Potential of Half Spherical Shell (v1) PhL started 10.15.09

(9.28.10) This is a failed attempt to solve the charged half-sphere bowl problem using the Stakgold integral equation method $\int_\sigma E(s|\xi)I(\xi)dS_\xi=1$. *I tried to imitate the way this method is applied to solve the full sphere problem, but I took a bad turn along the way which led to my Legendre My Problem doc sequence. I then found the problem solved in the Canonical pdf.*

These notes relate to Stakgold Volume II Chapter on potential theory, specifically I wrote these notes after studying the following Stakgold book section (before doing the examples)

A. Charged Conductor and Electrostatic Capacity (171)

I think the solid metal sphere is done OK here, but when I casually tried to apply the same method to the half shell of a sphere, something went wrong. Since I try to mimic the full sphere method, I have included those notes here first, to set an example.

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Overview (9.25.10, 6 pages)

In **Part A** I treat the full metal shell using the Stak's "integral equation method": $\int_\sigma E(s|\xi)I(\xi)dS_\xi=1$ for **s** on the spherical shell, an integral equation to solve for charge density here called I, V = 1 on the shell. Here are the steps:

$\int_\sigma E(s|\xi)I(\xi)dS_\xi=1$ // starting position

$E(s|\xi) = 1/4\pi R = (1/4\pi)\,\Sigma_{n'}\, P_{n'}(\cos\gamma)$ // expand 1/R on Legendres

$(1/4\pi)\,\Sigma_{n'} \int d\Omega_\xi\, P_{n'}(\cos\gamma)\, I(\Omega_\xi) = 1$ // insert into the above, I dep on Ω_ξ

$(1/4\pi) \int d\Omega_\xi \{ \Sigma_{n"}\, N_{0n"}\, \Sigma_{m"}\, (1/N_{m"n"})\, Y_{n"m"}(\Omega_\xi)\, Y^*_{n"m"}(\Omega_s)\}\ I(\Omega_\xi) = 1$ // expand $P_{n'}$

$(1/4\pi) \int d\Omega_\xi\, \Sigma_{n"}\, N_{0n"}\, \Sigma_{m"}\, (1/N_{m"n"})\, [\delta_{nn"}\delta_{mm"}\, N_{nm}]\ \ Y_{n"m"}(\Omega_\xi)\, I(\Omega_s) = 4\pi\, \delta_{n0}\delta_{m0}$ (*)

// apply $\int d\Omega_s\, Y_{nm}(\Omega_s)$ to both sides, use orthog on each side

$\int d\Omega_\xi\, Y_{nm}(\Omega_\xi)\, I(\Omega_\xi) \equiv N_{mn}\, I_{mn}$ // define projections I_{mn} of $I(\Omega)$

$2\pi \int dz_\xi\, Y_{n0}(z_\xi)\, I(z_\xi) \equiv N_{0n}\, I_{0n}$ // only non-zero I_{mn} values

$I(\Omega) = \Sigma_{mn}\, I_{mn} Y_{nm}(\Omega)$ // corresponding expansion

$(2n+1)^{-1}\, [N_{mn}\, I_{mn}] = 4\pi\, \delta_{n0}\delta_{m0}$ // just rewrite (*) in terms of I_{mn}

$I_{mn} = 4\pi\, \delta_{n0}\delta_{m0}\, (2n+1)/N_{mn} = \delta_{n0}\delta_{m0}\, 4\pi/N_{00} = \delta_{n0}\delta_{m0}$ // solve for I_{mn}

$I(\Omega) = \Sigma_{mn}\, I_{mn} Y_{nm}(\Omega) = Y_{00} = 1$ // do inverse transform to get I

Basically, we do the full Y-Y expansion of 1/R in sphericals (using Stak's Y), we expand $I(\Omega)$ on the Y's, we use orthogonality of the Y's, and we end up with $I_{mn} = \delta_{n0}\delta_{m0}$ and then $I(\Omega) = 1$. Not rocket science really. This is really just an SO(3) diagonalization, because we had a full $d\Omega$ SO(3) integration.

In **Part B** I naively try to use this same method for a bowl that is a half-spherical shell. I mimic the above sequence of steps, but I get to this point,

$(1/4\pi) \int_\sigma d\Omega_\xi \{ \Sigma_{n"}\, N_{0n"}\, \Sigma_{m"}\, (1/N_{m"n"})\, Y_{n"m"}(\Omega_\xi)\, Y^*_{n"m"}(\Omega_s) \}\, I(\Omega_\xi) = 1 = f(s)$

I note correctly that (1) the integral $d\Omega_\xi$ is only over the upper half shell, but since you know $I(\Omega)=0$ on the lower half shell, you could *regard* $d\Omega_\xi$ as a full sphere integral; (2) the equation is only valid when Ω_s is on the upper half shell. Due to (2), you cannot apply full $\int d\Omega_s\, Y_{nm}(\Omega_s)$ to both sides of this equation!!! If you apply $\int_{\text{upper half}} d\Omega_s\, Y_{nm}(\Omega_s)$ to both sides, then you cannot use orthogonality. You can define $\int_{\text{upper half}} d\Omega'\, Y_{nm}(\Omega')\, I(\Omega') \equiv N_{mn}\, I'_{mn}$ for this problem. Since you know $I(\Omega) = 0$ (no charge density in open space) on the lower half, you could conclude that $I'_{mn} = I_{mn}$ of Part A. At this point I have nowhere to go and I am confused. I make a few claims which I shall ignore for now because I am going to start over in the next Part!

In **Part C** I start over again doing Parts A and B "side by side", and I try to do this carefully in a numbered set of steps:

1-7. Derive $\Sigma_n\, P_n(z_s)\, I_n = 2$ for sphere and half sphere, verify full sphere.

1. Write the **Stak integral equation** for both cases.
2. Install our **1/R expansion** for 1/|ξ-s| .
3. In the resulting Stak equation, **interchange order** Σ_{nm} with $\int_\sigma d\Omega_\xi$.
4. Use result that **I = I(Ω)** so we end up only with $Y_{n0}(\Omega_\xi)$ in our integrands.
5. At this point, write results in terms of the P functions and we have then:

$(1/2)\,\Sigma_n\, P_n(z_s)\ \int_{-1}^{1} dz_\xi\ P_n(z_\xi)\, I(z_\xi) = 1$ whole sphere valid (-1,1) for z_s

$(1/2)\,\Sigma_n\, P_n(z_s)\ \int_{0}^{1} dz_\xi\ P_n(z_\xi)\, I'(z_\xi) = 1$ half sphere valid (0,1) for z_s

Since we know the solutions of these two problems will have different I(Ω), I distinguish them now by putting a prime on I for the second problem. *Assuming* I' = 0 on the lower half in the second case, as discussed at the end of Part B, we can rewrite the above equations so they are exactly the same, though validity ranges are not the same:

$(1/2)\,\Sigma_n\, P_n(z_s)\ \int_{-1}^{1} dz_\xi\ P_n(z_\xi)\, I(z_\xi) = 1$ whole sphere valid (-1,1) for z_s

$(1/2)\,\Sigma_n\, P_n(z_s)\ \int_{-1}^{1} dz_\xi\ P_n(z_\xi)\, I'(z_\xi) = 1$ half sphere valid (0,1) for z_s

6. We can regard the z integrals shown as the I_n projections in each case, so we **then have**

$\Sigma_n\, P_n(z_s)\, I_n = 2$ whole sphere valid (-1,1) for z_s

$\Sigma_n\, P_n(z_s)\, I'_n = 2$ half sphere valid (0,1) for z_s

I observe that $I_n = 2\,\delta_{n,0}$ is in fact *a* solution of *either* equation. For the full sphere case, I show using the usual P transform that if $I_n = 2\,\delta_{n,0}$, you end up with I(θ) = 1 for all θ, which is correct. But we can apply the same argument to the second problem and we find I'(θ) = 1 for all θ which we know is wrong, because we just assumed I'(θ) = 0 on the lower half a few lines above. I conclude that the second problem must have some *other* solution I'_n which *differs* from $I_n' = 2\,\delta_n$, and which yields I'(θ) = 0 on the lower half. But I have no idea at this point how to find this solution. [You can think of $\Sigma_n\, P_n(z_s)\, I'_n = \mathbf{P} \bullet \mathbf{I}'$ and knowledge of this dot product and P does not determine a unique **I'**]

Comment added: The half shell problem has some definite V on the lower half sphere, but we don't know what it is. Imagine this "correct V on lower half" is one entry in a list of a million candidate prescribed potentials for the lower half that we add to our V = 1 prescription on the upper half. We then have a **million valid Dirichlet problems [** a million interior ones, and a million exterior ones] . As we vary this unknown lower potential through this list, the condition above that $\Sigma_n\, P_n(z_s)\, I_n' = 2$ for z_s in (0,1) will always remain true! For each list candidate, there will be some solution I_n' which makes this true, and *also* gives the candidate prescribed potential on the lower half. The particular solution $I_n' = 2\,\delta_{n,0}$ is only the correct solution where the candidate potential on the lower half is the same constant as on the top half! Our equation $\Sigma_n\, P_n(z_s)\, I_n' = 2$ for z_s in (0,1) must therefore have some *other* solution [ie, other than $2\,\delta_{n,0}$] for our half shell problem! The solution to $\Sigma_n\, P_n(z_s)\, I_n' = 2$ is not unique, but is different for each

of our million Dirichlet problems. Of course only 1 of these million solutions I_n' causes $I'(\theta) = 0$ on the lower half sphere, and this is then the solution to our problem. [That is to say, when you use the "correct" lower sphere prescribed potential and solve the interior and exterior Dirichlet problems, you will find that the radial derivative of the potential has no jump across the lower sphere surface.] This just says that if you solve only *one* of a pair of dual integral (or dual series) equations, the solution is not unique and you have not solved the problem!

Another commented added: Now $\Sigma_n P_n(z_s) I_n' = 2$ can be regarded symbolically as $\int dn\, P_n(z_s) I'(n) = 2$ which we would call an "integral equation" for function I'(n). Write as $\int dn\, k(n',z_s) I'(n) = f(z_s) = 2$ which we rewrite as K I' = f = 2. This is a **first kind Fred**. I conjecture that the boundary conditions on the function I'(n) are not fully determined, and that is why this Fred 1 equation does not have a unique solution. We shall return to this subject below, but in θ space rather than n space. [I like my dot product argument given above better.]

7. Here I make use of that fact that (5a) above is valid for the full range of z_s so I can use orthogonality to find that $I_n = 2\,\delta_{n,0}$ and then $I(\theta) = 1$. Ie, this for the **full sphere**.

7A. Make the (incorrect) assumption that $\int_0^1 dx\ P_n(x)\, I(x) = \delta_{n0}$ for the half sphere. Let's look again at our two projection definitions for our two problems:

$$I_n = \int_{-1}^{1} dz_\xi\ P_n(z_\xi)\, I(z_\xi)$$

$$I_n' = \int_0^1 dz_\xi\ P_n(z_\xi)\, I'(z_\xi)$$

If we assume (wrongly) that $2\,\delta_{n,0}$ is the correct **solution** for both problems, we then have

$$\int_{-1}^{1} dz_\xi\ P_n(z_\xi)\, I(z_\xi) = 2\,\delta_{n,0} \qquad \text{full sphere problem}$$

$$\int_0^1 dz_\xi\ P_n(z_\xi)\, I'(z_\xi) = 2\,\delta_{n,0} \qquad \text{half sphere problem}$$

We can see at once that $I(z_\xi) = 1$ on full range of z_ξ solves the first equation set, since we can just use the orthogonality of the P_n functions. But we are forced to wonder: is there some function $I'(z_\xi)$ defined on the upper half sphere that satisfies the second equation set? We write this equation as two equations, dropping the prime for now, and changing to generic variable x:

$$\int_0^1 dx\ I(x) = 2$$

$$\int_0^1 dx\ P_n(x)\, I(x) = 0 \qquad n = 1,2,3.... \qquad // \text{ ie } \int_0^1 dx\ P_n(x)\, I(x) = \delta_{n0}$$

If we could find a solution I(x) defined on x in (0,1) that works in the above equation set, then that I(x) = I'(z_ξ) would be the solution to our problem, and it would have projections $I_n' = 2\,\delta_{n,0}$. So at this point, I sort of reformulated the half-sphere problem as a "search" for a function I(x) solving the above equations. It is clear that I(x) = 2 solves the first equation, but not the others, since the half range integrals of P_n are not all 0 certainly. So any solution I(x) must not be a constant. In addition to work below, I carried out this search in some docs entitled **"Legendre my problem XXX .doc"** stored in the Stakgold folder. My conclusion reached there after many attempted solutions is that no such function I(x) exists which solves the above equations! [See the last META doc in this series for a good summary of things.]

