Electrostatics Log PhL 6.24.10

My only reporting method so far has been in the day log, but that is a bit diffuse. The path is getting so complex now that I think I need to track it more clearly. I have updated my "physics work flow" doc so at least the log information is easier to get at.

Textbook Review.

1. **Purcell** was my first shot at the subject of electrostatics, 2nd semester sophomore year college. Purcell did his 100 pages on electric fields, field lines, Gauss's Law, divergence and curl of E, and very simple geometries. He did the charged disk in fact, but only got the potential on axis. Fine graphics and text. This was my first exposure to the math of divergence and curl.

2. **Bleaney and Bleaney** Junior, 2nd semester. Did this same stuff in first 60 pages, and dielectrics including the tensor, capacitors, then they did get into the Legendre stuff and images, line charges. But no Bessel stuff. This I think is called an "intermediate" book.

3. Jackson. His 100 pages is more "graduate level", first year grad school, Fall 1970. My second exposure to the Legendres and sphericals, but then he got also into cylindricals and Bessel functions, my first exposure to those guys because I never really did wave theory of drum heads. The Berkeley Wave book does not mention Bessel, eg. First exposure also to Green's general theorems and Green's Functions and delta function usage, completeness relations. Yesterday (6.27.10) I wrote a little Jackson electrostatics review, a mere 3 pages of high level notes concerning his three chapters and the "examples" he deals with, there are not very many. I have previous longer notes on each of these three chapters.

4. **Stakgold** was next, after a small gap of 39 years. Right now I have read through Chapter 6 on "Potential Theory", and have lots of notes. I did this Stak reading Jan-Aug 2009, but then when I hit conformal mapping I digressed into **Ahlfors** on complex variable theory. All along I did all kinds of back and fill work: integral theorems, parts and greens, proofs of divergence and Stokes theorems, projective transformations, infinite products, ellipses, fluid flow. Reviewing and updating my "TK" pages which I had carefully preserved (took kit). I then had my first encounter with the charged half sphere problem around Nov 1, 2009 and this influenced my direction quite a bit. Read Jackson on inversion. To get the half sphere, I needed the on-axis Green's function for a disk in order to use inversion. This led to oblate spheroidal coordinates in **Morse & Feshbach**, and ellipsoidal coordinates as well, including a look at **Kelvin's** original paper showing the σ on a charged ellipsoid, I even did Maple plots. Ran into the 3D wire issue and Jackson's recent papers. That took me to start of Dec 2009.

5. **Smythe, Morse and Feshbach**. I luckily was able to download these eye-opening books which I think would have to be placed in the "advanced" category. Heavier duty math appears in both, the whole subject of those other coordinate systems and their "atoms" and Smythian forms. I finally wrote my piece on tensors and curvilinear coordinates which I think was a landmark for me. Earlier I had done curvilinear stuff using Margenau and Murphy, so I was ready for all eleven separable coordinate systems at that point. So here began my strong interest in the potential and charge density on charged metal objects, and also in the related Green's Functions for such objects. I was following Green's life I think, 1828. Other books appeared in my download list like **Byerly**, but that was only for Lamé functions. **Whittaker and Watson** appeared in printed form, but as of 6/10 have not really done too much there. As part of my back

and fill, I spend many days pondering ODE's and their "special functions", which is what W&W are concerned about. I never had a teaching book before on special functions, just catalogs like **Bateman** and **A&S**, and QM texts**.** The online book availability has been absolutely crucial to all my efforts. Basically, I have been using the "half sphere problem", and more generally the "charged bowl problem" as motivation to pull me through a huge amount of collateral ork where I attempt to learn and stabilize large chunks of the math analysis world. It just happens that this analysis applies to electrostatics, but you can be sure that it applies to many other fields of interest.

