

Chapter 1: Introduction to Electrostatics

1.1 Electric field definition

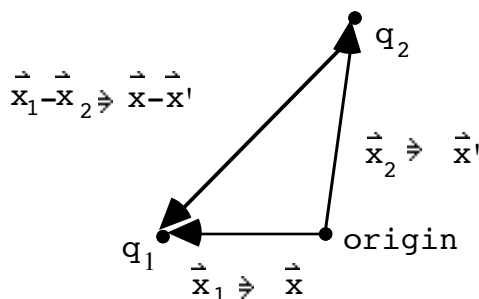
We begin at a very basic level. Definition of $\vec{E}$ in terms of a test charge, q :

$$\vec{F} = q \vec{E}. \quad (1.1)$$

Coulomb's law in the Gaussian system of units:

$$\vec{F}_{12} = q_1 q_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}. \quad (1.2)$$

(force of 2 on 1) Picture:



$$\vec{E}_2(\vec{x}_1) = q_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} \quad \text{or} \quad \vec{E}(\vec{x}) = q \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \quad (1.3)$$

For a charge density:

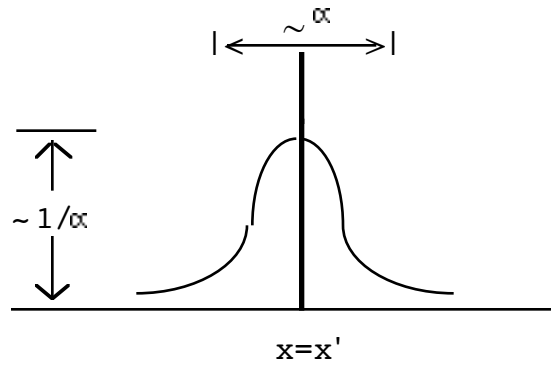
$$\vec{E}(\vec{x}) = \int d^3x' \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}. \quad (1.4)$$

(Similarly for surface charge or line charge densities.) Of course, if we choose $\rho = q \delta(\vec{x} - \vec{x}')$, we get back the previous discrete expression.

1.2 Dirac delta function properties

One way to define Dirac delta functions:

$$\delta(x-x') \equiv \lim_{\alpha \rightarrow 0} \frac{1}{\alpha\sqrt{2\pi}} \exp\left[-\frac{1}{2\alpha^2} (x-x')^2\right],$$



We get

$$\int_{-\infty}^{\infty} \delta(x-x') d(x-x') = \frac{1}{\alpha\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-1/(2\alpha^2)x^2} = 1.$$

(Easy way of seeing the last integral:

$$I \equiv \int_{-\infty}^{\infty} dx e^{-ax^2},$$

$$I^2 = \int dx \int dx' e^{-a(x^2+x'^2)} = \int r d\phi dr e^{-ar^2},$$

$$= 2\pi \int_0^{\infty} dr r e^{-ar^2} = 2\pi \frac{1}{2a} = \frac{\pi}{a} .)$$

In 3-D space:

$$\delta(\vec{x}-\vec{x}') \equiv \lim_{\alpha \rightarrow 0} \left(\frac{1}{\alpha\sqrt{2\pi}} \right)^3 \exp\left[-\frac{1}{2\alpha^2} [(x-x')^2 + (y-y')^2 + (z-z')^2] \right].$$

Generalize it to any other orthogonal coordinate system:

$$\text{old:} \quad ds^2 = dx^2 + dy^2 + dz^2$$

$$\text{new:} \quad ds^2 = \frac{du^2}{U^2} + \frac{dv^2}{V^2} + \frac{dw^2}{W^2}$$

