Potential of half-spherical shell v2 , but should really be:
Charge distribution for Half Spherical Shell (v2) PhL 11.1-2.09

(9.28.10) This is a second failed attempt to solve the charged half-sphere bowl problem using the Stakgold integral equation method $\int_\sigma E(s|\xi)I(\xi)dS_\xi=1$. *Basically I show that this can be written as* $(16\pi/\sqrt{2}) \int_0^1 dz\, I(z)\, d^{-1/2} K(\sqrt{2b/d}) = 1$, *but I cannot solve this integral equation for* $\sigma = I(z)$.

Well, I have butchered this problem so badly now, it is time to start over, knowing now what I did not know before. This past error is commented on below. I am able to do the dφ integration for I(z) charge distribution, but it makes a messy elliptic K function, and then I cannot do the z integration to see whether or not the claimed I(z) really works, and then I stop.

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Overview (9.26.10 3 pages)

History Note: When I wrote this doc, I had just concluded from my Legendre My Problem series of docs that the "order interchange" required to obtain the $\Sigma_n P_n(z_s) I_n = 2$ equation was "illegal" and that this series "did not exist" as a stable coefficient series. That provided an explanation to me as to why the integral equation system $\int_0^1 dx\ P_n(x) I(x) = \delta_{n0}$ seemed to have no solution. It turns out the series is just fine and the integral equation is irrelevant, but I did not know that at the time. So in this doc, I carefully do not do the order interchange, and I try to solve $\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1$ without involving this supposedly illegal series.

In **Section 1** "setting up" I install the usual 1/R expansion into the Stak integral equation and then I give a long argument about why I think you can interchange the dφ integration with the 1/R Σ_n. I then do this order interchange (it is the dz one that is illegal I argue) to get equation (*) below which still involves index m:

$$(1/4\pi) \int_0^1 dz\ \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y^*_{nm}(\Omega_s) \left[\int_0^{2\pi} d\varphi\, Y_{nm}(\Omega) I(z,\varphi) \right] = 1 \qquad (*)$$

I then use I = I(z) to remove the m's and this becomes

$$\int_0^1 dz\, k(z_s,z)\, I(z) = 1 \qquad\qquad k(z_s,z) = (1/2)\, \Sigma_{n=0}^{\infty}\, P_n(z_s)\, P_n(z)$$

which I note also appears in Canonical PDF for the half sphere, where z_s lies only on the bowl. I repeat the PDF claim that this is a Fred 1 integral equation, but I say nothing about the nature of the kernel. I then repeat my discussion of why, if you allow the order interchange with z to get $2 = \Sigma_{n=0}^{\infty}\, I_n\, P_n(z_s)$, this series does not exist and therefore that is an illegal step to take.

In **Section 2** called " Plan A: Doing the dφ ring integration", I seem to completely ignore the above section's results. I only note that the PDF claims the solution to the Fred 1 is $I(x) = (2/\pi)\, 1/\sqrt{x}$, but I don't at this point try to verify that fact. Instead, I start over with a new plan called Plan A. [This is not what the PDF says, in fact. It is only the first of two terms. I must have made a goof, but I never use this result for anything.]

a. Set up to do the dφ integration. In this new plan, instead of expanding 1/R in the usual way, I write it in this manner (s and ξ are both on the sphere)

$$R = \sqrt{2}\ \sqrt{1-\cos(\gamma)} \qquad\qquad \cos\gamma = \cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos(\varphi-\varphi_s)$$

When I insert this into $\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1$ I get this result

$$(\,4\pi/\sqrt{2}\,)\ \int_0^1 dz\, I(z) \int_0^{2\pi} d\varphi\ [\ 1 - \cos\theta\cos\theta_s - \sin\theta\sin\theta_s\cos(\varphi-\varphi_s)\ ]^{-1/2} = 1$$

where I give the dφ integral the name I_1 which I then write as [so that $(\,4\pi/\sqrt{2}\,)\ \int_0^1 dz\, I(z)\, I_1 = 1$]

$$I_1 = \int_0^{2\pi} d\varphi\ [\ a - b\cos(\varphi-\varphi_s)]^{-1/2} = 2\int_0^{\pi} d\varphi\ [\ a - b\cos(\varphi)]^{-1/2}$$

where $a = 1 - \cos\theta\cos\theta_s$ and $b = \sin\theta\sin\theta_s$. Nowadays, this is a standard integral for me, but it was not at the time of writing. I know now that [and this involved an email to Dan Z re a swap in GR]

$$\int_0^{\pi} dx(1/\sqrt{a \pm b\cos x}\,) = (2/\sqrt{a+b}\,)\, K(\sqrt{2b/(a+b)}\,) \qquad a > b > 0$$

b. Carry out the dφ integration and compare with Maple So not knowing the above result, I then *laboriously* work on this integral. I look first in Maple and it gives a result that looks similar to the above, but involves i and a-b. I guess I did not trust this, so I convert the integral I_1 to one which has $1-k^2\sin^2\theta$ below decks so it really is in the std elliptic function form and then I find that $I_1 = 4\, c^{-1/2}\, K(i\sqrt{2b/c}\,)$

where $c = 1/\sqrt{a-b}$ and I note that this agrees with the Maple result, though I don't like the "i" sitting in there and there seems to be a branch cut ambiguity.