8. Apply $\int_0^1 dz_s\, P_m(z_s)$ to get a Matrix Equation for I_n . Here (just as I do later in Legendre My Problem vol 2 Section 6) I validly apply $\int_0^1 dz_s\, P_m(z_s)$ to both sides of my equation $\Sigma_n P_n(z_s) \int_0^1 dx\, P_n(x)\, I(x) = \Sigma_n P_n(z_s)\, I_n = 2$, and this leads to a matrix equation which one might solve for the "vector" I_n. In a note I added today 9.25.10, I observe that this matrix equation is $R\,\mathbf{I} = R\hat{\mathbf{e}}_1$ where $R_{nm} \equiv \int_\beta^1 dz_s\, P_m(z_s)P_n(z_s)$, and that, when $\beta \neq -1$, I conjecture that detR = 0 and that this equation can have solutions for I_n *other than* the known solution $I_n = \delta_{n0}$.

But then I continue on assuming that the solution has to be $I_n = \delta_{n0}$ (ignoring factor 2), and I then take a stab at solving the equation $\int_0^1 dx\, P_n(x)I(\theta) = 2\,\delta_{n,0}$.

I use $\varphi_n(x) = (2n+1)^{1/2}\, P_n(-1+2x)$ as a *bona fide* complete set on (0,1). I then (strangely) expand $\varphi_n(x) = \Sigma_{m=0}^{\infty}\, k_{nm}\, P_m(x)$. [In light of what I learned very much later in Leg My Prob vol 2 part II Theorem 1A, this is an "existent" series since the P_m form a complete set an in interval that encloses (0,1).] I turn the crank some more and I find this series as my "solution": $I(\theta) = 2\,\Sigma_{n=0}^{\infty}\, k_{n0}\, \varphi_n(\cos\theta)$. But I find that the coefficients k_{n0} (known and calculable) seem to diverge as n increases, suggesting that this method is not generating a solution and maybe there is no solution ! I defer further study of this method.

9. Try expanding $P_n(z)$ and I(z) on complete φ_n defined on (0,1). Here I repeat the above work but in a perhaps more logical manner, using $P_n(x) = \Sigma_{m=0}^{\infty}\, c_{nm}\, \varphi_m(x)$ in the range (0,1). [I return later to this exact same approach in Legendre My Problem vol 1 Section 6 which is called "Plan C".] I then find the solution series to be $I(x) = \Sigma_{n=0}^{N}\, J_n\, \varphi_n(x)$ and I do a few Maple plots and I think maybe this is some kind of unpleasant "distribution", I(x) does not seem like a normal function.

10. Try some simple delta-function forms for I(z). Here I try a few delta-function distribution combinations, blindly searching for a simple solution to $\int_0^1 dx\, P_n(x)\, I(x) = \delta_{n0}$ (later to get the name Legendre My Problem). I even try some differential operators L_0 applied to distributions as possible solutions. I get nowhere doing this. My reason for trying deltas is that in the previous section, it seemed that I(x) was some kind of distribution.

11. Search on line for solution to the half sphere problem. I throw up my hands, I cannot solve

$\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1$, I decide the time has come to look up the solution of the half sphere "in the literature". Up to this point, I was trying to do it all on my own!

12. Discovery of the Canonical PDF. At this point I discover and download the Russian "Canonical" pdf which contains this problem. I am pretty excited to find the solution somewhere! I do not at this point fully trace the solution, but I am able to derive the "dual series equations" from a general Smythian form for V in terms of some a_n coefficients. I then quote the result for the a_n and for $\sigma = I = g$ for the general bowl. Note that the a_n are for the potential, they are not the σ_n I have been messing with above. I set $\theta_0 = \pi/2$ for the half sphere and simplify the results. I seem to find in this case that $\sigma(z) = (2/\pi)1/\sqrt{z}$ but I have never studied this result.

13. Test this Canonical solution. Given this expression for $I = \sigma$, and still assuming $\int_0^1 dx\ P_n(x)\ I(x) = \delta_{n0}$, it is pretty clear that we are not going to get $\int_0^1 dx\ P_n(x)/\sqrt{x} = 0$ for n = 1,2,3... . This increased my doubts about this δ_{n0} result even more. I formally conclude (bold letters) that the δ_{n0} result must in fact be wrong.

I then again consider $\Sigma_n\ P_n(z_s)\ I_n = 2$ where $I_n = \int_0^1 dz_\xi\ P_n(z_\xi)\ I(z_\xi)$. Since $I(z) = (2/\pi)/\sqrt{z}$ according to Canonical, I should be able to compute the I_n and add up the series and it should sum to 2 for any value of z_s in the range (0,1). I have Maple do this for the first 50 terms. The sum is reasonably constant, varying from 2.2 to 2.5 for 10 equally spaced values of z_s , but the number seems to be the wrong number. For the time being, I decided to let that discrepancy ride.

Next, I rederive $\Sigma_n\ P_n(z_s)\ [\ \int_0^1 dx\ P_n(x)\ I(x)\ ] = 2$ for the umpteenth time, and I write it in the alternate obvious form which is $\int_0^1 dx\ [(1/2)\Sigma_n\ P_n(z_s)P_n(x)]\ I(x) = 1$ and I am happy to see that this also appears in the Canonical doc, where the bracket thing is called $k(z_s,x)$. Remember that I am still uncertain about the δ_{n0} equation. I wondered perhaps for the first time if my stuff=2 equation was wrong due to an illegal order interchange of sum and integral.

I next insert their $I(z) = (2/\pi)1/\sqrt{z}$ and tell Maple to try to verify $\int_0^1 dx\ (1/\sqrt{x})\ \Sigma_n\ P_n(z)\ P_n(x) = \pi$. With 20 terms in the Σ_n the LHS is very non constant, but with 50 terms it is fairly constant again with my value of about 2.5, not π. Even today 9.25.10 I don't know what is going on here.

I then ponder the nature of the kernel $\Sigma_{n=0}^{\infty}\ P_n(z)\ P_n(x)$. I try to plot it, and I conclude without proof that " the sum $\Sigma_{n=0}^{\infty}\ P_n(z)\ P_n(x)$ is in fact a distribution, similar to $\Sigma_{n=0}^{\infty}\ (n+1/2)\ P_n(z)\ P_n(x)$." At this point, I finally give up and cease and desist. But of course this just means I went to the v2 document.
[in v2, I show that $\Sigma_{n=0}^{\infty}\ P_n(z)\ P_n(x)$ may be a perfectly reasonable non-distributional type function!]

Notes in blue at the end: First 6" is nothing. In fact the whole blue section looks pretty trashy, so I will just ignore it. It does things I have already done, probably these were obs notes before I rewrote certain sections.

A. Review of the Full Charged Spherical Shell Problem

This is a closed surface and we are interested in the exterior problem. The integral equation is this, where we are simply computing the potential by superposition and setting it arbitrarily to u = 1 on the surface:

$\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$ s and ξ are points on the spherical shell of radius 1

$E(s|\xi) = 1/4\pi R = (1/4\pi) \Sigma_{n'} P_{n'}(\cos\gamma)$ γ is the angle between our two rays (r = 1)

Put point ξ at Ω' and put point s at Ω. Then we have

$(1/4\pi) \Sigma_{n'} \int d\Omega' P_{n'}(\cos\gamma) I(\Omega') = 1$

Install the full Y expansion to get this saying

$(1/4\pi) \int d\Omega' \{ \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) Y_{n"m"}(\Omega') Y^*_{n"m"}(\Omega)\} I(\Omega') = 1$ // see p 126

Now apply $\int d\Omega\, Y_{nm}(\Omega)$ to both sides. The RHS is then $\int d\Omega\, Y_{nm}(\Omega)\, Y^*_{00}(\Omega)$ since in Stak norm we have $Y_{00} = 1$. On page 395 this integral is $\delta_{n0}\delta_{m0} N_{00} = 4\pi\, \delta_{n0}\delta_{m0}$.

The LHS has $\int d\Omega\, Y_{nm}(\Omega)\, Y^*_{n"m"}(\Omega) = \delta_{nn"}\delta_{mm"} N_{nm}$ so we get

$(1/4\pi) \int d\Omega' \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) [\delta_{nn"}\delta_{mm"} N_{nm}] \; Y_{n"m"}(\Omega') I(\Omega') = 4\pi\, \delta_{n0}\delta_{m0}.$

$(1/4\pi) \int d\Omega' N_{0n} (1/N_{mn}) [N_{nm}] \; Y_{nm}(\Omega') I(\Omega') = 4\pi\, \delta_{n0}\delta_{m0}.$

$(1/4\pi) \int d\Omega' (4\pi/(2n+1)) Y_{nm}(\Omega') I(\Omega') = 4\pi\, \delta_{n0}\delta_{m0}.$

$(2n+1)^{-1} \int d\Omega' Y_{nm}(\Omega') I(\Omega') = 4\pi\, \delta_{n0}\delta_{m0}.$

$(2n+1)^{-1} [N_{mn} I_{mn}] = 4\pi\, \delta_{n0}\delta_{m0}.$ // using p 396 A.6 projection

From this we conclude that

$I_{mn} = 4\pi\, \delta_{n0}\delta_{m0} (2n+1)/N_{mn} = \delta_{n0}\delta_{m0}\, 4\pi/N_{00} = \delta_{n0}\delta_{m0}$

Then we have found that

$I(\Omega) = \Sigma_{mn}\, I_{mn} Y_{nm}(\Omega) = Y_{00} = 1.$

So we have now "solved our integral equation" and we find that $I(\Omega) = 1$. We can then normalize this so that the total charge on the sphere is Q'. Just going through the steps here.

B. Attempt of the Half Charged Spherical Shell Problem ("bowl")

By wandering off trying to do something "on my own", I got in big trouble here. I wandered in the wilderness and not only could not solve this problem, but opened more cans of worms. See conclusions at the end of this little section. But I can't *always* just stay on the beaten path. I did learn more about Maple and got to review some earlier Stakgold ODE facts for Legendre.

So here is how I *thought* I was solving this half-shell problem. Let's just try to mimic the steps above as used for the full sphere.

Part 1: Set things up.

The thin half-shell is treated as an "open surface".

$\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$ s and ξ are points on the spherical half shell of radius 1

where now $I(\xi)$ is the sum of the charge densities on the two sides of our thin half-shell.

$E(s|\xi) = 1/4\pi R = (1/4\pi) \Sigma_{n'} P_{n'}(\cos\gamma)$ γ is the angle between our two rays

Put point ξ at Ω' and put point s at Ω. Then we have

$(1/4\pi) \Sigma_{n'} \int_\sigma d\Omega' P_{n'}(\cos\gamma) I(\Omega') = 1$ where now $\int_\sigma d\Omega'$ is *not* a full 4π integral!