U,V,W are called "scale factors" (they are intrinsically positive)

$$\text{old:} \quad d^3x = dx dy dz$$

$$\text{new:} \quad d^3x = \frac{du \, dv \, dw}{U \cdot V \cdot W}$$

((UVW)⁻¹ is just the Jacobian of the transformation, where U⁻¹ = $\left| \frac{\partial \vec{x}}{\partial u} \right|$, etc.) General idea:

$$\delta(\vec{x} - \vec{x}') = \lim_{\alpha \rightarrow 0} \left(\frac{1}{\sqrt{2\pi} \, \alpha} \right)^3 \exp \left[\frac{-1}{2\alpha^2} \left(\begin{array}{c} \text{extended} \\ \text{infinitesimal} \\ \text{length element} \end{array} \right) \right]$$

$$\delta(\vec{x} - \vec{x}') = \lim_{\alpha \rightarrow 0} \left(\frac{1}{\sqrt{2\pi} \, \alpha} \right)^3 \exp \left[\frac{-1}{2\alpha^2} \left(\frac{(u-u')^2}{U^2} + \frac{(v-v')^2}{V^2} + \frac{(w-w')^2}{W^2} \right) \right]$$

$$\Rightarrow \delta(\vec{x} - \vec{x}') = \delta\left(\frac{u-u'}{U}\right) \delta\left(\frac{v-v'}{V}\right) \delta\left(\frac{w-w'}{W}\right).$$

Given (useful in many other situations)

$$\delta(f(x)) = \sum_i \frac{1}{\left| \frac{df}{dx}(x_i) \right|} \delta(\vec{x} - \vec{x}_i),$$

so

$$\delta(ax) = \frac{1}{|a|} \delta(x),$$

$$\Rightarrow \delta(\vec{x} - \vec{x}') = U \, V \, W \, \delta(u-u') \, \delta(v-v') \, \delta(w-w').$$

Makes sense since this now gives

$$\int \delta(\vec{x} - \vec{x}') d^3x = \int U V W \delta(u - u') \delta(v - v') \delta(w - w') \times \frac{du dv dw}{U V W},$$

$$= 1. \quad \checkmark$$

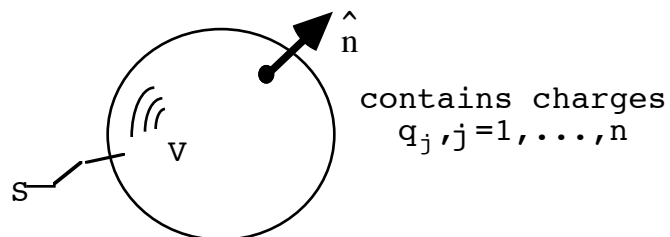
In particular, can show that the repⁿ of the 3-D delta function in spherical coord. is

$$\delta(\vec{r} - \vec{r}') = \frac{1}{r^2} \delta(r - r') \delta(\cos\theta - \cos\theta') \delta(\phi - \phi').$$

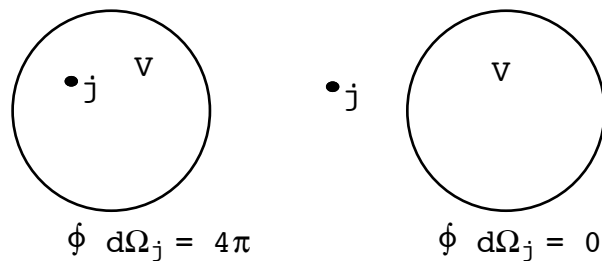
These three dimensional delta functions can be "smeared" or integrated over some dimensions or objects, producing new lower dimensional delta functions. For example, the "line delta function" seen in the introduction is nothing but a 2 dimensional delta function integrated over the line in question. Then, an integration across the line produces unity. These delta functions can also be "directed"; that is, the result can depend on the direction of integration. We will see these types of objects repeatedly in Chs.5 and 6.