c. Interpret, and transform from K(ix) to something better Before continuing, I note that you can think of the metal bowl as a set of metal rings at varying z, so $dV \equiv (4\pi/\sqrt{2})\, (dz\, I(z))\, I_1$ is then the contribution to the total V = 1 from a particular ring. Now I don't really see how this observation helps much.

I then show, using $k = i\sqrt{2b/c}$ and K(ik/k') = k' K(k), that the integral can be written in the new form $I_1 = 4\, d^{-1/2}\, K(\sqrt{2b/d})$ where d = a+b, and this then agrees with the standard form quoted above. So I have now converted $\int_\sigma E(s|\xi) I(\xi) dS_\xi = 1$ to this form

$$(16\pi/\sqrt{2}) \int_0^1 dz\, I(z)\, d^{-1/2}\, K(\sqrt{2b/d}) = 1$$

$$b = \sqrt{1-z^2}\,\sqrt{1-z_s^2} \qquad d = 1 - z\, z_s + \sqrt{1-z^2}\,\sqrt{1-z_s^2}$$

I note that the PDF says $I(z) = (2/\pi)\, 1/\sqrt{z}$ and that one then ought to be able to verify that this solution works in the above equation, which would require showing $\int_0^1 dz/\sqrt{z}\; d^{-1/2}\, K(\sqrt{2b/d}) = \sqrt{2}/3$. I have no idea how to do this integral, so I then see if I can verify it numerically. [In the previous doc section 13 I attempted to compute the I_n and verify $\Sigma_n P_n(z_s)\, I_n = 2$ using Maple, so that was a little different.]

d. Attempt a single variable numerical integral of the above with Maple. I enter that above equation's LHS into Maple and select a random value $z_s = .324$ and try to make it do a numerical integration to see if the result is $\sqrt{2}/3$. Maple struggles for 20 minutes even with integration limits at 0.2 and 0.6. I should try this again to see why it could not do it. The mws filename is stated, but I cannot find it on Alta C or D!

Comments on the double P sum. I then compare my two results so far

$$(16\pi/\sqrt{2}) \int_0^1 dz\, I(z)\, d^{-1/2}\, K(\sqrt{2b/d}) = 1$$

$$\int_0^1 dz\, k(z_s,z)\, I(z) = 1 \qquad k(z_s,z) = (1/2) \Sigma_{n=0}^{\infty} P_n(z_s)\, P_n(z)$$

and this then suggests that the following *might* be true:

$$k(z_s,z) = (1/2) \Sigma_{n=0}^{\infty} P_n(z_s)\, P_n(z) = (16\pi/\sqrt{2})\, d^{-1/2}\, K(\sqrt{2b/d})$$
$$b = \sqrt{1-z^2}\,\sqrt{1-z_s^2} \qquad d = 1 - z\, z_s + \sqrt{1-z^2}\,\sqrt{1-z_s^2}$$

from which we could conclude that $\Sigma_n P_n(z_s)\, P_n(z)$ is a perfectly reasonable function and not some kind of weird distribution. I say "might be true" for this reason: only the dz *integrals* of the two functions are equal, but they are equal for all values of z_s in the range (0,1), so this certainly makes it seem to be equal. I cannot find this sum in GR. PBM vol 3 on page 342 has some generally like this one. Canonical talks about it and pulls an integral representation for it out of a hat, but no function is written down. There is a

formula called the C-D formula for the finite sum of two P's, so you would have to take a limit somehow, see GR.

e. Consider a 2D numerical integration with Maple. I consider trying to do a numerical 2D integration on the bowl to verify $\int_\sigma E(s|\xi)I(\xi)dS_\xi = 1$. I tried a bit, but Maple could not really do anything useful.

f. Conclusion. I conclude that maybe doing the dφ integral first to get the K function is like picking a wrong coordinate system to solve a problem. I then onto the v3 doc.