Install the full Y expansion to get

$$(1/4\pi) \int_\sigma d\Omega' \{ \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) Y_{n"m"}(\Omega') Y^*_{n"m"}(\Omega) \} I(\Omega') = 1 \quad \text{// p 126} \qquad (*)$$

Now apply full $\int d\Omega\, Y_{nm}(\Omega)$ to both sides.

STOP!! Let's rewrite (*) this way

$$(1/4\pi) \int_\sigma d\Omega_\xi \{ \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) Y_{n"m"}(\Omega_\xi) Y^*_{n"m"}(\Omega_s) \} I(\Omega_\xi) = 1 = f(s)$$

where integral $\int_{\sigma} d\Omega_{\xi}$ is only over the upper half sphere *and*

where the equation is valid ONLY when s ~ Ω lie on the upper half sphere.

This equation is only valid for point s on the upper half sphere, because that is the surface on which our integral equation is forcing self-consistency. So it is only valid for Ω_s on the upper half sphere. It is NOT valid for s on the lower half sphere. Therefore, you cannot apply full $\int d\Omega\ Y_{nm}(\Omega)$ to both sides of this equation!!! If you assume the equation is true on the lower half sphere as well, then you are doing the problem of the full sphere, which we have already done and found I = constant.

The most you can do, if you want, is apply $\int_{\text{upper half}} d\Omega_s\ Y_{nm}(\Omega_s)$ to both sides, but then of course you cannot use full Y orthogonality.

Before realizing this STOP!! fact, I ended up getting the I = constant contradiction for the half sphere, so I am now going to delete notes on that subject below since they are resolved.

Here then is what we have so far: [σ refers to the surface of the half-sphere]

$$(1/4\pi) \int_{\sigma=\text{uhs}} d\Omega_{\xi} \{ \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) Y_{n"m"}(\Omega_{\xi}) Y^*_{n"m"}(\Omega_s) \} I(\Omega_{\xi}) = 1$$

$$(1/4\pi) \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) Y^*_{n"m"}(\Omega_s) \{ \int_{\sigma=\text{uhs}} d\Omega_{\xi} Y_{n"m"}(\Omega_{\xi}) I(\Omega_{\xi}) \} = 1$$

I will define the projection of I in the standard way as in A.6.

$$I_{n"m"} N_{n"m"} = \int_{\sigma} d\Omega_{\xi} Y_{n"m"}(\Omega_{\xi}) I(\Omega_{\xi}) = \int_{\sigma=\text{uhs}} d\Omega_{\xi} Y_{n"m"}(\Omega_{\xi}) I(\Omega_{\xi})$$

WARNING: This is a "standard projection" (left =) only if we assume $I(\Omega_{\xi}) = 0$ on the bottom half of the sphere. I think this is a reasonable thing to assume since I is the discontinuity of $\partial_n u$. However, to avoid a possible error, let's agree here to just DEFINE I_{nm} by the right =. Let's not assume it is a "standard" projection. We shall then make no assumptions whatsoever about $I(\Omega_{\xi})$ at locations other than on σ.

Defining I_{nm} in this way, we get

$$(1/4\pi) \Sigma_{n"} N_{0n"} \Sigma_{m"} (1/N_{m"n"}) Y^*_{n"m"}(\Omega_s) I_{n"m"} N_{n"m"} = 1$$

$$(1/4\pi) \Sigma_{n"} N_{0n"} \Sigma_{m"} Y^*_{n"m"}(\Omega_s) I_{n"m"} = 1$$

But let's go ahead now and assume $I(\Omega_{\xi})$ has no azimuth dependence. Then we know

$$I_{nm} N_{nm} = \int_{\sigma=\text{uhs}} d\Omega_{\xi} Y_{nm}(\Omega_{\xi}) I(\Omega_{\xi}) = \delta_{m,0} I_{n0} N_{n0} = 2\pi \delta_{m,0} \int_0^1 dz\ P_n(z) I(\theta)$$

$$I_{n0} = (2\pi/ N_{n0}) \int_0^1 dz\, P_n(z)\, I(\theta)$$

Our integral equation, which started as $\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$, is now

$(1/4\pi) \Sigma_n N_{0n} P_n(\cos\theta_s) I_{n0} = 1$ // valid only for s on the half sphere ???

$\Sigma_{n=0} I_n P_n(z)/ (2n+1) = 1$ $I_n = I_{n0}$ valid only for $0 \le z \le 1$

COMMENT: Right at this point we are dead meat! The solution is $I_0 = 1$ and $I_n = 0$, and Maple of course confirms that below. So something is WRONG with this equation, no need to go below the line below .

This solution of course makes the equation also be true for θ_s on the lower sphere half, and then we might as well have started with the full sphere integral equation.

C. Let's now do cases A and B "side by side"

Do it all again, comparing whole sphere to half sphere, maybe that will bring out my error.

1-7. Derive $\Sigma_n P_n(z_s) I_n = 2$ for sphere and half sphere, verify full sphere.

1. We start with the **integral equation**:

$\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$ σ = whole sphere

$\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$ σ = half sphere

2. We **install the YY expansion** for 1/R which I believe is true for two points ξ and s on the sphere:

$1/|\xi - s| = \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y_{nm}(\Omega_\xi) Y^*_{nm}(\Omega_s)$ p 126 A

This is the same for either problem. It's just that in the half sphere problem, we will only apply it for point pairs on the upper half-sphere. So we now have [$dS_\xi = r^2 d\Omega_\xi = d\Omega_\xi$]

$(1/4\pi) \int_\sigma d\Omega_\xi \{ \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y_{nm}(\Omega_\xi) Y^*_{nm}(\Omega_s) \} I(\Omega_\xi) = 1$ σ = whole sphere

$(1/4\pi) \int_\sigma d\Omega_\xi \{ \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y_{nm}(\Omega_\xi) Y^*_{nm}(\Omega_s) \} I(\Omega_\xi) = 1$ σ = half sphere

3. Now we **change order of integration** $d\Omega_\xi$ and the summations m and n to get

$(1/4\pi)\ \Sigma_n\ N_{0n}\ \Sigma_m\ (1/N_{mn})\ Y^*_{nm}(\Omega_s)\ \int_\sigma d\Omega_\xi\ Y_{nm}(\Omega_\xi)\ I(\Omega_\xi) = 1$ $\quad\quad$ σ = whole

$(1/4\pi)\ \Sigma_n\ N_{0n}\ \Sigma_m\ (1/N_{mn})\ Y^*_{nm}(\Omega_s)\ \int_\sigma d\Omega_\xi\ Y_{nm}(\Omega_\xi)\ I(\Omega_\xi) = 1$ $\quad\quad$ σ = half

4. Now we **assume $I(\Omega_\xi)$ = indep of φ_ξ** , which is the correct symmetry axis, to get

$(1/4\pi)\ \Sigma_n\ N_{0n}(1/N_{0n})\ Y^*_{n0}(\Omega_s)\ \int_\sigma d\Omega_\xi\ Y_{n0}(\Omega_\xi)\ I(\Omega_\xi) = 1$ $\quad\quad$ σ = whole

$(1/4\pi)\ \Sigma_n\ N_{0n}(1/N_{0n})\ Y^*_{n0}(\Omega_s)\ \int_\sigma d\Omega_\xi\ Y_{n0}(\Omega_\xi)\ I(\Omega_\xi) = 1$ $\quad\quad$ σ = half

5. We can just **write in our P functions** at this point: and do $\int_\sigma d\Omega_\xi = \int dz_\xi d\varphi_\xi = 2\pi \int dz_\xi$ and cancel the N_{0n} matching factors:

$(1/2)\ \Sigma_n\ P_n(z_s)\ \int_{-1}^{1}\ dz_\xi\ P_n(z_\xi)\ I(z_\xi) = 1$ $\quad\quad$ whole sphere $\quad$ valid (-1,1) for z_s

$(1/2)\ \Sigma_n\ P_n(z_s)\ \int_{0}^{1}\ dz_\xi\ P_n(z_\xi)\ I(z_\xi) = 1$ $\quad\quad$ half sphere $\quad$ valid (0,1) for z_s

These appear to be different equations, which we like to see. Now move factor 1/2 and use $z_\xi = x$:

$\Sigma_n\ P_n(z_s)\ \int_{-1}^{1}\ dx\ P_n(x)\ I(x) = 2$ $\quad\quad$ whole sphere $\quad$ valid (-1,1) for z_s $\quad$ (5a)

$\Sigma_n\ P_n(z_s)\ \int_{0}^{1}\ dx\ P_n(x)\ I(x) = 2$ $\quad\quad$ half sphere $\quad$ valid (0,1) for z_s $\quad$ (5b)

These are two distinct "integral equations" and we expect them to have different solutions I(x).

6. If we **DEFINE projections** by the integrals shown here, we restate the above as

$\Sigma_n\ P_n(z_s)\ I_n^{(whole)} = 2$ $\quad\quad$ whole sphere $\quad$ valid (-1,1) for z_s

$\Sigma_n\ P_n(z_s)\ I_n^{(half)} = 2$ $\quad\quad$ half sphere $\quad$ valid (0,1) for z_s

At this point, our two equations look very close, so you wonder how they could have different solutions. In either equation, the obvious solution is $I_0 = 2$ and $I_n = 0$ for n>0.

For the first equation, we can use the theory of the complete Legendre poly set which says:

$f(\theta) = \Sigma_n\ f_n\ P_n(\cos z)$ $\quad\quad$ $f_n = (2n+1)/2\ \int_{-1}^{1}\ dx\ P_n(x)\ f(\theta)$

$f(\theta) = \Sigma_n\ (n+1/2)\ F_n\ P_n(\cos z)$ $\quad\quad$ $F_n = \int_{-1}^{1}\ dx\ P_n(x)\ f(\theta)$

So finding $I_n^{(whole)} = \delta_{n,0}\, 2$ we conclude that $I^{(whole)}(\theta) = (1/2)*2 = 1$, which agrees with our previous solution of the whole sphere problem above.

But what corresponding thing can we say for the second equation? We have no "expansion theorem" for the half sphere problem, so the above line is NOT true for the half sphere. [Wrong! The expansion theorem is still valid. The problem is that if we assume $I_n^{(half)} = \delta_{n,0}\, 2$, this valid expansion theorem tells us that $I^{(half)}(\theta) = 1$ for all θ, and we know this is wrong for the half problem. Thus, $I_n^{(half)} = \delta_{n,0}\, 2$ must be "the wrong solution" for the half problem, though it is certainly *a* solution.]

So let's continue along:

7. For the **whole sphere equation**, we can apply $\int_{-1}^{1} dz_s\, P_m(z_s)$ to (5a) and use orthog to get

$$\Sigma_n \int_{-1}^{1} dz_s\, P_m(z_s)\, P_n(z_s) \int_{-1}^{1} dx\, P_n(x)\, I(x) = 2 \int_{-1}^{1} dz_s\, P_m(z_s)$$

$$\Sigma_n\ \delta_{n,m}\, 2/(2n+1) \int_{-1}^{1} dx\, P_n(x)\, I(x) = 2\, \delta_{m,0}\, 2$$

$$2/(2m+1) \int_{-1}^{1} dx\, P_m(x)\, I(x) = 2\, \delta_{m,0}\, 2$$

$$\int_{-1}^{1} dx\, P_m(x)\, I(x) = 2\, \delta_{m,0}$$

If we try our known solution $I(x) = 1$, this last equation is true. We just replicated the above.

7a. Make the (incorrect) assumption that $\int_0^1 dx\ P_n(x)\, I(x) = \delta_{n0}$ for the half sphere.

This is Legendre My Problem.