1.3 Gauss' law and solid angles

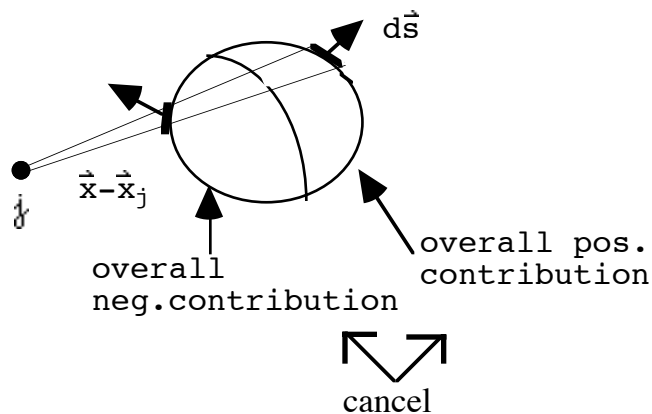
The integral of the normal component of $\vec{E}$ due to point charges is given as follows.



Interpretation: solid angle subtended by $d\vec{s}$ as seen from location of j^{th} charge. There are now 2 possible situations:



Reason (simple object):



Hence

$$\oint_S d\vec{a} \cdot \hat{n} \cdot \vec{E} = \sum_{j \text{ in } V} 4\pi q_j. \quad (1.9)$$

This is a statement about the global behavior of $\vec{E}$. We need to get statements about the local behavior of $\vec{E}$; that is, we need differential equations (supplemental by BC's). Can always imagine a charge density as being due to point charges in infinitesimal volumes. Then we may replace

$$\sum_{j \text{ in } V} q_j \rightarrow \int_V d^3x \rho(\vec{x}). \quad (1.10)$$

Now, our flux statement becomes (by Gauss)

$$4\pi \int_V d^3x \rho(\vec{x}) = \oint_S da \hat{n} \cdot \vec{E} = \int_V d^3x \vec{\nabla} \cdot \vec{E}(\vec{x}). \quad (1.11)$$

True for an arbitrary volume, V . Result:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho(\vec{x}). \quad (1.12)$$

We notice that

$$\vec{E}(\vec{x}) = \int d^3x' \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = -\vec{\nabla}\Phi(\vec{x}), \quad (1.13)$$

where $\Phi(\vec{x})$ (the "scalar potential") is given by

$$\Phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}. \quad (1.14)$$

Comparison with the above informs us that

$$\nabla^2\Phi(\vec{x}) = -4\pi\rho(\vec{x}), \quad (1.15)$$

and therefore that

$$\nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} = -4\pi \delta(\vec{x} - \vec{x}'). \quad (1.16)$$

(We also see that $\vec{\nabla} \times \vec{E} = 0$.) [There are also many direct ways of showing (1.16).] A more general statement is in fact $(\vec{x}' - \vec{x}) \rightarrow \vec{x}$ for convenience)

$$\nabla_i \nabla_j \frac{1}{r} = \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} - \frac{4\pi}{3} \delta_{ij} \delta(\vec{x}) \quad (1.17)$$

($r = |\vec{x}|$) This equation will be justified later in the semester in Ch.5.

1.4 Deviations from inverse square law

Now that we have an expression for $\Phi(\vec{x})$, we can talk about searching for deviations from the inverse square law.

Assume

$$\vec{F}_{12} = q_1 q_2 \frac{\hat{n}_{12}}{r_{12}^{2+\delta}} \quad \leftarrow \text{from 2 to 1, } \delta \ll 1 \quad (1.18)$$

Regard q_1 as a test charge. Then

$$\vec{E}_2(\vec{x}_1) = q_2 \frac{\hat{n}_{12}}{r_{12}^{2+\delta}}, \quad (1.19)$$

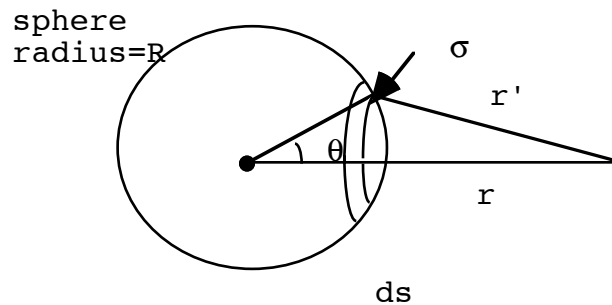
$$\Rightarrow \Phi_2(\vec{x}_1) = \frac{q_2}{(1+\delta)r_{12}^{1+\delta}}. \quad (1.20)$$