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1. Setting up. Our integral equation is this, where σ = the upper half sphere:

$$\int_\sigma E(s|\xi)I(\xi)dS_\xi \quad = 1$$

We then insert the usual expansion for 1/R

$$(1/4\pi) \int_\sigma d\Omega_\xi \{ \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y_{nm}(\Omega_\xi) Y^*_{nm}(\Omega_s) \} I(\Omega_\xi) \quad = 1$$

$$(1/4\pi) \int_\sigma d\Omega \{ \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y_{nm}(\Omega) Y^*_{nm}(\Omega_s) \} I(\Omega) \quad = 1$$

$$(1/4\pi) \int_0^1 dz \int_0^{2\pi} d\varphi \{ \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y_{nm}(\Omega) Y^*_{nm}(\Omega_s) \} I(z,\varphi) \quad = 1$$

Now we are going to be very careful, since we are very gun shy after 6 days of pain. I want to explain clearly WHY I think we are allowed to move the dφ integration "to the right". In legendre my problem v2 we considered this generic situation, and I quote: (I have added constant K_n)

$$\int_a^b dx \ \{\Sigma_n K_n \ f_n(y) \ g_n(x)\} \ I(x) \ =?= \ \Sigma_n K_n \ f_n(y) \ \int_a^b dx \ g_n(x) \ I(x) \ \equiv \Sigma_n K_n \ f_n(y) \ I_n$$

If the functions $f_n(y)$ do not form a basis on (a,b), then this series does not exist, and we have to replace the symbol =?= with ≠ !!! So in this situation, interchanging the order of integration and summation is not allowed !!! . There exists no set of numbers I_n such that $h(y) = \Sigma_{n=0}^{\infty} f_n(y) I_n$ on (a,b).

Let's try to apply that to our current situation with φ. We have

$$\int_0^{2\pi} d\varphi \ \{\Sigma_n \ f_n^*(\varphi_s) \ g_n(\varphi)\} \ I(\varphi) =?= \ \Sigma_n \ f_n^*(\varphi_s) \ \int_0^{2\pi} d\varphi \ g_n(\varphi) \ I(\varphi) \ \equiv \Sigma_n \ f_n^*(\varphi_s) \ I_n$$

where I have sort of carelessly added back in the * operation since now have complex functions. Here we have $f_n^*(\varphi_s) = e^{\pm in\varphi s}$ (one sign or other, don't care now). The main idea is that since the sum we end up

with has basis functions which are complete on $(0,2\pi)$, the series we get "exists" and we are permitted to move the $d\varphi$ integration to the right. So let's do this above to get our next step

$$(1/4\pi) \int_0^1 dz \int_0^{2\pi} d\varphi \{ \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y_{nm}(\Omega) Y^*_{nm}(\Omega_s) \} I(z,\varphi) = 1$$

$$(1/4\pi) \int_0^1 dz \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y^*_{nm}(\Omega_s) [\int_0^{2\pi} d\varphi Y_{nm}(\Omega) I(z,\varphi)] = 1 \qquad (*)$$

Notice that we have NOT moved the dz integral to the right. Now we assume no φ dependence of I, and we get the next steps

$$(1/4\pi) \int_0^1 dz \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y^*_{nm}(\Omega_s) [\delta_{m,0} \int_0^{2\pi} d\varphi Y_n(\Omega) I(z)] = 1$$

$$(1/4\pi) \int_0^1 dz \Sigma_n N_{0n} \Sigma_m (1/N_{mn}) Y^*_{nm}(\Omega_s) [\delta_{m,0} (2\pi) P_n(z) I(z)] = 1$$

$$(1/2) \int_0^1 dz \Sigma_n N_{0n} (1/N_{0n}) Y^*_{n0}(\Omega_s) P_n(z) I(z)] = 1$$

$$\int_0^1 dz [(1/2)\Sigma_{n=0}^{\infty} P_n(z_s) P_n(z)] I(z) = 1 \qquad k(z_s,z) = (1/2) \Sigma_{n=0}^{\infty} P_n(z_s) P_n(z)$$

This then is our integral equation simplified, and this is how it appears in the PDF paper. In our half shell problem, z is only being integrated over the upper half sphere and z_s is also on that half. Nothing is on the lower half. The above equation is NOT VALID if you take z_s on the lower half. The claim is made that this is a Fredholm one-variable integral equation of the first kind and there is the kernel k.