Go back to our **half-sphere situation**

$\Sigma_n P_n(z_s)\, I_n^{(half)} = 2$ half sphere valid (0,1) for z_s

$\int_0^1 dx\, P_n(x)\, I(x) = I_n^{(half)}$ // this is the definition of $I_n^{(half)}$ shown in (5b)

Now let's "try out" the candidate solution $I_m^{(half)} = 2\delta_{m,0}$. We then have these equations

$$\int_0^1 dx\, I(x) = 2$$

$$\int_0^1 dx\, P_n(x)\, I(x) = 0 \qquad \text{for } n = 1,2,3....$$

and this does seem to take us back to our original disaster in "Legendre my problem.doc", except I now have a 2 instead of a 1 in the first RHS. This constant is arbitrary anyway since once we solve the problem, we have to scale so that first integral above is Q.

8. Apply $\int_0^1 dz_s\, P_m(z_s)$ to get a Matrix Equation R I = R$\hat{e}_1$ for I_n.

Now go back to the half-sphere integral equation (5b),

$$\Sigma_n\, P_n(z_s)\ \int_0^1 dx\, P_n(x)\, I(x) = 2 \qquad \text{half sphere} \qquad \text{valid } (0,1) \text{ for } z_s$$

We can certainly apply $\int_0^1 dz_s\, P_m(z_s)$ to both sides if we like,

$$\Sigma_n\ \int_0^1 dz_s\, P_m(z_s) P_n(z_s)\ \int_0^1 dx\, P_n(x)\, I(x) = 2 \int_0^1 dz_s\, P_m(z_s)$$

If we make some obvious definitions, this says

$$\Sigma_n\ R_{mn}\ \{\ \int_0^1 dx\, P_n(x)\, I(x)\ \} = 2\, R_{m1} \qquad R_{mn} \equiv \int_0^1 dz_s\, P_m(z_s) P_n(z_s)$$

We can certainly *define* the integral {} to be I_n for this problem, and we then have

$$\Sigma_{n=0}^{\infty}\ R_{mn} I_n = 2 R_{m1} \qquad R\, I = 2\, C \qquad \text{solution } I = 2\ 1\ !$$

where C is the first column of the matrix R. This seems to give a matrix equation one could solve for I.

$$I = 2\, R^{-1} C$$

But I think this is just going to tell us what we already know, that $I_n = \delta_{n,0}\, 2$.

Note: I "rediscovered" the above matrix equation in Legendre My Problem, vol 2, Section 6.

Note Added 9.25.10. Rescaling I by a factor of 2, the matrix equation above is R **I** = $\mathbf{R_1}$ = R$\hat{\mathbf{e}}_1$. It is true that the equation R **I** = R$\hat{\mathbf{e}}_1$ has *a* solution **I** = $\hat{\mathbf{e}}_1$. The real question is: are there perhaps other solutions? If detR ≠ 0, then we have **I** = $R^{-1}R\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_1$ as the only solution. But suppose det R = 0. Then we ask whether the equation R (**I** - $\hat{\mathbf{e}}_1$) = 0 has a solution other than (**I** - $\hat{\mathbf{e}}_1$) = 0. One way to state the answer is that there can be other solutions (**I** - $\hat{\mathbf{e}}_1$) ≠ 0 if the operator (matrix) R has a non-trivial nullspace! Then R **J** = 0 has solutions **J** ≠ 0. Matrix R has a nullspace if it is less than full rank. This of course would also mean that detR = 0. My conclusion (no proof) is that the matrix $R_{nm} \equiv \int_\beta^1 dz_s\, P_m(z_s) P_n(z_s)$ with β ≠ -1

is in fact a matrix of less than full rank, and this means det R = 0, and that means other solutions **I** are possible. Perhaps the dimension of the nullspace is 1 and there is only one "other solution". In the special case that $\beta = -1$, we have $R_{nm} \sim \delta_{nm}$ so $R \sim 1$ and this matrix *does* have full rank and has det $R \neq 0$, so when $\beta = -1$, the *only* solution is $\mathbf{I} = \hat{\mathbf{e}}_1$. For $\beta \neq -1$, I don't offhand know how you would *prove* that det R = 0 or that R has less than full rank. R is a matrix with discrete indices but is of infinite dimension. I am more used to having continuous indices for an infinite dimensional matrix, as with the kernel of an integral equation k(x,y). I have never wondered about det k in this case, since this differs from the usual Fredholm determinant. My proof would have to be indirect, perhaps by finding an explicit solution for I_n. If you solve the charged bowl problem, you will then have your I_n , and you will then have shown that det R = 0.

The real problem we have is how to construct I(θ) given that $I_n = \delta_{n,0}\, 2$. We need some kind of expansion theorem. Here is a candidate expansion theorem for (0,1) taken from below:

$$f_n = \int_0^1 dx\, \varphi_n(x) f(x) \qquad f(x) = \Sigma_{n=0}^{\infty} f_n \varphi_n(x) \qquad x = \cos\theta$$

where

$$\varphi_n(x) = (2n+1)^{1/2}\, P_n(-1+2x) \qquad \varphi_0(x) = P_0(-1+2x) = 1$$

$$\int_0^1 dx\, \varphi_n(x)\varphi_m(x) = \delta_{n,m} \qquad \delta(x-y) = \Sigma_n\, \varphi_n(x)\varphi_m(y)$$

But the projection WE are dealing with is this:

$$I_n = \int_0^1 dx\, P_n(x) I(\theta) = 2\, \delta_{n,0}$$

which is not the right expansion for this proposed expansion theorem which wants to see

$$J_n = \int_0^1 dx\, \varphi_n(x) I(\theta)$$

Maybe we can related these two sets of projections. We would need this:

$$\varphi_n(x) = \Sigma_{m=0}^{\infty}\, k_{nm}\, P_m(x) \qquad \text{at least valid on (0,1)}$$

Then

$$f_n = \int_0^1 dx\, \varphi_n(x) f(x) = \int_0^1 dx\, \Sigma_{m=0}^{\infty}\, k_{nm}\, P_m(x) f(x) = \Sigma_{m=0}^{\infty}\, k_{nm} \int_0^1 dx\, P_m(x) f(x)$$

$$= \Sigma_{m=0}^{\infty}\, k_{nm}\; I_m = \Sigma_{m=0}^{\infty}\, k_{nm}\; 2\, \delta_{m,0} = 2\, k_{n0}$$

If we could find this, then we would have

$$I(\theta) = \Sigma_{n=0}^{\infty} f_n \varphi_n(x) \;=\; 2\,\Sigma_{n=0}^{\infty}\, k_{n0}\;\varphi_n(\cos\theta)$$

Now let's try to compute the matrix element k_{nm}

$$\varphi_n(x) = \Sigma_{m=0}^{\infty}\, k_{nm}\, P_m(x)$$

$$(2n+1)^{1/2}\, P_n(-1+2x) \;=\; \Sigma_{m=0}^{\infty}\, k_{nm}\, P_m(x)$$

Let's TRY (a bit dubious perhaps) applying the full $\int_{-1}^{1} dx\, P_i(x)$ to both sides

$$\int_{-1}^{1} dx\, P_i(x)\,(2n+1)^{1/2}\, P_n(-1+2x) \;=\; \Sigma_{m=0}^{\infty}\, k_{nm} \int_{-1}^{1} dx\, P_i(x) P_m(x)$$

$$=\; \Sigma_{m=0}^{\infty}\, k_{nm}\; 2\,(2i+1)^{-1}\delta_{i,m} \;=\; k_{ni}\; 2\,(2i+1)^{-1}$$

This suggests that

$$k_{ni} = (1/2)\,(2i+1)\,(2n+1)^{1/2} \int_{-1}^{1} dx\, P_i(x)\, P_n(-1+2x)$$

In particular we have

$$k_{n0} = (1/2)\,(2n+1)^{1/2} \int_{-1}^{1} dx\, P_n(-1+2x)$$

Maple tells us however that these integrals get huge for larger n, at the $P_n(-3)$ end of the range. So this gives a series

$$I(\theta) = \Sigma_{n=0}^{\infty} f_n \varphi_n(x) \;= 2\,\Sigma_{n=0}^{\infty}\, k_{n0}\;\varphi_n(\cos\theta)$$

which seems to me to diverge for some constant in-range θ! Our next "little problem" for the day.

9. Try expanding $P_n(z)$ and $I(z)$ on complete φ_n defined on (0,1)

Suppose we try going the other direction: Not $\varphi_n(x) = \Sigma_{m=0}^{\infty}\, k_{nm}\, P_m(x)$ but instead

$$P_n(x) = \Sigma_{m=0}^{\infty}\, c_{nm}\, \varphi_m(x) \qquad\qquad x \text{ in } (0,1)$$

In a sense this is "better" because we are expanding on the complete set of eigenfunctions for our correct interval, and we are nowhere involving x elsewhere. Can we now compute these c_{nm} ?

Apply $\int_0^1 dx\ \varphi_i(x)$ to both sides to get

$$\int_0^1 dx\ \varphi_i(x)\, P_n(x) = \Sigma_{m=0}^{\infty} c_{nm} \int_0^1 dx\ \varphi_i(x)\, \varphi_m(x) = c_{ni}$$

That was pretty simple. What do these things look like? Here is some Maple:

```
> with(orthopoly):
> phi := n -> (2*n+1)^(1/2)* P(n,-1+2*x):
> phi(2):
> cc := (n,i) -> int(phi(i)*P(n,x),x=0..1):
> N := 6:
> c := matrix(N,N,cc);
```

$$c := \begin{bmatrix} \frac{1}{6}\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4}\sqrt{3} & \frac{1}{20}\sqrt{5} & 0 & 0 & 0 & 0 \\ \frac{1}{8}\sqrt{3} & \frac{1}{8}\sqrt{5} & \frac{1}{56}\sqrt{7} & 0 & 0 & 0 \\ -\frac{1}{24}\sqrt{3} & \frac{1}{8}\sqrt{5} & \frac{1}{16}\sqrt{7} & \frac{1}{144}\sqrt{9} & 0 & 0 \\ -\frac{1}{16}\sqrt{3} & \frac{1}{32}\sqrt{5} & \frac{3}{32}\sqrt{7} & \frac{1}{32}\sqrt{9} & \frac{1}{352}\sqrt{11} & 0 \\ \frac{1}{64}\sqrt{3} & -\frac{3}{64}\sqrt{5} & \frac{1}{16}\sqrt{7} & \frac{1}{16}\sqrt{9} & \frac{1}{64}\sqrt{11} & \frac{1}{832}\sqrt{13} \end{bmatrix}$$

```
> c := evalf(matrix(N,N,cc));
```

$$c := \begin{bmatrix} .2886751347 & 0 & 0 & 0 & 0 & 0 \\ .4330127020 & .1118033989 & 0 & 0 & 0 & 0 \\ .2165063510 & .2795084973 & .04724555913 & 0 & 0 & 0 \\ -.07216878367 & .2795084973 & .1653594569 & .02083333333 & 0 & 0 \\ -.1082531755 & .06987712431 & .2480391854 & .09375000000 & .009422229517 & 0 \\ .02706329388 & -.1048156865 & .1653594569 & .1875000000 & .05182226234 & .004333595283 \end{bmatrix}$$

At least these matrix elements seem to be getting smaller as you go down and out.