My Electrostatics Final Exam

I finished Stakgold Chapter 6 on Potential Theory at the start of Nov 2009, and I have been taking this "final exam" (as I jokingly call it) for the last 8 months, it now being roughly the start of July 2010. In fact what I did was give myself an advanced course in electrostatics, building on the Stakgold and Jackson stuff, and getting into the advanced books of Smythe and M&F. As part of this course, I wrote perhaps 40 documents filling in the many weak spots in my mathematical analysis arsenal. Here is a partial list of these collateral topics:

conformal mapping
integration in general, and certain integrals in particular
Kelvin source material reading
Maple plotting methods
study of internal coding inside Maple
using the Maxima computer algebra program
tensors and curvilinear coordinates
ODE theory in general, and application to PDE theory
Sturm-Liouville theory of 1D ODE's, completeness of eigenfunctions
document recording the various transforms that I constantly need.
gamma function properties
review of orbitals in chemistry
ellipsoidal coordinates and Lamé functions
oblate coordinates and Legendre functions of imaginary arguments
cylindrical coordinates and Bessel functions
toroidal coordinates and special Q functions
application of Stakgold's integral equations to potential problems
coordinate systems with origin on a piece of metal
extremely detailed study of Legendre functions, including read p and q versions
Bateman readings on hypergeometrics and Legendres.
1/R expansions in various coordinate systems
infinite products (well, this came up a little earlier during Stakgold, but in same bin)
ellipses.doc and the various ellipse theorems.
divergence of J and V stuff
group-theoretic diagonalization of integral equations

Explanation of the Electrostatics Flow 2009-2010

This would not be obvious to someone just looking at the flow log document.

When I finished Stak chapter 6, I did want to take a look at Smythe. I have always been intrigued by those strange looking sums of P and Q functions of imaginary arguments, they were a big mystery to me. They are part of the atoms of the oblate spheroidal coordinates, so I learned a lot about them. I was fascinated that you could solve the problem of the potential of a charged ellipsoid, charged spheroid, charged elliptical disk, and charged circular disk so easily in these coordinates. For the disk, I was aware of the painful dual integral equation method and cylindrical Smythian form that Jackson used. So for me, the big payoff of oblates was being able to solve problems with the spheroid and disk. I never did much with the prolate spheroid, but imagine nothing much different happens there. Jim did once tell me that Richard Price did these problems in oblates. It was interesting that Lord Kelvin wrote papers on the ellipsoid stuff that I could more or less read, dated 1842.

Somehow soon after finishing Stak Chap 6 in Oct 2009 I gained an interest in the "half sphere" problem. I called it a half-spherical shell, and I wanted to know the charge distribution on it. This is of course just a special case of the spherical bowl. That is, what is the charge density on a charged half-sphere? I thought I could solve this using the Stak integral equation method, and thought I had a solution, but this led to a whole sea of strange problems including non-convergent series. I will review this matter later and elsewhere, but this problem has been an ongoing "draw" for me to whack through the advanced electrostatics jungle. A stick in the sand placed at an early date.

I had looked at the canonical PDF which treats the charged bowl problem, but I could not understand its method of solution, some Mehler integral representation and other methods. Then perhaps with a comment from Jim, I decided the best way to do the charged bowl was to invert the on-axis Green's function of a disk! After all, a disk is flat and should be simple.

Since the disk is the limit of a spheroid, I was ready to go with my newly learned oblate coordinate tools. Perhaps this was some or all of the motivation for reading Smythe's oblate discussion and examples. For whatever reason, I did a lot of reading in Smythe Chapter 5, and wrote up a full walk through, and did the hole in plate problem, which I would later call an iris, in a parallel E field. I also did a sort of fictitious problem of hole in a charged infinite plate. Although such a thing cannot really exist, it can exist in the central region of a very large disk (I later learned this fact).

So I learned the Smythian method, and did the cone, then the Neumann spheroid, and the Dirichlet spheroid. I then wanted to do the Oblate Spheroid Green's Function problem on axis, since that was going to solve my charged bowl problem by inversion! But it was just as simple to do the general solution to the oblate spheroid Green's problem, so that is what I did. Then I took the limit where spheroid becomes a disk and I had expressions for the potential and charge density. I was able to check this result because Smythe in problem 85 gave the on-axis result.

This was my first encounter with a problem that was going to recur. Using the Smythian Form method I found a complete result for the oblate spheroid Green's problem, and in particular I had the on-axis and disk limit results. But this result was a single sum of various P and Q functions. I knew that if I were to invert this result, I would get some messy sum result for the charged half-sphere (and bowl) problem. But I had seen the solution to *that* problem, and it involves just some elementary functions more or less, no obscure sums of Q functions of imaginary arguments. So I just drew a blank. I did all that work to find the Oblate Spheroid Green's function, but the result was so messy I would not dream of inverting with it. Unless I had a nice catalog of expansion theorems for these Q P expressions, I would never be able to obtain an elementary function result.