At this point, we PONDER (but we do not accept) doing an interchange of ordering:

$$2 = \Sigma_{n=0}^{\infty} P_n(z_s) \int_0^1 dz P_n(z) I(z) =?= \Sigma_{n=0}^{\infty} I_n P_n(z_s) \qquad I_n = \int_0^1 dz P_n(z) I(z)$$

But as we examine the resulting equation $2 = \Sigma_{n=0}^{\infty} I_n P_n(z_s)$ for z_s only in (0,1), we declare that this infinite series does not exist. What we mean is this. If we consider the equation $2 = \Sigma_{n=0}^{N} I_n P_n(z_s)$ on the interval $z_s = (0,1)$, we could find some coefficients I_n that would make a best fit. As we increase N, all these coefficients "move". The sequence of functions $f_N(z_s) = \Sigma_{n=0}^{N} I_n P_n(z_s)$ does not converge to anything at all, much less "2". In other words, the series does not exist and the reason is that $P_n(z_s)$ do not form a basis for (0,1). If they did, then the coefficients would not "move" and the series would exist (though it might still diverge).

Therefore, although we have pondered the above integration/summation interchange, we know we cannot do it. That is a good thing, because I just spend three days showing that if you *could* do it, you end up finding that there exists no function I(z) which solves the integral equation.

2. Plan A: Doing the dφ ring integration. So we have this Fredholm equation,

$$1 = \int_0^1 dx\, k(z_s,x)\, I(x) \qquad\qquad k(z_s,x) = (1/2)\, \Sigma_{n=0}^{\infty}\, P_n(z_s)\, P_n(x)$$

and we think the solution is this [I have omitted the second term of Canonical PDF result, no matter]

$$I(x) = (2/\pi)\, 1/\sqrt{x}$$

It is a tribute I think to the difficulty of this problem that, even when you know the solution, it is still very hard to show it is the right solution.

a. Set up to do the dφ integration. We could go back to an earlier form

$$\int_\sigma E(s|\xi) I(\xi) dS_\xi \qquad = 1$$

$$\int_\sigma [\, 4\pi/\, |s-\xi|\,) I(\xi) d\Omega_\xi \qquad = 1$$

We need something to put in for that distance. If we have s and ξ both on a unit sphere with R being the distance between the tips of the arrows, we have a triangle and law of cosines says

$$R^2 = 1^2 + 1^2 - 2\ 1\ 1\ \cos(\gamma) = 2 - 2\cos(\gamma) = 2\,[\, 1 - \cos(\gamma)]$$

$$R = \sqrt{2}\ \sqrt{1-\cos(\gamma)}$$

where γ is the angle between the two arrows. And we know that (Stak p 397 and my notes where I derive this fact)

$$\cos\gamma = \cos\theta\, \cos\theta_s + \sin\theta\, \sin\theta_s\, \cos(\varphi-\varphi_s)$$

So we can write our integral equation in this way

$$(\, 4\pi/\sqrt{2}\,) \int_0^1 dz \int_0^{2\pi} d\varphi\ [\, 1 - \cos\theta\, \cos\theta_s - \sin\theta\, \sin\theta_s\, \cos(\varphi-\varphi_s)\,]^{-1/2}\, I(\theta,\varphi) = 1$$

Now let's assume that I(θ,φ) = I(θ) which we sometimes write (wrongly) as I(z). Then we can try to do the dφ integration:

$$I_1 = \int_0^{2\pi} d\varphi\, [\, a - b\, \cos(\varphi-\varphi_s)]^{-1/2} \qquad a = 1 - \cos\theta\, \cos\theta_s \qquad b = \sin\theta\, \sin\theta_s$$

On our half sphere, we have cosθ varying from 1 at the top to 0 on the equator. Always positive. Therefore a and b are both positive and in the range (0,1). Also, for the record,

$$c \equiv a - b = 1 - \cos\theta\cos\theta_s - \sin\theta\sin\theta_s = 1 - \cos(\theta-\theta_s)$$

The angle $\theta-\theta_s$ can take the full range $(-\pi/2,\pi/2)$, but on that entire range $\cos(\theta-\theta_s)$ is positive. Therefore the quantity a-b also lies in the range (0,1). So a,b,c all have this same property: (0,1) range.

b. Carry out the dφ integration and compare with Maple. If we change variables to $\varphi' = \varphi-\varphi_s$ with $d\varphi = d\varphi'$, we find that

$$I_1 = \int_0^{2\pi} d\varphi' \, [\, a - b\cos(\varphi')]^{-1/2}$$

and so does not depend on φ_s. The integrand is even in φ' so we have

$$I_1 = 2\int_0^{\pi} d\varphi \, [\, a - b\cos(\varphi)]^{-1/2}$$

$a = 1 - \cos\theta\cos\theta_s$ $\quad$ $b = \sin\theta\sin\theta_s$

I have just spent 4 minutes looking for this integral in GR, I thought it was there and I had used it in fact recently. Wrong. But I know I have seen this before, somewhere. Reminds me of elliptic function integrals, but those are more $\sin^2\theta$ in the square root. Maple says it has an indefinite integral. So now I look in the GR indefinite section and it is on page 154 #4 which gives an "F function". I have studied these elliptic functions here D:\Work\My Interests\Physics\Mechanics\Pendulum\elliptic functions.doc. Maple is claiming this for the integral