Now does this help us solve our problem? Here is our valid expansion theorem

$$J_n = \int_0^1 dx\ \varphi_n(x) I(x) \qquad I(x) = \Sigma_{n=0}^{\infty} J_n\, \varphi_n(x) \qquad x = \cos\theta$$

and we have

$$P_n(x) = \Sigma_{m=0}^{\infty} c_{nm}\, \varphi_m(x) \qquad (*)$$

and we have

$$I_n = \int_0^1 dx\, P_n(x) I(x) = 2\,\delta_{n,0} \qquad J_n = \int_0^1 dx\, \varphi_n(x) I(x)$$

So let's apply $\int_0^1 dx\, I(x)$ to both sides of (*),

$$\int_0^1 dx\, I(x)\, P_n(x) = \Sigma_{m=0}^{\infty} c_{nm} \int_0^1 dx\, I(x)\, \varphi_m(x)$$

$$I_n = \Sigma_{m=0}^{\infty} c_{nm}\, J_m \qquad I = c\,J \qquad \Rightarrow \qquad J = c^{-1} I \qquad \Rightarrow \qquad J_n = \Sigma_{m=0}^{\infty} c^{-1}{}_{nm}\, I_m$$

We know $I_n = 2\,\delta_{n,0}$ and we need to solve for the J_m so we can put them in our series. But that means we need c^{-1}. Then we have

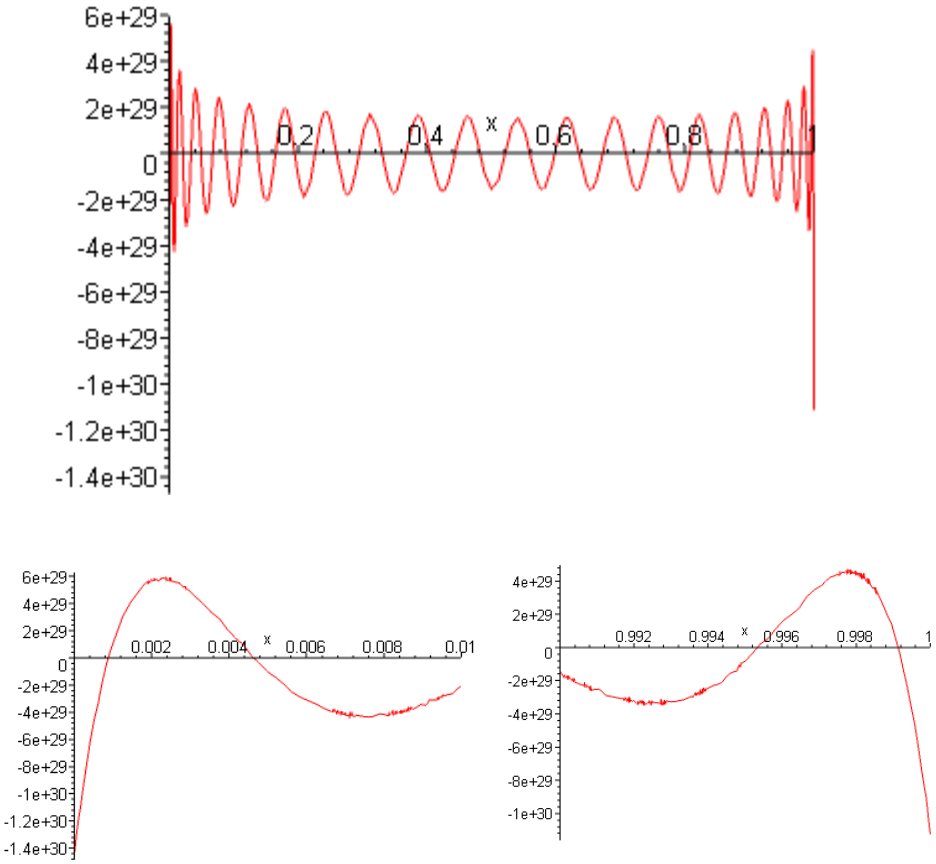
$$J_n = \Sigma_{m=0}^{\infty} c^{-1}{}_{nm}\, I_m = \Sigma_{m=0}^{\infty} c^{-1}{}_{nm}\, (2\,\delta_{m,0}) = 2\,[c^{-1}]_{n0} = 2 * \text{first column of } c^{-2}$$

```
col1 := col(cinv,1);
```

col1 := [3.464101614, -13.41640786, 63.49803147, -311.9999999, 1572.080150, -8069.223747]

The numbers alternate sign and increase rapidly, just as we saw with the k_{nm} approach (which no doubt is exactly the same approach). If you have Maple compute $I(x) = \Sigma_{n=0}^{N} J_n\, \varphi_n(x)$ you get plots like this one, similar to our "Legendre my problem.doc" plots.

```
> restart: with(orthopoly):with(linalg):
> phi := n -> (2*n+1)^(1/2)* P(n,-1+2*x):
> phi(2):
> cc := (n,i) -> int(phi(i)*P(n,x),x=0..1):
> N := 40:
> c := evalf(matrix(N,N,cc)):
> cinv := evalf(inverse(c)):
> J := col(cinv,1):
> IofX := sum(2*J[n]*phi(n),n=1..N):
> plot(IofX,x=0..1);
```



where I show blowups of the left and right ends. Remember that the left end is x = 0 or $\theta = \pi/2$ which is the edge of our half sphere. and x = 0 is the smooth round top of it. $I(\theta,\varphi)$ is the charge density in this problem. The charge in a piece of area is $d\Sigma = I(x)d(\cos\theta)d\varphi = I(x)\,dx\,d\varphi$. The charge in a ring around the sphere is then $d\Sigma_{ring} = 2\pi\, I(x)\, dx$. Everything I have done seems to be saying that the charge is peaking up at both the sharp rim and at the smooth center top, which just seems wrong to me. I cannot seem to find the answer on the web.

Physical arguments? The charges are repelled from each other. They want to go to the edge, but maybe the center point would pile up for this reason: charge is repelled equally from all directions toward the center point, so it peaks up there. But in the 2D finite wire we did not see any piling up at the center.

10. Try some simple delta-function forms for I(z)

I think I really believe that this is correct

$$\int_0^1 dx\, P_n(x) I(x) = 2\,\delta_{n,0}$$

Let's try to go backwards from this to the integral equation we started with. The above solves this:

$$\Sigma_n\, P_n(z_s) \int_0^1 dz_\xi\, P_n(z_\xi)\, I(z_\xi) = 2 \qquad \text{half sphere} \qquad \text{valid } (0,1) \text{ for } z_s$$

$$(1/4\pi)\,\Sigma_n\, N_{0n}(1/N_{0n})\, Y^*_{n0}(\Omega_s) \int_\sigma d\Omega_\xi\, Y_{n0}(\Omega_\xi)\, I(\Omega_\xi) = 1 \qquad \sigma = \text{half}$$

$$(1/4\pi)\,\Sigma_n\, N_{0n}\,\Sigma_m\,(1/N_{mn})\, Y^*_{nm}(\Omega_s) \int_\sigma d\Omega_\xi\, Y_{nm}(\Omega_\xi)\, I(\Omega_\xi) = 1 \qquad \sigma = \text{half}$$

$$(1/4\pi) \int_\sigma d\Omega_\xi\, \{\, \Sigma_n\, N_{0n}\,\Sigma_m\,(1/N_{mn})\, Y_{nm}(\Omega_\xi)\, Y^*_{nm}(\Omega_s)\,\}\, I(\Omega_\xi) \quad = 1 \qquad \sigma = \text{half sphere}$$

$$\int_\sigma E(s|\xi) I(\xi) dS_\xi \; = 1 \qquad \sigma = \text{half sphere}$$

So is there some distributional solution so these equations:

$$\int_0^1 dx\, I(x) \; = 2$$

$$\int_0^1 dx\, P_n(x) I(x) \; = 0 \qquad n = 1,2,3...$$

Something like this

$$I(x) = A\delta(x-0) + B\delta(x-1)$$

Then our equations above become:

$$A/2 + B/2 = 2$$
$$A/2P_n(0) + B/2P_n(1) = 0$$

$$A/2 \quad + B/2 = 2$$
$$A/2P_n(0) + B/2 = 0$$

$$A/2\,[\, 1 - P_n(0)\,] = 2$$

But $P_n(0) = 0$ for odd n, and = f(n) for even n. Then our solution for A would be A(n) for even n and would be ∞ for odd n. Not too good.

How about this

$$I(x) = a\,\delta(x-1)$$

Then equations say

$$a/2 = 2$$
$$a/2 = 0$$

so sorry. You might try something fancier like this:

$$I(x) = L_0\delta(x-0) + L_1\delta(x-1)$$

where the L_i are differential operators. So consider again:

$$\int_0^1 dx\, I(x) = 2$$

$$\int_0^1 dx\, P_n(x)I(x) = 0 \qquad n = 1,2,3...$$

At the x = 1 end, if we just take $L_1 = B$, then that end generates B/2 for both lines which is not dependent on n. Can we find some L_0 that makes the x=1 end independent of n?

$$\int_0^1 dx\, P_n(x)\,[\, L_0\delta(x-0)] = \text{independent of n.}$$

If L_0 is self-adjoint, I think we can write the above as

$$\int_0^1 dx\, [L_0P_n(x)]\,\delta(x-0) = (1/2)\,[L_0P_n(x)]|^{x=0}$$

For example, if $L_0 = -(1-x^2)\partial_x^2 + 2x\partial_x$ then $[L_0P_n(x)]|^{x=0} = n(n+1)P_n(0)$ // Schaum p 146.

but this is not independent of n. At the x=1 end we get $n(n+1)P_n(1)$.

11. Search on line for solution to the half sphere problem.

Conclusion: (10.31.09 11:30 AM) There is nothing more I can do with this problem. I have tried solving $\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1$ as best I could. I used Maple, I tried different methods, I keep getting some kind of strange distributional result that does not make any sense. So now is the time when we have to find the answer from someone else.

How would I go about finding the solution to this problem? Let's log some web searches:

search: "hemispherical shell" conductor "charge distribution" 128 hits

12. Discovery of the Canonical PDF (10.31.09)

I found a large PDF and downloaded it from scribd, "canonical structures in potential theory" is part of the title (did this 10.31.09). It has a chapter specifically on spherical shells. This looks very promising. They will do both the spherical cap (more general than my hemispherical shell) and also a spherical barrel. This is the exact right source, I was lucky. Let's follow the notes a bit. Well, maybe this is just Dirichlet for the cap with some prescribed potential on it which he calls $U^0(\theta,\varphi)$. I have in mind this is 0

of course. He does the usual Y expansion but calls this a Fourier-Legendre series expansion where α_{nm} are coefficients. The math here is pretty hairy, but he claims to find a general solution. Then he says that if the cap is at a constant potential, things simplify and he did that already in Section 1.4, which I now eagerly look at.

Here is some fascinating opening text:

1.4 Dual equations: a classification of solution methods

It is perhaps well known that a comprehensive theory to solve dual equations does not exist, and that many treatments have been developed to obtain solutions to such equations. Essentially, these treatments can be grouped into three basic methods: the definition-extension method described by W.E. Williams, [76] the substitution method described by B. Noble, [46] and the multiplying factor method also described by B. Noble [47].