Verify last statement:

$$\vec{E}_2 = -\vec{\nabla}_1 \Phi_2 = -\frac{q_2}{(1+\delta)} \vec{\nabla}_1 \frac{1}{|\vec{x}_1 - \vec{x}_2|^{1+\delta}} \quad (1.21)$$

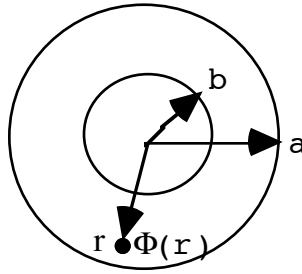
$$= -\frac{q_2}{(1+\delta)} \left\{ -\frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^{3+\delta}} (1+\delta) \right\}. \quad (1.22)$$

Picture:



$$d\Phi = \frac{\sigma ds}{(1+\delta)r'^{1+\delta}} \quad (q = \sigma S, S = 4\pi R^2) \quad (1.23)$$

Imagine



What one can show is that

$$\Phi(r) = \Phi_{\text{in}}(r) + \Phi_{\text{out}}(r), \quad (1.24)$$

where

$$\Phi_{\text{in}} = \frac{q_a}{(1-\delta^2)} \frac{1}{2ar} \left[(a+r)^{1-\delta} - (a-r)^{1-\delta} \right], \quad (1.25)$$

$$\Phi_{\text{out}} = \frac{q_b}{(1-\delta^2)} \frac{1}{2br} \left[(r+b)^{1-\delta} - (r-b)^{1-\delta} \right]. \quad (1.26)$$

Now imagine connecting a wire between the spheres:

$$\Rightarrow \Phi(a) = \Phi(b). \quad (1.27)$$

Can show that this requires

$$q_b \approx \frac{q_a \delta}{2(b-a)} \left[a \ln\left(\frac{a-b}{a+b}\right) + b \ln\left(\frac{4a^2}{a^2-b^2}\right) \right], \quad (1.28)$$

so that q_b vanishes when $\delta=0$ (inverse square law.)

1.5 Surface charge and dipole layers

We learned in the introduction that

$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_{21} = 4\pi\sigma, \quad (1.29)$$

$$(\vec{E}_2 - \vec{E}_1) \times \hat{n}_{21} = 0. \quad (1.30)$$

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If both volumes are vacuum, then

$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{n}_{21} = 4\pi\sigma \quad ((E_2 - E_1)_n = 4\pi\sigma), \quad (1.31)$$

and the potential is

$$\Phi(\vec{x}) = \int_s \frac{\sigma(\vec{x}') d\vec{a}'}{|\vec{x} - \vec{x}'|}. \quad (1.32)$$

This eqⁿ is consistent with the statement about the discontinuity in $\vec{E}$. Can see it as follows. Imagine a flat surface sheet of charge:

$$\vec{E} = -\vec{\nabla}\Phi = -\int_s d\vec{a}' \sigma(\vec{x}') \vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|}, \quad (1.33)$$

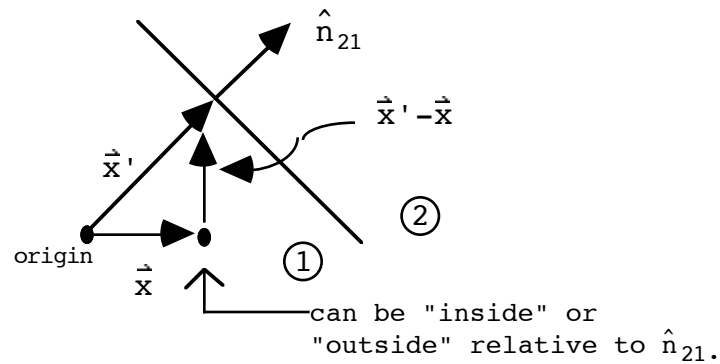
$$\Rightarrow E_n = -\int_s d\vec{a}' \sigma(\vec{x}') \hat{n}_{21} \cdot \vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|}. \quad (1.34)$$