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Int((a-b*cos(phi))^(-1/2),phi = 0..Pi);
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$$\int_0^{\pi} \frac{1}{\sqrt{a - b\cos(\phi)}}\, d\phi$$

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int((a-b*cos(phi))^(-1/2),phi = 0..Pi);
```

$$2\,\frac{\text{EllipticK}\left(\sqrt{2}\sqrt{-\frac{b}{a-b}}\right)\sqrt{\frac{a+b}{a-b}}}{\sqrt{a+b}}$$

where EllipticK is the function I usually call bolded **K**(x). Recall from those elliptic notes

$$F(\varphi,k) = \int_0^{\varphi} d\theta \, \frac{1}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin(\varphi)} dt \, \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} \qquad \text{first kind} \qquad \mathbf{K}(k) = F(\pi/2,k)$$

So this is claiming that

$$\mathbf{K}(k) = \int_0^{\pi/2} d\theta \; \frac{1}{\sqrt{1-k^2\sin^2\theta}}$$

This makes me think there is a simple shuffle. Go back to

$$I_1 = 2 \int_0^{\pi} d\varphi \; [\, a - b\cos(\varphi)]^{-1/2}$$

Change variables to $\varphi = 2\theta$ which has $d\varphi = 2d\theta$ and runs $\theta = (0, \pi/2)$ so

$$I_1 = 4 \int_0^{\pi/2} d\theta \; [\, a - b\cos(2\theta)]^{-1/2}$$

Now throw in $\cos(2\theta) = 1 - 2\sin^2\theta$ and inside [] we then have

$$a - b\cos(2\theta) = a - b\,(1 - 2\sin^2\theta) = a - b + 2b\sin^2\theta$$

$$= (a-b)\,[1 - k^2\sin^2\theta\,] \qquad \text{where } -k^2(a-b) = 2b \qquad k^2 = 2b/(b-a)$$

So we have then arrived at

$$I_1 = 4 \int_0^{\pi/2} d\theta \; [(a-b)]^{-1/2} \; [1 - k\sin^2\theta\,]^{-1/2} = 4[(a-b)]^{-1/2} \int_0^{\pi/2} d\theta \; [1 - k\sin^2\theta\,]^{-1/2}$$

$$= 4[(a-b)]^{-1/2} \; \mathbf{K}(k) \qquad k^2 = 2b/(b-a) \quad \Rightarrow \quad I_1 = 4\, c^{-1/2}\, K(i\sqrt{2b/c}\,) \qquad c = (a-b)^{-1/2}$$

and *this agrees with the Maple result* if you ignore the branch issues.

<u>c. Interpret, and transform from K(ix) to something better.</u> Now we need to ponder this result for a moment. We started with this integral equation

$$(\, 4\pi/\sqrt{2}\,) \int_0^{1} dz\, I(\theta) \int_0^{2\pi} d\varphi \; [\, 1 - \cos\theta\cos\theta_{\mathbf{s}} - \sin\theta\sin\theta_{\mathbf{s}}\cos(\varphi-\varphi_{\mathbf{s}})\,]^{-1/2} = 1$$

where [..] = R(**s**,**r**). If we consider a particular "ring" of width dz, on which the charge is all constant, and if we observe from location **s** which is on a *different ring*, the contribution to the potential at s from the ring at z is given by

$$dV = (\, 4\pi/\sqrt{2}\,)\,(dz\, I(z)) \int_0^{2\pi} d\varphi \; [\, 1 - \cos\theta\cos\theta_{\mathbf{s}} - \sin\theta\sin\theta_{\mathbf{s}}\cos(\varphi-\varphi_{\mathbf{s}})\,]^{-1/2}$$

$$= (\, 4\pi/\sqrt{2}\,)\,(dz\, I(z))\; I_1$$

where $d^2Q(z) = dz\, I(z)d\varphi$ is the charge on a piece of the ring being integrated. This is then

$dV = 4\pi\,(dz\ I(z))\quad I_1 = 4\pi\,dz\ I(z)\ 4c^{-1/2}\ \mathbf{K}(k)$ $\qquad k^2 = -2b/c$

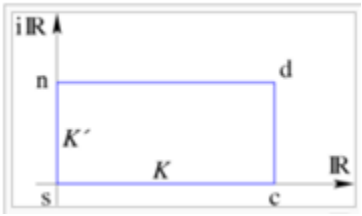
$a = 1 - \cos\theta\ \cos\theta_s$
$b = \sin\theta\ \sin\theta_s$
$c \equiv a - b = 1 - \cos\theta\ \cos\theta_s - \sin\theta\ \sin\theta_s = 1 - \cos(\theta - \theta_s)$

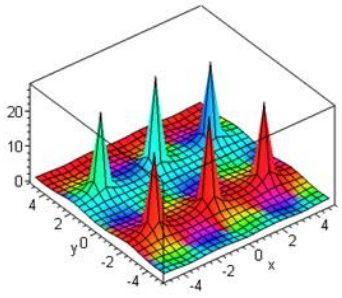
We know the charge is all positive, so somehow this dV is positive and real. My concern here is that the quantity k here seems imaginary:

$k = i\sqrt{2b/c}$

and this is not really "normal" for the K function which likes an argument 0,1.