The common and distinctive feature of all these methods is the utilization, in one form or another, of Abel's integral equation (or transform) technique. Let us illustrate these methods with the simple problem of determining the potential of a charged spherical cap when its surface is held at a constant unit value of potential. This problem produces the following dual series equations involving Legendre polynomials $P_n(\cos\theta)$,

$$\sum_{n=0}^{\infty} a_n P_n(\cos\theta) = 1, \ \theta \in (0, \theta_0), \qquad (1.\ 100)$$

$$\sum_{n=0}^{\infty} (2n+1) a_n P_n(\cos\theta) = 0, \ \theta \in (\theta_0, \pi), \qquad (1.\ 101)$$

where the unknown desired set of coefficients $\{a_n\}_{n=0}^{\infty}$ must belong to the Hilbert functional space l_2. The concrete form of condition (1. 85) that imposes this constraint on the coefficients is

$$\sum_{n=0}^{\infty} \frac{n+1}{2n+1} |a_n|^2 < \infty. \qquad (1.\ 102)$$

So, where are these two equations coming from? These guys are doing a sort of "form fit" method, so what happens if I try that. Then

$u(r,\theta) = \Sigma_n a_n r^n P_n(z)$ inside r = 1

$u(r,\theta) = \Sigma_n a_n r^{-n-1} P_n(z)$ outside r = 1

This form makes u be continuous at r = 1 for all z. This applies to the free space dotted part of the boundary, and also to the spherical cap metal. So setting $u(1,\theta) = 1$ gives 1.100 above, but where is the second equation coming from? On the dotted portion of the sphere, the derivative must be continuous, so we have

$\partial_r u(r,\theta) = \Sigma_n a_n n r^{n-1} P_n(z)$ inside r = 1

$\partial_r u(r,\theta) = \Sigma_n a_n (-n-1) r^{-n-2} P_n(z)$ outside r = 1

The difference of these is 0 on the dotted boundary so we have

$\Sigma_n a_n [n + n + 1] P_n(z) = 0$ on dotted boundary

which is 1.101. These are called "dual series equations".

His next step is to imagine there is some function g(θ) which provides a RHS for 1.101 for the entire range, and happens to be 0 in the high range:

$\Sigma_n (2n+1) a_n P_n(\cos\theta) = g(\theta)$ $0 \le \theta \le \pi$ g = 0 on high part

I think this g(θ) is the total surface charge density since it is the difference of the gradients, and it is of course 0 on the dotted boundary. So this is I(θ) in my notation!

He then constructs an integral equation for $g^*(\theta) \equiv g(\theta)\sin\theta$ [I have seen this idea before]

$\int_0^{\theta_0} d\theta' \, g^*(\theta') K(\theta',\theta) = 1$ $K(\theta',\theta) = (1/2) \Sigma_n P_n(\cos\theta) P_n(\cos\theta')$

I think this is exactly my integral equation $\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$!!! Probably the P_n thing is a way to write 1/R, it certainly looks reasonable to me. He points out: first kind Fred, but no name is assigned to this integral equation. At this point, he uses a method involving "Abel integral equations". Amazingly, he finds the following solution:

A double application of the inversion formulae to this iterated Abel integral equation (see the next section) yields the solution for the function g in closed form:

$$g(\vartheta) = \frac{\sqrt{2}}{\pi} \left\{ \frac{2\cos\frac{1}{2}\theta_0}{\sqrt{\cos\vartheta - \cos\theta_0}} + \frac{\pi}{2} - \arcsin\left(\frac{\cos\frac{1}{2}\theta_0}{\cos\frac{1}{2}\theta} \right) \right\}. \qquad (1.\ 111)$$

The substitution of this expression for the function g in (1. 104) gives, after elementary integration, the final solution for the Fourier coefficients:

$$a_n = \frac{1}{\pi} \left[\frac{\sin n\theta_0}{n} + \frac{\sin(n+1)\theta_0}{n+1} \right]. \qquad (1.\ 112)$$

The function g coincides with the surface charge density on the cap, and possesses the expected singularity of order $-\frac{1}{2}$ as $\vartheta \to \theta_0$ (Formula (1. 111)).

And there is says it, g = Σ and **I have found the solution to my proble**m. Now for the half-sphere, set $\theta_0 = \pi/2$. Then

$\theta_0 = \pi/2$

$\cos\theta_0 = 0$

$\cos(\theta_0/2) = \cos(\pi/4) = 1/\sqrt{2}$

$$g(\theta) = (\sqrt{2}/\pi)\ \{ \sqrt{2}/\sqrt{\cos\theta} + \pi/2 - \arcsin(1/[\sqrt{2}\cos(\theta/2)) \}$$

If I draw a standard orientation right triangle with vertical edge 1, diagonal $\sqrt{2}\cos(\theta/2)$, then the angle in the triangle at the lower left is θ. The horizontal edge is then

$$x^2 = 2\cos^2(\theta/2) - 1^2 = 1 + \cos\theta - 1 = \cos\theta$$

But this tells us that $\tan\theta = 1/\cos\theta$ which seems odd. That seems to say $\sin\theta = 1$ and $\theta = \pi/2$. This must be a strange limit due to doing an exact half sphere. Then the above becomes simply

$$g(\theta) = (\sqrt{2}/\pi)\ \{ \sqrt{2}/\sqrt{\cos\theta} + \pi/2 - \pi/2 \} = (2/\pi)\ 1/\sqrt{\cos\theta}$$

At $\theta = 0$ where $\cos\theta = 1$, nothing strange happens (as I expected). It blows up in the usual square root manner at the rim edge. So this seems a pretty reasonable solution, no fancy horrible distributions.

The potential meanwhile has

$$a_n = (1/\pi)\ [\ \sin(n\pi/2)/n + \sin((n+1)\pi/2)/(n+1)\ ]$$

$$u(r,\theta) = \Sigma_n\ a_n\ r^n P_n(z) \qquad \text{inside } r = 1$$
$$u(r,\theta) = \Sigma_n\ a_n\ r^{-n-1} P_n(z) \qquad \text{outside } r = 1$$

and the problem is now completely solved!

So the solution to my problem seems to be this:

$$I(\theta) = (2/\pi)\ 1/\sqrt{\cos\theta}$$

according to the PDF I found (and have carefully saved!!!).

13. Test this Canonical solution.

The first questions is this:

$$\int_0^1 dx\ P_n(x) I(x) = 0 \qquad n = 1,2,3...$$

[Since $I_n = \delta_{n0}$, one would expect the above to be true. But by now I know that this is not the right I_n for this problem, so I am then not surprised to see that the above is not true.]

I don't think this is going to be true

$$\int_0^1 dx\ P_n(x)/\sqrt{x}$$

This appears p 796 GR with $\sigma = -1/2$ which is in the allowed range. Integral is

$$\sqrt{\pi}\ 2^{-1/2}\,\Gamma(1/2) / \{\,\Gamma(1 - 1/4 - n/2)\,\Gamma(-1/4 + n/2 + 3/4)\,\}$$

$$= \sqrt{\pi}\ 2^{-1/2}\,\Gamma(1/2) / \{\,\Gamma(3/4 - n/2)\,\Gamma(n/2 + 1/2)\,\}$$

I enter $\Gamma(3/4 - n/2)\,\Gamma(n/2 + 1/2)$ in Maple and try some integer n values, all different. Also can directly evaluate the integral shown, it is different for each n.

Conclusion: My claim that $\int_0^1 dx\, P_n(x) I(x) = 0$ for n = 1,2,3.... must be wrong. [it is!] Not too big of a surprise, since it is this equation that leads to the distributional nightmare.

So let's back up a bit. What happens with this expansion:

$$\Sigma_n\, P_n(z_s) \int_0^1 dx\, P_n(x)\, I(x) = 2 \qquad \text{half sphere} \qquad \text{valid (0,1) for } z_s$$

If I insert $I(x) = (2/\pi)\, 1/\sqrt{x}$ we get

$$\Sigma_{n=0}^{\infty} P_n(z_s)\left[\,(2/\pi) \int_0^1 dx\, P_n(x) /\sqrt{x}\,\right] = 2$$

$$\Sigma_{n=0}^{\infty} P_n(z_s)\left[\, \int_0^1 dx\, P_n(x) /\sqrt{x}\,\right] = \pi$$

$$\Sigma_{n=0}^{\infty} P_n(z_s)\, I_n = \pi$$

The RHS is 3.14156. I can make Maple compute the I_n and do the sums for different z_s and with the sum going to some N. If I set N = 50, here is what I get for the LHS setting z_s = 0, .1, .2, ... 1.0

$$0$$
$$s := 2.510880813$$
$$.1$$
$$s := 2.554401915$$
$$.2$$
$$s := 2.514444199$$
$$.3$$
$$s := 2.470709910$$
$$.4$$
$$s := 2.426774482$$
$$.5$$
$$s := 2.385541014$$
$$.6$$
$$s := 2.348015106$$
$$.7$$
$$s := 2.313805956$$
$$.8$$
$$s := 2.281038884$$
$$.9$$
$$s := 2.249775684$$
$$1.0$$
$$s := 2.221633500$$

First of all, we don't get π. Second, we don't get a constant result, but it is slowly varying, I will admit that much.

I am thinking now more about $1/\sqrt{\cos\theta}$. Can I write this as a P_n double sum? Jackson p 62 reminds us of this fact for two points on a sphere.

$1/|\xi\text{-}s| = \Sigma_n P_n(\cos\gamma)$ γ is the angle between the two vectors

This then leads to the more general form with the Y's.

Can I show that g(θ) from the paper solves anything at all? The most important would be to show this:

$\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1$ // independent of location s on half sphere

All I am doing in the next step is inserting the tried and true expansion for 1/R:

$(1/4\pi) \int_\sigma d\Omega_\xi \{ \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y_{nm}(\Omega_\xi) Y^*_{nm}(\Omega_s) \} I(\Omega_\xi) = 1$ σ = half sphere

then we do the change of order

$$(1/4\pi)\,\Sigma_n\, N_{0n}\,\Sigma_m\,(1/N_{mn})\; Y^*_{nm}(\Omega_s)\int_\sigma d\Omega_\xi\; Y_{nm}(\Omega_\xi)\; I(\Omega_\xi) = 1 \qquad \sigma = \text{half}$$

Next we assume no ξ azimuthal dependence so the last integral is $\delta_{m,0}\int_\sigma d\Omega_\xi\; Y_{n0}(\Omega_\xi)\; I(\Omega_\xi)$. Hence

$$(1/4\pi)\,\Sigma_n\; Y^*_{n0}(\Omega_s)\int_\sigma d\Omega_\xi\; Y_{n0}(\Omega_\xi)\; I(z_\xi) = 1 \qquad \sigma = \text{half}$$

$$(2\pi/4\pi)\,\Sigma_n\; P_n(z_s)\int_0^1 dz_\xi\; P_n(z_\xi)\; I(z_\xi) = 1 \qquad \sigma = \text{half}$$

$$\Sigma_n\; P_n(z_s)\int_0^1 dx\; P_n(x)\; I(x) = 2 \qquad \sigma = \text{half}$$

$$\int_0^1 dx\; [(1/2)\Sigma_n\; P_n(z_s)P_n(x)]\; I(x) = 1$$

It seems to me that all these equations MUST be true. If this is the correct solution

$$I(x) = (2/\pi)\; 1/\sqrt{x}$$

then it MUST solve the above equation. Let's look at the PDF version of the integral equation:

$$\int_0^{\theta_0} d(\cos\theta')\; g(\theta')\; K(\theta',\theta) = 1 \qquad K(\theta',\theta) = (1/2)\,\Sigma_n\; P_n(\cos\theta)\; P_n(\cos\theta')$$

This agrees exactly with my integral equation above! I have $\theta_0 = \pi/2$ as my special case of interest.
So we really do have the same integral equation.

Maybe the order interchange of integration and summation is illegal? To me, it seems such an obvious thing to do, given the PDF line above, to say this:

$$(1/2)\,\Sigma_n\; P_n(\cos\theta)\int_0^{\theta_0} d(\cos\theta')\; P_n(\cos\theta')\; g(\theta') = 1$$

$$\Sigma_n\; P_n(\cos\theta)\; g_n = 2 \qquad g_n = \int_0^{\theta_0} d(\cos\theta')\; P_n(\cos\theta')\; g(\theta')$$

$$g_n = 2\,\delta_{n,0}$$

I am doing something wrong right in the above 3 lines!! I have a very *fundamental* misunderstanding of things I am afraid. [yup!] [nope !]

Can I show that *their* solution solves *their* integral equation?