$\uparrow$
 constant vector

But remember

$$d\vec{a}' \cdot \hat{n}_{21} \cdot \vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|} = d\vec{s}' \cdot \frac{(\vec{x}' - \vec{x})}{|\vec{x} - \vec{x}'|^3} = d\Omega'. \quad (1.35)$$

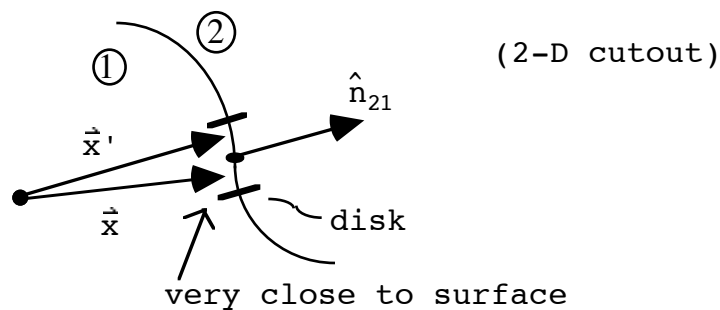
Observation position is now from $\vec{x}$ (field position) rather than $\vec{x}'$ (source position as it was before). Picture:



Also, remember that $d\Omega' > 0$ "inside", $d\Omega' < 0$ "outside."

$$\Rightarrow E_n(\vec{x}) = -\int_s \sigma(\vec{x}') d\Omega', \text{ flat surfaces (all } \vec{x}). \quad (1.36)$$

Now imagine we are very close to an arbitrary surface. We can imagine partitioning the surface into an infinitesimal disk and a remainder:



Because the fields are linear

$$\vec{E} = \vec{E}_{\text{remainder}} + \vec{E}_{\text{disk}}. \quad (1.37)$$

But because we are very close to the disk's surface, it is clear that $\sigma(\vec{x}')$ and $\hat{n}_{21}$ are nearly constant over the integration over the disk, so that

$$(E_{2,\text{disk}})_n \approx 2\pi\sigma, \quad (1.38)$$

$$(E_{1,\text{disk}})_n \approx -2\pi\sigma. \quad (1.39)$$

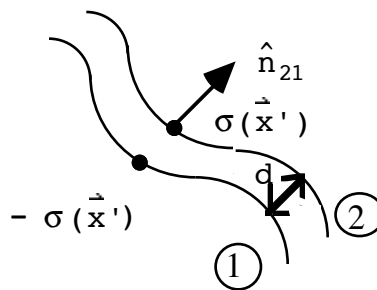
1.12

very near the surface. This means there is a discontinuity in E_n near the surface given by

$$(E_2 - E_1)_n = 4\pi\sigma, \quad (1.40)$$

as stated above. (The E_n due to the remainder is obviously continuous.) Since the contribution of the remainder to E_{1n} & E_{2n} is not known, E_{1n} & E_{2n} are separately unknown.

Another type of surface distribution is the dipole layer.



Define

$$D(\vec{x}') = \lim_{d(\vec{x}') \rightarrow 0} \sigma(\vec{x}') d(\vec{x}'). \quad (1.41)$$

One can show (see the derivation preceeding prob. 1.7; the surface $S = S_2$)

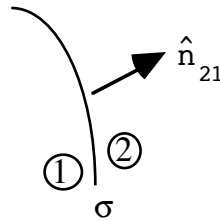
$$\begin{aligned} & \text{not a const. vector} \\ & \downarrow \\ \Phi(\vec{x}) & \xrightarrow{d \rightarrow 0} \int_S D(\vec{x}') \underbrace{\hat{n}_{21} \cdot \vec{\nabla}' \frac{1}{|\vec{x} - \vec{x}'|}}_{-d\Omega'(\text{from field point})} da' \\ \Rightarrow \Phi(\vec{x}) & = - \int_S D(\vec{x}') d\Omega'. \end{aligned} \quad (1.42)$$