Remember that there are 12 Jaco'bi functions like sn(u,k) where the two letter name of the function derives from this rectangle picture (left)





where k is a parameter. These functions all have quarter period K(k) in the real direction and half period K'(k) in the imaginary direction. They have a zero at the corner of the first letter, and a pole at the corner of the second letter. The right picture above is a plot of sn(u,k) with k = 0.5 from -5,5 both ways. You see the double periodicity and you see the poles. So this gives a little interpretation of the meaning of K(k) and the "normal range" is k in (0,1). These other symbols are often seen:

$m = k^2$ $\qquad m_1 = 1 - k^2 = k'^2$ $\qquad$ k = modulus $\qquad$ m = parameter

OK, I think I am "up to speed" on the elliptic world again. GR p 908 shows transformations on the K function which allow you to get things in the right range. In particular:

$K(ik/k') = k'\,K(k)$ $\qquad$ where k and k' are related as shown above

So let $x = k/k'$. Then $k' = k/x$ and $k = x/\sqrt{1+x^2}$ hand calc, so $k/x = 1/\sqrt{1+x^2}$

$K(ix) = (k/x)\,K(k) = [1/\sqrt{1+x^2}\,]\,K(x/\sqrt{1+x^2}\,)$

In our case we have $x = \sqrt{2b/c}$ $\quad x^2 = 2b/c$, $1+x^2 = 1 + 2b/c = (c + 2b)/c = d/c$

$a = 1 - \cos\theta\ \cos\theta_s$
$b = \sin\theta\ \sin\theta_s$

$c \equiv a - b = 1 - \cos\theta \cos\theta_s - \sin\theta \sin\theta_s = 1 - \cos(\theta - \theta_s)$
$d \equiv c + 2b = 1 - \cos\theta \cos\theta_s + \sin\theta \sin\theta_s = 1 - \cos(\theta + \theta_s)$ // = a + b (added 9.26.10)

The quantity $\theta + \theta_s$ can range from 0 to π on our half sphere, so the cos can be either sign, which means d ranges from 0 to 2 and is positive therefore. So we have

$$K(i\sqrt{2b/c}) = \sqrt{c/d}\ K(\sqrt{c/d} * \sqrt{2b/c}) = \sqrt{c/d}\ K(\sqrt{2b/d})$$

where now everything is real. I am hopeful that the arg of K is in the 0-1 range, but have not proven this fact. If not, more transformations are needed. And we now have

$$I_1 = 4\, c^{-1/2} K(i\sqrt{2b/c}) = 4\, c^{-1/2} \sqrt{c/d}\ K(\sqrt{2b/d}) = 4\, d^{-1/2} K(\sqrt{2b/d})$$

Our ring integral is now

$$dV = (4\pi/\sqrt{2})\,(dz\, I(z))\ I_1 = (16\pi/\sqrt{2})\, I(z) dz\ d^{-1/2} K(\sqrt{2b/d})$$

where

$b = \sin\theta \sin\theta_s$
$d \equiv c + 2b = 1 - \cos\theta \cos\theta_s + \sin\theta \sin\theta_s = 1 - \cos(\theta + \theta_s)$

And then our integral equation is this:

$$V = \int_0^1 dV = (16\pi/\sqrt{2}) \int_0^1 dz\, I(z)\, d^{-1/2} K(\sqrt{2b/d}) = 1\ !$$

$$b = \sqrt{1-z^2}\sqrt{1-z_s^2}$$

$$d = 1 - z\, z_s + \sqrt{1-z^2}\sqrt{1-z_s^2}$$

Now of course we want to claim that this equation is solved by

$$I(z) = (2/\pi)\, 1/\sqrt{z}$$

which means the following should be true:

$$(16\pi/\sqrt{2})\,(2/\pi) \int_0^1 dz/\sqrt{z}\ d^{-1/2} K(\sqrt{2b/d}) = 1$$

$$(32/\sqrt{2}) \int_0^1 dz/\sqrt{z}\ d^{-1/2} K(\sqrt{2b/d}) = 1$$

$$\int_0^1 dz/\sqrt{z}\ d^{-1/2} K(\sqrt{2b/d}) = \sqrt{2}/32$$

$$b = \sqrt{1-z^2}\,\sqrt{1-z_s^2}$$
$$d = 1 - z\,z_s + \sqrt{1-z^2}\,\sqrt{1-z_s^2}$$

It would take me a very long time to deal with such an integral! So Plan A has not panned out. We are stuck with a horrible mess for an integral, just to verify something!

d. Attempt a single variable numerical integral of the above with Maple.

So here is my setup [this code is saved in file elliptic 11_2_09.mws] [cannot find anything like this in a search on 9.26.10, it is lost, luckily code appears below and could be re-entered.]

