext}(x) = \theta(x)f(x)$ on (-1,1). This is too many conditions on the problem and there is no solution f(x). In our two docs, we learned the hard way that f(x) does not exist.

Restate once again. We imagine a function $f_{ext}(x)$ that is 0 on (-1,0) and is some f(x) on (0,1). Can the Legendre expansion coefficients be δ_{n0} for this function $f_{ext}(x)$? Well, those coefficients imply that the function is $f_{ext}(x) = (1/2)$ on (-1,1). But we assumed $f_{ext}(x)$ is 0 on (-1,0). So the answer is: No, δ_{n0} cannot be the coefficients of any such function $f_{ext}(x)$. In other words, no such function $f_{ext}(x)$ exists which has δ_{n0} as coefficients. In turn, this means that f(x) does not exist where $f_{ext}(x) = \theta(x)f(x)$. That means that our infinite set of integral equations $\int_{0}^{1} dx\ f(x)\, P_n(x) = \delta_{n0}$ has no solution f(x).