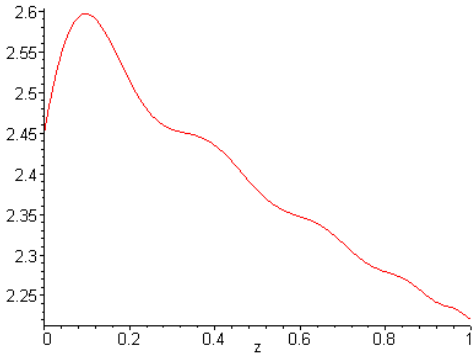
$$\int_0^1 dx\; [(2/\pi)\; 1/\sqrt{x}\;]\; [(1/2)\;\Sigma_n\; P_n(z)\; P_n(x)] = 1$$

$$\int_0^1 dx\ (1/\sqrt{x})\ \Sigma_n\ P_n(z)\ P_n(x) = \pi$$

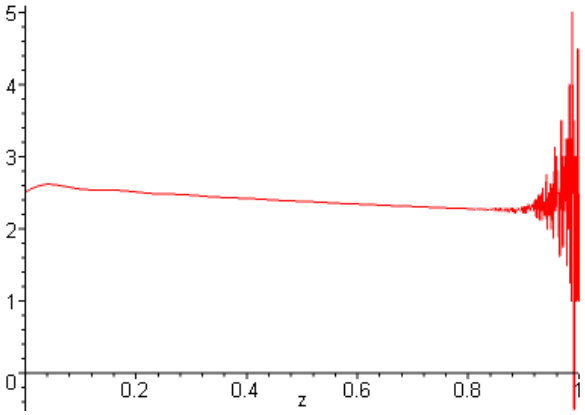
The PDF processes this with some fancy integral representations. Can I get Maple to help at all ???

```
> restart: with(orthopoly):
> f := sum(P(n,z)*P(n,x), n = 0..20):
> g := int((x)^(-1/2)*f, x=0..1):
> plot(g, z = 0..1);
```

This gives a plot which is not very constant,



But if I increase the sum to 50 terms, then I get this plot

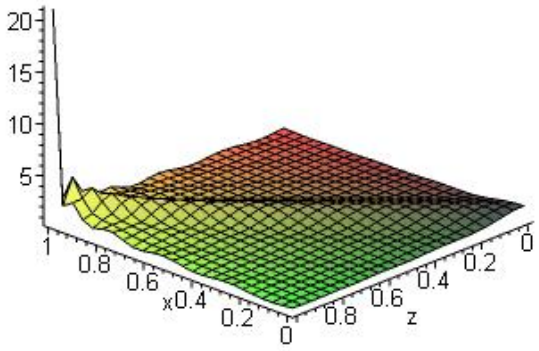


which looks like it is trying to be flat. But why is it flat at 2.3 instead of π ??? I think this is a complete artifact.

What can I say about the convergence of this series:

$$\Sigma_{n=0}^{\infty}\ P_n(z)\ P_n(x)$$

First of all, I would say it was "not very convergent". There are no 1/n type factors. So we really have to worry about large n behavior of the $P_n(x)$. What does this function even look like if you plot it??? If I do a number of terms like maybe 20, it seems to be a ridge of some sort:




```
> restart;with(orthopoly):Digits := 20:
> f := sum(P(n,z)*P(n,x), n = 0..20):
> plot3d(f,x=0..1,z=0..1);
```

I know from Schaum p 147 that

$\delta(x-z) = \Sigma_{n=0}^{\infty} (n+1/2)\, P_n(z)\, P_n(x)$ web confirm: $\sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(x') P_l(x) = \delta(x - x')$

so our sum is not quite a delta function, but you can see it will peak up on the ridge since it is then adding a series of positive terms. Our sum must be ∞ when x = z.

Conclusion: the sum $\Sigma_{n=0}^{\infty} P_n(z)\, P_n(x)$ is in fact a distribution, similar to $\Sigma_{n=0}^{\infty} (n+1/2)\, P_n(z)\, P_n(x)$. Therefore, we are going to have numerical problems with it. Certainly when x = 1 and z = 1 our sum is going to blow up, you see it peaking at the corner of the square above. Consider now:

$$\Sigma_{n=0}^{\infty} P_n(x)\, P_n(x) = \Sigma_{n=0}^{\infty} (P_n(\cos\theta))^2$$

Will this blow up or not? Our PDF paper suggests that this sum would be

$$(1/2\pi) \int_0^{\theta} d\varphi \,/\, (\cos\varphi - \cos\theta)$$

```
int( (cos(phi) - cos(theta))^(-1), phi = 0..theta)
simplify(%);
```

$$-2 \frac{\arctan\left(\frac{\sin(\theta)}{\sqrt{-\sin(\theta)^2}}\right)}{\sqrt{-\sin(\theta)^2}}$$

The arctan thing becomes $\tan^{-1}(-i)$ so that $\tan\alpha = -i$ and $i\tan\alpha = 1$ so $\tanh(i\alpha) = 1$ so $\alpha = -i\infty$ so I would say the numerator was infinite at any θ. We are integrating right into a pole where $\cos\varphi = \cos\theta$.

$$(1/2\pi) \int_0^{\theta} d\varphi \,/\, (\cos\varphi - \cos\theta)$$

[Note: in v2, I show that $\Sigma_{n=0}^{\infty} P_n(x)\, P_n(y)$ is a multiple of a K function, so it not a distribution.]

Obsolete old notes

The integral equation failed to diagonalize, and we have merely replaced it with a sum equation which we don't know how to solve. At least we have gotten rid of our inconsistencies!

Now go back to the original integral equation

$$(1/4\pi) \int_\sigma d\Omega' \{ \Sigma_n N_{0n} \Sigma_m (1/N_{mn})\, Y_{nm}(\Omega')\, Y^*_{nm}(\Omega) \}\, I(\Omega') = 1 \qquad \text{// p 126} \qquad (*)$$

and assume as before that $I(\Omega') = I(\theta',\varphi') = I(\theta')$ only, Then in the above we see

$$\int_\sigma d\Omega'\, Y_{nm}(\Omega')\, I(\theta') = 2\pi\delta_{m,0} \int_0^1 dz'\, P_n(z')\, I(\theta') = \delta_{m,0}\, I_{n0} N_{n0}$$

Therefore, we can slightly simplify (*) in this way

$$(1/4\pi) \Sigma_n N_{0n} \Sigma_m (1/N_{mn})\, Y^*_{nm}(\Omega)\, [\delta_{m,0}\, I_{n0} N_{n0}] = 1$$

$$(1/4\pi) \Sigma_n N_{0n}\, (1/N_{0n})\, Y^*_{n0}(\Omega)\, I_{n0} N_{n0} = 1$$

$$(1/4\pi) \Sigma_n N_{0n} P_n(\cos\theta)\, I_{n0} = 1$$

and we just arrive at our same result as quoted above.

Part 2: Solve the sum equation.

So now we have to face this thing and solve it!

$$(1/4\pi) \Sigma_n N_{0n} P_n(\cos\theta)\, I_{n0} = 1 \qquad \text{valid only for } 0 \le \theta \le \pi/2$$

$$\Sigma_{n=0} I_n\, P_n(z)/(2n+1) = 1 \qquad I_n = I_{n0} \qquad \text{valid only for } 0 \le z \le 1$$

If this were valid on the entire -1,1 range, we would quickly say this: $P_n(z)$ are a complete orthogonal set, so match both sides and conclude that $I_0 = 1$ and $I_n = 0$ for $n > 0$. That is exactly what you would do for the problem of the full sphere, and that would be the correct conclusion.

But here, this thing is NOT valid on (-1,1) so we cannot do that.

Notice, by the way, that this does *not* align with what a while ago I called my problem Legendre

$$\int_0^1 dx\, I(x)\, P_0(x) = 1$$

$$\int_0^1 dx\, I(x)\, P_n(x) = 0 \qquad n = 1,2,3,4...\infty \qquad \text{My Problem}$$

You could *consider* a possible solution of the equation $\Sigma_{n=0}\ (2n+1)\ I_n\ P_n(z) = 1$ that had this form, which is to say $I_0 = 1$ and $I_n = 0$ for $n > 0$, but you find that such a solution does not exist. So forget that whole document Legendre my problem.doc.

Now, once again, back to

$$\Sigma_{n=0}\ (2n+1)^{-1}\ I_n\ P_n(x) = 1 \qquad I_n = I_{n0} \qquad \text{valid only for } 0 \le x \le 1$$

What we really need at this point is a complete set of basis functions on this interval. Consider

$$[\ (2n+1)/2]\ \int_{-1}^1 dx\, P_n(x)\, P_m(x) = \delta_{n,m}$$

Let $x = -1+2x'$ so that $dx = 2dx'$ so we have

$$(2n+1)\int_0^1 dx'\, P_n(-1+2x')\, P_m(-1+2x') = \delta_{n,m}$$

$$(2n+1)\int_0^1 dx\, P_n(-1+2x)\, P_m(-1+2x) = \delta_{n,m}$$

We then have this set of basis functions

$$\varphi_n(x) = (2n+1)^{1/2}\, P_n(-1+2x) \qquad \varphi_0(x) = P_0(-1+2x) = 1$$

$$\int_0^1 dx\, \varphi_n(x)\varphi_m(x) = \delta_{n,m}$$

We think that the projection/transform going with this would be

$$f_n = \int_0^1 dx\, \varphi_n(x) f(x)$$

$$f(x) = \Sigma_{n=0}^{\infty} f_n \varphi_n(x)$$

Go back now to our equation from above

$$\Sigma_{n=0}\ (2n+1)^{-1}\ I_n\ P_n(x) = 1 \qquad I_n = I_{n0} \qquad \text{valid only for } 0 \le x \le 1$$

Apply $\int_0^1 dx\ \varphi_m(x)$ to both sides to get

$$\Sigma_{n=0}\ (2n+1)^{-1}\ I_n\ \int_0^1 dx\ \varphi_m(x)\ P_n(x)\ =\ \int_0^1 dx\ \varphi_m(x)\ 1\ =\ \delta_{m,0}$$

$$\Sigma_{n=0}\ (2n+1)^{-1}\ I_n\ K_{mn} = \delta_{m,0} \qquad\qquad K_{mn} \equiv\ \int_0^1 dx\ \varphi_m(z)\ P_n(z)$$

Define $(2n+1)^{-1}\ K_{mn} = A_{mn}$ then we have

$$\Sigma_{n=0}\ A_{mn}\ I_n\ = \delta_{m,0}$$

$$A\ I\ = 1$$

$$\Rightarrow \qquad I = A^{-1}\ 1$$

So all we have to do is invert an ∞ x ∞ dimensional matrix of known quantities. Probably as you move out into this matrix, quantities get smaller and you hope the solution stabilizes.