```
restart;
b := sqrt(1-z^2)*sqrt(1-zs^2);
```

$$b := \sqrt{1-z^2}\,\sqrt{1-zs^2}$$

```
d := 1 - z*zs + b;
```

$$d := 1 - z\,zs + \sqrt{1-z^2}\,\sqrt{1-zs^2}$$

```
e := sqrt(2*b/d);
```

$$e := \sqrt{2}\,\sqrt{\frac{\sqrt{1-z^2}\,\sqrt{1-zs^2}}{1-z\,zs+\sqrt{1-z^2}\,\sqrt{1-zs^2}}}$$

```
V := Int( (z*d)^(-1/2) * EllipticK(e));
```

$$V := \mathrm{Int}\left(\frac{\mathrm{EllipticK}\left(\sqrt{2}\,\sqrt{\frac{\sqrt{1-z^2}\,\sqrt{1-zs^2}}{1-z\,zs+\sqrt{1-z^2}\,\sqrt{1-zs^2}}}\right)}{\sqrt{z\,(1-z\,zs+\sqrt{1-z^2}\,\sqrt{1-zs^2})}}\right)$$

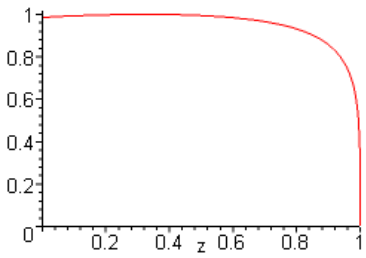
```
zs := 0.324;
```

$$zs := .324$$

```
V;
```

$$\mathrm{Int}\left(\frac{\mathrm{EllipticK}\left(.9726546564\,\sqrt{2}\,\sqrt{\frac{\sqrt{1-z^2}}{1-.324\,z+.9460570807\,\sqrt{1-z^2}}}\right)}{\sqrt{z\,(1-.324\,z+.9460570807\,\sqrt{1-z^2})}}\right)$$