For constant D this means that the potential is dependent only on the perimeter of an open surface. (For a closed

surface see for prob. 1.19(b)). This implies (as before take a small disk)

$$\Phi_2^d = 2\pi D, \quad \Phi_1^d = -2\pi D. \quad (1.43)$$

Rest is continuous $\Rightarrow (\Phi_2 - \Phi_1) = 4\pi D$. Really a local statement at every point on surface. An easier way of seeing this last result is



Previous result:

$$E_{1n}^d = -2\pi\sigma \quad (\text{single layer}).$$

Therefore, we have $E_n^{\text{total}} = -4\pi\sigma$ between the two layers, so that

$$\Delta\Phi = -\int \vec{E} \cdot d\vec{\ell} = 4\pi\sigma d \xrightarrow{d \rightarrow 0} 4\pi D. \quad (1.44)$$

1.6 Boundary conditions and uniqueness of solutions

General question: When are the solutions to the eqⁿs of electrostatics,

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi\rho, & \vec{\nabla} \times \vec{E} &= 0, \\ (\text{or } \nabla^2\Phi &= -4\pi\rho, & \vec{E} &= -\vec{\nabla}\Phi) \end{aligned}$$

unique? An application of Gauss ("Green's first identity"):

$$\int_V (\Phi \nabla^2 \psi + \vec{\nabla} \psi \cdot \vec{\nabla} \Phi) \, d^3x = \oint_S da \Phi \frac{\partial \psi}{\partial n}. \quad (1.45)$$

Suppose we have two solutions $(\Phi_{1,2})$ whose B.C.'s are:

$$\text{Dirichlet:} \quad \Phi_1|_s = \Phi_2|_s = f(x,y,z) \quad (1.46)$$

$$\text{Neumann:} \quad \frac{\partial \Phi_1}{\partial n}|_s = \frac{\partial \Phi_2}{\partial n}|_s = g(x,y,z) \quad (1.47)$$

$$\text{Mixed:} \quad \left. \begin{array}{l} \Phi_1|_{s'} = \Phi_2|_{s'} \\ \frac{\partial \Phi_1}{\partial n}|_{s''} = \frac{\partial \Phi_2}{\partial n}|_{s''} \end{array} \right\} \quad s' \cup s'' = s \quad (1.48)$$

$\uparrow_{\text{"union"}} \quad \uparrow_{\text{whole surface (no overlap)}}$

Let $U = \Phi_1 - \Phi_2$. Then $\nabla^2 U = 0$ in V and

$$U|_s = 0 \quad , \quad \text{Dirichlet} \quad (1.49)$$

$$\frac{\partial U}{\partial n}|_s = 0 \quad , \quad \text{Neumann} \quad (1.50)$$

$$U|_{s'} = 0, \quad \frac{\partial U}{\partial n}|_{s''} = 0, \quad \text{Mixed.} \quad (1.51)$$

Let $\psi = \phi = U$ in Green's first identity. Result:

$$\int_V (U \nabla^2 U + (\vec{\nabla} \cdot U)^2) d^3x = \oint_s da U \frac{\partial U}{\partial n} |_s . \quad (1.52)$$

For all three cases, we have the rhs vanishing.

$$\Rightarrow \int_V d^3x (\vec{\nabla} U)^2 = 0, \quad (1.53)$$

$$\Rightarrow \vec{\nabla} U = 0 \text{ in } V,$$

$$\Rightarrow U = \text{const.}$$

(For Dirichlet & Mixed cases we know that $U|_s = 0$, so that the constant is zero there.)