Now I ran into exactly this situation in that my problem doc. First, we state our matrix situation using start-with-0 indices:

$$\Sigma_{n=0}\ A_{mn}\ I_n\ = \delta_{m,0}$$

$$A_{mn}\ = (2n+1)^{-1}\ K_{mn}\ =\ (2n+1)^{-1}\ \int_0^1 dx\ \varphi_m(z)\ P_n(x)\ =$$

$$=\ (2n+1)^{-1}\ \int_0^1 dx\ \{\ (2m+1)^{1/2}\ P_m(-1+2x)\ \}\ P_n(x)$$

$$= (2n+1)^{-1}\ (2m+1)^{1/2}\ \int_0^1 dx\ P_m(-1+2x)\ P_n(x)$$

If we want help from Maple matrix machinery, we have to restate this in start-with-1 indices:

$$B_{mn} = A_{m-1,n-1} \qquad\qquad \text{so}\quad B_{11} = A_{00}$$

$$J_n = I_{n-1} \qquad\qquad \text{so } J_1 = I_0$$

Then our matrix equation of interest is this:

$$\Sigma_{n=1}^{\infty}\ B_{mn}\ J_n\ = \delta_{m,1} \qquad\qquad BJ = 1 \qquad\qquad J = B^{-1}\ 1\ = \text{first column of } B^{-1}$$

where

$$B_{mn} = A_{m-1,n-1} = (2n-1)^{-1}\,(2m-1)^{1/2} \int_0^1 dx\, P_{m-1}(-1+2x)\, P_{n-1}(x) \qquad m,n \geq 1$$

When I insert this into Maple with a 6x6 matrix, I find that the first column of B^{-1} is just 1. That seems wrong to me, but if it were correct, it would imply that

$J = (1,0,0,0....)$

$I = (1,0,0,0,0...)$

$I_0 = 1 \qquad I_n = 0 \quad n \geq 1$

But this is exactly the solution I don't want! Something (as usual, it happens daily) is wrong.
Here is the Maple work. First we make sure the first index is row, with this test:

```
CC :=(m,n) -> m^2 + n;

C = matrix(6,6,CC);
```

$$CC := (m,n) \to m^2 + n$$

$$C = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 11 & 12 & 13 & 14 & 15 \\ 17 & 18 & 19 & 20 & 21 & 22 \\ 26 & 27 & 28 & 29 & 30 & 31 \\ 37 & 38 & 39 & 40 & 41 & 42 \end{bmatrix}$$

For example, m = 2 would be second row, n = 1 first column, expect $2^2 + 1 = 5$, is correct. Recall from above

$$B_{mn} = A_{m-1,n-1} = (2n-1)^{-1}\,(2m-1)^{1/2} \int_0^1 dx\, P_{m-1}(-1+2x)\, P_{n-1}(x) \qquad m,n \geq 1$$

Now we go ahead and do the real thing:

```
> restart;
> with(orthopoly):
> with(linalg):
Warning, new definition for norm
Warning, new definition for trace
> BB :=(m,n) -> (2*n-1)^(-1)* (2*m-1)^(1/2) *
  int(P(m-1,-1+2*x)*P(n-1,x),x=0..1);
```

$$BB := (m,n) \to \frac{\sqrt{2m-1}\int_0^1 \mathrm{P}(m-1,-1+2x)\,\mathrm{P}(n-1,x)\,dx}{2n-1}$$

```
> B := matrix(6,6,BB);
```

$$B := \begin{bmatrix} 1 & \frac{1}{6} & 0 & \frac{-1}{56} & 0 & \frac{1}{176} \\ 0 & \frac{1}{18}\sqrt{3} & \frac{1}{20}\sqrt{3} & \frac{1}{56}\sqrt{3} & -\frac{1}{216}\sqrt{3} & -\frac{1}{176}\sqrt{3} \\ 0 & 0 & \frac{1}{100}\sqrt{5} & \frac{1}{56}\sqrt{5} & \frac{1}{72}\sqrt{5} & \frac{1}{352}\sqrt{5} \\ 0 & 0 & 0 & \frac{1}{392}\sqrt{7} & \frac{1}{144}\sqrt{7} & \frac{3}{352}\sqrt{7} \\ 0 & 0 & 0 & 0 & \frac{1}{1296}\sqrt{9} & \frac{1}{352}\sqrt{9} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3872}\sqrt{11} \end{bmatrix}$$

```
> BINV := inverse(B);
```

$$BINV := \begin{bmatrix} 1 & -\sqrt{3} & 3\sqrt{5} & -11\sqrt{7} & 45\sqrt{9} & -197\sqrt{11} \\ 0 & 6\sqrt{3} & -18\sqrt{5} & 72\sqrt{7} & -312\sqrt{9} & 1422\sqrt{11} \\ 0 & 0 & 20\sqrt{5} & -100\sqrt{7} & 500\sqrt{9} & -2500\sqrt{11} \\ 0 & 0 & 0 & 56\sqrt{7} & -392\sqrt{9} & 2352\sqrt{11} \\ 0 & 0 & 0 & 0 & 144\sqrt{9} & -1296\sqrt{11} \\ 0 & 0 & 0 & 0 & 0 & 352\sqrt{11} \end{bmatrix}$$

and you see at once the problem that the first column is 1. Here is the Maple code text:

```
> restart;
> with(orthopoly):
> with(linalg):
> BB :=(m,n) -> (2*n-1)^(-1)* (2*m-1)^(1/2) * int(P(m-1,-1+2*x)*P(n-1,x),x=0..1):
> B := matrix(6,6,BB):
> BINV := inverse(B):
> CC :=(m,n) -> m^2 + n:
> C = matrix(6,6,CC):
```

So now once again I am stumped. This is the 4th full day I have spent on this half-sphere problem! Today at least I eliminated some of the total conflicts I was having last time I worked on this.

****

I think all this stuff below the above line no longer applies to his half-sphere problem, but I will leave these notes here for now.

Part 2. Reduce the problem to $\int_0^1 dz\, I(z)\, P_n(z) = 0 \quad n = 1,2,3...$

So let's go back to

$$\int_\sigma d\Omega'\, Y_{nm}(\Omega')\, I(\Omega') = (2n+1)4\pi\, \delta_{n0}\delta_{m0}.$$

which really tells us that

$$\int_\sigma d\Omega'\, I(\Omega') = 4\pi \qquad n = m = 0$$

$$\int_\sigma d\Omega'\, Y_{nm}(\Omega')\, I(\Omega') = 0 \qquad \text{for all } n,m \neq 0,0$$

The first is sort of a normalization thing. But how do you extract I(Ω') from the second condition set? OK, let's write this out as (integral only over upper the half shell)

$$\int_0^{2\pi} d\varphi' \int_0^{\pi/2} \sin\theta'\, d\theta'\, Y_{nm}(\theta',\varphi')\, I(\theta',\varphi') = 0 \qquad \text{for all } n,m \neq 0,0$$

where we have put our z axis as the symmetry axis of our half spherical shell. We certainly expect I to not depend on φ, so we then have

$$\int_\sigma d\Omega'\, Y_{nm}(\Omega')\, I(\Omega') = \int_0^{\pi/2} \sin\theta'\, d\theta'\, I(\theta') \int_0^{2\pi} d\varphi'\, Y_{nm}(\theta',\varphi')$$

$$= \delta_{m,0}\, 2\pi \int_0^{\pi/2} \sin\theta'\, d\theta'\, I(\theta')\, Y_{n0}(\theta',\varphi')$$

$$= \delta_{m,0}\, 2\pi \int_0^{\pi/2} \sin\theta'\, d\theta'\, I(\theta')\, P_n(\cos\theta') = 0$$

For m=0 we then get

$$\int_\sigma d\Omega'\, Y_{n0}(\Omega')\, I(\Omega') = 2\pi \int_0^{\pi/2} \sin\theta'\, d\theta'\, I(\theta')\, P_n(\cos\theta') = 0$$

or

$$\int_0^1 dz\, I(z)\, P_n(z) = 0 \qquad n = 1,2,3,4 \qquad z = \cos\theta$$

Now we have at least simplified our confusion. Note that the P_n are not a complete set on (0,1). We also have that

$$\int_\sigma d\Omega'\, I(\Omega') = 4\pi = \int_0^{\pi/2} \sin\theta'\, d\theta'\, I(\theta') \int_0^{2\pi} d\varphi'\, 1 = 2\pi \int_0^{\pi/2} \sin\theta'\, d\theta'\, I(\theta')$$

$$\Rightarrow \int_0^1 dz\, I(z) = 2 = \int_0^1 dz\, I(z) P_0(z)$$

But the 1 in our original integral equation $\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$ was the arbitrary potential on the boundary, we could have set it to 1/2 and then we would get $\int_0^1 dz\, I(z) P_0(z)$.

Part 3: Attack My Problem

SO, we seem to arrive at **My Problem** which is to solve these equations for I(z):

$$\int_0^1 dz\, I(z) P_0(z) = 1 \qquad n=0$$

$$\int_0^1 dz\, I(z)\, P_n(z) = 0 \qquad n = 1,2,3,4 \qquad z = \cos\theta$$

When I first arrived at this point in these notes, I thought this would be an easy problem to solve and I would have deduced I(z) for this geometry, I(z) being the sum of the charge on the two surfaces.

Question: given the above condition on I(z), can we deduce what general form I(z) might have?

Comment: it is extremely easy for me to generate questions I cannot answer!

For example, could I(z) be a constant? That would imply that $\int_0^1 dz\, P_n(z) = 0$ for all n > 0. But I know that fact is not true (Maple), so no, I(z) cannot be a constant. We could try to expand I(z) on $P_{n'}(z)$ in the range (0,1) [but warning: not a complete set on that interval] and then we get

$$I(z) = \Sigma_{n'} a_{n'} P_{n'}(z) \qquad 0 = \int_0^1 dz\, [\Sigma_{n'} a_{n'} P_{n'}(z)]\, P_n(z)$$

$$= \Sigma_{n'} a_{n'} \int_0^1 dz\, P_{n'}(z)\, P_n(z)$$

$$= \Sigma_{n'} a_{n'}\, K_{n'n}$$

where $K_{n'n}$ are some known (symmetric) constants we can write down. We then have

$$\Sigma_{n'=1}^{\infty} K_{nn'}\, a_{n'} = 0 \qquad \text{for } n = 1,2,3...\infty$$

This is an ∞ x ∞ matrix problem Ka = 0. If det(K) = 0, there is a possible non-trivial solution. I have no idea what to do at this point, so I am down a blind alley (what else is new?). Note: the $P_{n'}(z)$ do not form a complete set on the interval (0,1) so the expansion $I(z) = \Sigma_{n'} a_{n'} P_{n'}(z)$ is probably invalid.

Conclusion: I do not know how to solve this half-shell problem.

At this point, I spent about 3 DAYS thinking about My Problem as stated above, and I wrote up these ponderings in a document Legendre my problem.doc . I concluded in that document that the solution to My Problem is essentially that f(z) = 0 as some kind of distribution similar in nature to $f_N(z) = N\cos(Nz)$ as $N \to \infty$. So after all this work, I have no idea how charge distributes itself on the half-shell. Perhaps I(z) is a distribution and so maybe some step in my development leading up to My Problem is invalid. I then realized there are a lot simpler problems where I don't know how charge distributes itself, such as the 2D unit disk, and maybe the same disk in 3D. It seems logical to go figure out these seemingly simpler problems and *then* maybe some day come back to the half sphere problem. Conveniently to this end, Stak is now going to talk about 2D conductors. Another problem would be how charge distributes itself on a finite wire in 1D!

I don't think charge distribution problems are as cookbook as Dirichlet problems. I have no idea on this general how charge distributes itself question, despite doing **Jackson**. Most of Jackson's *three* chapters on electrostatics is Green's Function and Dirichlet and support stuff the newbie needs at this point in his/her physics development. But as I peruse Jackson right now, at the *very end* of Electrostatics II he does an example of this kind of problem: the thin disk in 3D space. He shows in 3.179 that the charge density is finite at the center, and increases as you go to the edge, being infinite at the edge, but in a manner that is still integrable. There is no delta function, but there is an infinity, and somehow that has probably caused problems with my work above. Jackson does see fit to comment in a very long paragraph on page 91 that even this simple geometry was historically a tough problem, involving dual integral equations, and he treats it as a mixed BC problem in cylindrical coords. He mentions no less than five historical methods with people's names. So maybe I should feel better that I could not do it in my 3 day odyssey. I wonder if he has added more in his later edition(s), mine being 1st Ed 1962, 1967 printing.

I suspect that if I were to better catalog all the examples *Stakgold* has done in this potential theory chapter, I would find one or more examples that apply to this problem, and I would find my mistake made in the half-spherical-shell development above.