```
plot(e,z=0..1);
```




```
VV := evalf(int( (z*d)^(-1/2) * EllipticK(e), z= 0.2..0.6));
```

I try a limited range first for the integral just to see how it does, started at 9:43. Notice that it does in fact seem true that the K argument is staying in range 0-1, but brushes a lot along the top! I doubt Maple is going to have success with this kind of integration, but I am just doing due diligence.

While waiting, let's compare

$$(16\pi/\sqrt{2}) \int_0^1 dz\, I(z)\, d^{-1/2}\, K(\sqrt{2b/d}) = 1$$

$$\int_0^1 dz\, k(z_s,z)\, I(z) = 1 \qquad k(z_s,x) = (1/2) \Sigma_{n=0}^{\infty} P_n(z_s)\, P_n(x)$$

This seems to suggest that

$$k(z_s,x) = (1/2) \Sigma_{n=0}^{\infty} P_n(z_s)\, P_n(x) = (16\pi/\sqrt{2})\, d^{-1/2}\, K(\sqrt{2b/d})$$
$$b = \sqrt{1-z^2}\,\sqrt{1-z_s^2}$$
$$d = 1 - z\, z_s + \sqrt{1-z^2}\,\sqrt{1-z_s^2}$$

so then we have our Legendre sum as at least an explicit function, albeit of horrible argument. This kernel once again is the potential as seen from a point s on the z_s ring, of an entire charge ring located at z, and of course in general these two rings are different and both lie on the sphere.

Maple is just chugging along at 10:04, so a full 20 minutes is not enough to do the integral just for a single value of z_s . I imagine it is struggling with the singular areas of the integral.

e. Consider a 2D numerical integration with Maple

$$1 = \int_0^1 dx\, k(z_s,x)\, I(x) \qquad k(z_s,x) = (1/2) \Sigma_{n=0}^{\infty} P_n(z_s)\, P_n(x)$$

and we think the solution is this

$$I(x) = I(x) = (2/\pi)\, 1/\sqrt{x}$$

It is (even more so) a tribute to the difficulty of this problem that, even when you know the solution, it is still very hard to show it is the right solution !!

Can I make Maple show that this solution is at least plausible? I have already shown that you don't get far taking a finite sum in the kernel. This is a distribution-like sum and it does not converge in any sense that the first thousand terms bear any resemblance to the entire sum.

It would probably be better to attempt a 2D numeric integration on the half sphere:

$$(4\pi/\sqrt{2}) \int_0^1 dz \int_0^{2\pi} d\varphi\, [\, 1 - \cos\theta \cos\theta_s - \sin\theta \sin\theta_s \cos(\varphi-\varphi_s)\,]^{-1/2}\, I(\theta) = 1$$

$$(8/\sqrt{2}) \int_0^1 dz \int_0^{2\pi} d\varphi\, [\, 1 - z\, z_s - \sqrt{1-z^2}\,\sqrt{1-z_s^2}\, \cos(\varphi-\varphi_s)\,]^{-1/2}\, (1/\sqrt{z}) = 1$$

I think we can claim no φ_s dependence as we did above due to the rearrangement theorem, so

$$(8/\sqrt{2}) \int_0^1 dz \int_0^{2\pi} d\varphi \; [\, 1 - z\, z_s - \sqrt{1-z^2}\,\sqrt{1-z_s^2}\;\cos(\varphi)\,]^{-1/2}\,(1/\sqrt{z}\,) = 1$$

and then we claim the dφ integral could be written (-π,π) and then cos(φ) is even, so

$$(16/\sqrt{2}) \int_0^1 dz \int_0^{\pi} d\varphi \; [\, 1 - z\, z_s - \sqrt{1-z^2}\,\sqrt{1-z_s^2}\;\cos(\varphi)\,]^{-1/2}\,(1/\sqrt{z}\,) = 1$$

Now let's use $z_s = y$, some fixed value, and z = x, to get

$$(16/\sqrt{2}) \int_0^1 dx/\sqrt{x} \int_0^{\pi} d\varphi \; [\, 1 - x\, y - \sqrt{1-x^2}\,\sqrt{1-y^2}\;\cos(\varphi)\,]^{-1/2} = 1$$

This is at least easy to set up in Maple, to wit

```
restart;
y := .3;
```

$$y := .3$$

```
ig := (x^(-1/2))*(1 - x*y - sqrt(1-x^2)*sqrt(1-y^2)*cos(phi))^(-1/2);
```

$$ig := \frac{1}{\sqrt{x}\,\sqrt{1 - .3\,x - .9539392014\,\sqrt{1-x^2}\,\cos(\phi)}}$$

```
ig1 := Int(Int(ig,phi=0..Pi),x=0..1);
```

$$ig1 := \int_0^1 \int_0^{\pi} \frac{1}{\sqrt{x}\,\sqrt{1 - .3\,x - .9539392014\,\sqrt{1-x^2}\,\cos(\phi)}}\, d\phi\, dx$$

But getting it to actually *do* a numeric integration is another issue. First of all, it is very slow. You pick some value of y and tell it to go. It runs for 2 minutes then complains

```
> ig1 := evalf(int(  (int(ig,phi=0..Pi)    ,x=0..1) ));
Error, (in series/O) too many levels of recursion
```

Well, now that I think about the numerical problem, I don't want to try it. Probably much of the integration contribution comes from the neighborhood of point s where R = 0. We are integrating over a singular point, and an even mesh is not going to handle that properly if at all. I could waste a lot of hours going down this path, though as usual, it is an interesting problem in general.

f. Conclusion: I suspect that doing the dφ integration first is like choosing a bad set of coordinates to solve a problem. It probably gives the right answer eventually, but I think the PDF paper is going to show us a better coordinate system, so to speak. I just wanted to at least TRY doing at least this problem verification "on my own" before going down the fancy Mehler-Dirichlet road.

So it is true: this problem (at least when approached from the integral equation) is so hard that even a solid day's effort cannot even verify whether the claimed solution is right or not! I am rather amazed at this fact. Of course there may be some other way to do this.

Nov 5. 2009. There *is* another way to do this problem. Jim suggested it, then I learned it from Jackson. It is the method of inversion and I will switch now to v3 and start clean. Lord Kelvin solved the problem this way in 1847, says Jackson.