While this disappointment was simmering, I noticed problem 38 in Smythe which was the start of a sequence of problems that seemed to lead to a solution of the charged spherical bowl, so I was motivated. Problem 38 asks us to find the charge on an iris due to a point charge in the hole, and the answer he gives is extremely simple. The σ on a charged disk is also quite simple, but not easy to derive. And this iris

thing was a Green's function, which is usually harder than a charged object problem, but the result was still simple. So, my plan here was this: I had just done the Oblate Spheroid Green's, why not spend a little extra effort and do the Oblate *Hyperboloid* ("Bloid") Green's, then take the limit as the bloid becomes an iris, and take the charge into the hole, and I would find that simple Smythe problem 39 result. Besides, I though this bloid thing would be interesting in its own right.

This led to an incredible month-long saga (Feb 2010) of doing this oblate bloid Green's problem. Elsewhere I will outline all the problems I had, reality of p and q functions and all that stuff, but I eventually got what I think was the correct result.

But again, I had the same problem I had with the on-axis disk Green's. The in-hole iris Green's potential and charge are nasty *double* sums this time of P and Q functions, not just simple square roots of things as Smythe problem 38 was showing at least for σ. So once again I was burned. I spent another month pondering this iris situation, digressing into rings, and trying to find some other way to do the on-axis disk Green's problem that might not give a messy P Q sum answer.

I was thinking at this point maybe I just have the wrong coordinate system, and maybe toroidals was the right system. I had just read a PDF on using toroidals to solve ring of charge problems, and I was thinking maybe we can make a disk out of a set of rings. But then along came the "red flag superposition theorem" which sort of knocks out superposition in this sense (multiple pieces of metal).

I then started wondering about possible P Q summation theorems which could simplify my results for the on-axis disk Green's, and for the in-hole iris Green's. This started as my "Cape Cod" idea. I wrote up the 1/R expansion in various coordinate systems that might be relevant, but could not really construct a table of "useful sums", though that might still be doable.

While on this tangent, I just *happened* to notice Problem 2.10 in green Jackson which is a special case of Smythe's problem 38 about the charge on the iris with Green's charge in hole. Unlike Smythe, Jackson clearly stated that you get the result by doing an inversion, not by messing with oblate bloids. So this quickly put me back on the inversion warpath. In June 2010 I figured out how to do a "double inversion" so that you start with a charged disk, and you end up with the iris Green's with charge in hole. You first go from offset disk + constant potential to a Green's bowl situation with Green's charge on the complementary surface. Then you rotate the bowl, and then the second inversion takes it into the iris Green's problem. Somehow I never thought of doing this double inversion before. I spent a long time getting the details right, and in the end came up with Smythe's simple Problem 38 result. This was for me a big breakthrough. I was not even sure whether or not to even believe his elementary functions result until I was able to derive it.

So this then convinced me that somehow the oblate bloid sum result I got must agree with the Smythe simple result, but I have not tried to show this. It would be one of those "addition theorem" type things involving P and Q functions.

The lesson here is that inversion always gives a simple result, and oblate coordinate expansions always give messy sum results (but at least you get results!). It turns out that when you take limits like spheroid goes to disk and point goes on axis (or charge goes in hole), things in the summation world just don't simplify very much. I felt surely they would when I started. I felt the disk potential was messy, but the disk charge is simple, so that ought to happen in these other problems. But I need some kind of summation tricks to see this. I have to wonder if others historically had this same problem.

Finally I was able to "march" through Smythe's important (to me) sequence of problems:

Problem 38: Green's for iris with point charge in hole
Problem 39: Green's for iris with ring charge in hole
Problem 40: Potential on cap of a charged bowl.

Problem 41: Capacitance of a charged bowl.
Problem 42: Surface charge on a charged bowl

This was of course another milestone for me, because I started long ago trying to find the σ on the inner and outer surfaces of a charged bowl, which were graphically reported in Kelvin's paper. I think this same sequence of problems would also give the potential of the charged bowl in terms of elementary functions if you can do the required integrals.

After this, I realized that simple disk-to-disk inversions could give the Green's function for (a) iris with charge in hole; (b) disk with charge in its plane. I did both these problems this simpler way. The iris one duplicated results from the double inversion method, so I got happy here. Along the way I did a cleaned up version of the Jackson inversion notes, made an "inversion formulas" doc.

So what comes next?