1.7 Dirichlet and Neumann Green functions

Rather than trying to solve for each charge distribution as a special case, there is a more powerful method which allows us to solve a class of problems with a given set of

BC's. Called Green functions. Let us say we are trying to solve

$$\nabla^2 \Phi(\vec{x}') = -4\pi\rho(\vec{x}'), \quad (1.54)$$

in V . (Can have boundaries at infinity or in finite regions.) The trick is first to solve

$$\nabla^2 G(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}'), \quad (1.55)$$

for the same volume. $G(\vec{x}, \vec{x}')$ is then the potential of a point charge at $\vec{x}'$, the BC of which are as yet totally unspecified. (I probably should have introduced (1.55) as $\nabla^2 G(\vec{x}', \vec{x}) = -4\pi\delta(\vec{x} - \vec{x}')$ to conform with the usage that $\vec{x}'$ is regarded as the source and $\vec{x}$ as the field coordinate. In any case you can regard the coordinates inside as $G(\text{source}, \text{field})$.) We will use this freedom of choice of BC to simplify our expressions.

"Green's second identity":

$$\int_V d^3x' (\Phi \nabla^2 \psi - \psi \nabla^2 \Phi) = \oint_S da' \left[\Phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \Phi}{\partial n'} \right]. \quad (1.56)$$

Let $\Phi = G(\vec{x}, \vec{x}')$, $\psi = \Phi(\vec{x}')$. Then

$$\begin{aligned} \Rightarrow \int d^3x' [G(\vec{x}, \vec{x}') \nabla^2 \Phi(\vec{x}') - \Phi(\vec{x}') \nabla^2 G(\vec{x}, \vec{x}')] \\ = \oint_S da' \left[G(\vec{x}, \vec{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \right], \end{aligned} \quad (1.57)$$

$$\text{LHS} = -4\pi \int d^3x' G(\vec{x}, \vec{x}') \rho(\vec{x}') + 4\pi \Phi(\vec{x}). \quad (1.58)$$

($\hat{n}$ = outward normal) We now use the freedom of choice of BC for $G(\vec{x}, \vec{x}')$ to specify

$$G_D(\vec{x}, \vec{x}') = 0, \quad \vec{x}' \text{ on } S \quad (\text{Dirichlet}). \quad (1.59)$$

Then

$$\Phi(\vec{x}) = \int d^3x' G_D(\vec{x}, \vec{x}') \rho(\vec{x}') - \frac{1}{4\pi} \oint_S da' \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}. \quad (1.60)$$

Physically, this Green function is the solution for the potential of a +unit point charge at $\vec{x}'$ in the presence of grounded conductors.

For Neumann BC, we must be more careful. One must realize that since

$$\nabla'^2 G_N(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x}-\vec{x}'), \quad (1.61)$$

$$\Rightarrow \int_V d^3x' \vec{\nabla}' \cdot \vec{\nabla}' G_N(\vec{x}, \vec{x}') = -4\pi, \quad (1.62)$$

$$\Rightarrow \oint_S da' \frac{\partial G_N(\vec{x}, \vec{x}')}{\partial n'} = -4\pi. \quad (1.63)$$

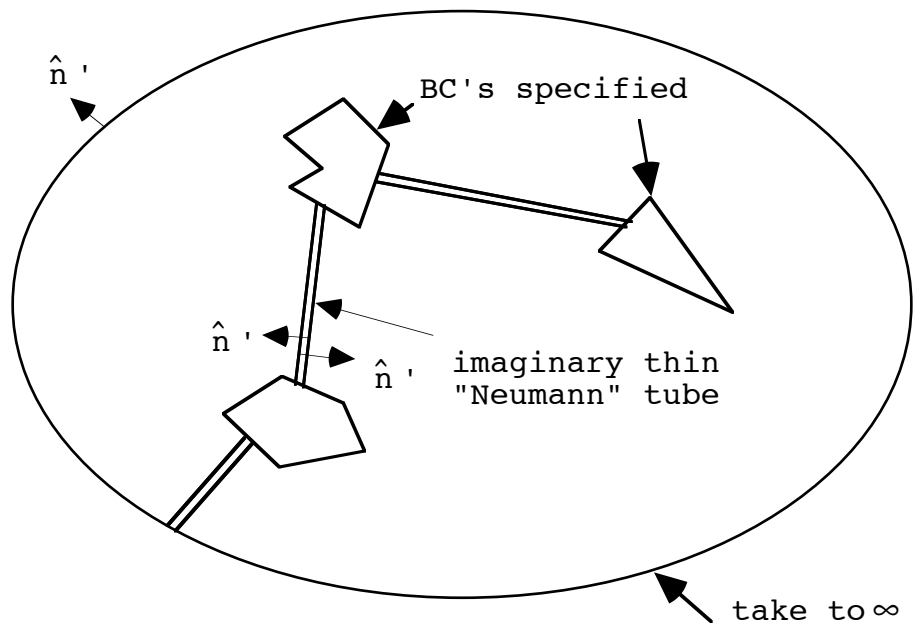
One can not take $\frac{\partial G_N}{\partial n'} = 0$ on S in general. Easiest thing to do:

$$\frac{\partial G_N}{\partial n'} = -\frac{4\pi}{S} \text{ for } \vec{x}' \text{ on } S. \quad (1.64)$$

S = total surface area. Then

$$\begin{aligned} \Phi(\vec{x}) &= \frac{1}{S} \int_S da' \Phi(\vec{x}') + \int d^3x' G_N(\vec{x}, \vec{x}') \rho(\vec{x}') \\ &\quad \underbrace{\hspace{1.5cm}} \\ &\equiv \langle \Phi \rangle_S + \frac{1}{4\pi} \oint_S da' G_N(\vec{x}, \vec{x}') \frac{\partial \Phi}{\partial n'}. \end{aligned} \quad (1.65)$$

A possible situation is to find the fields outside of:
("exterior problem")



If we add the imaginary surfaces shown, then we have a single surface, S , which has infinite area as we take the outer part to ∞ . Clearly, we then have

$$\langle \Phi \rangle_S = 0, \quad (1.66)$$

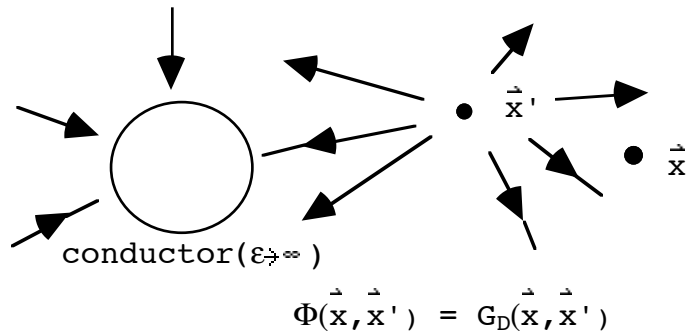
and we may consistently take

$$\frac{\partial G_N}{\partial n'} = 0, \quad (1.67)$$

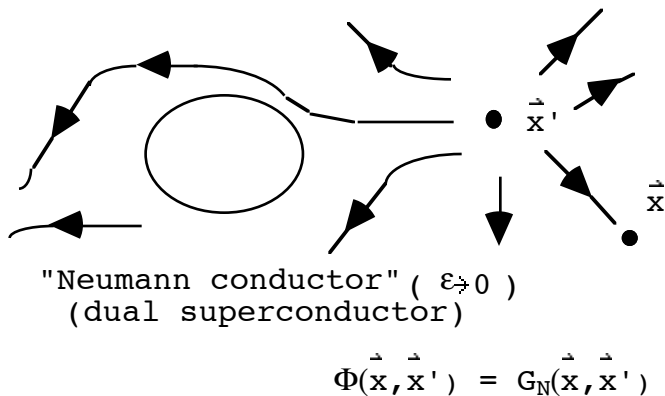
on the finite volumes. (Doesn't contradict $\oint_S \frac{\partial G_N}{\partial n'} da' = -4\pi$.)

Thus, $G_N(\vec{x}, \vec{x}')$ is the potential for a point charge for zero normal electric field on the boundaries for the exterior problem.

Can think of Dirichlet as (conductor has surface charge density which excludes $\vec{E}$ norm.)



Can think of Neumann as ("Neumann conductor" has dipole density which excludes $\vec{E}$ tang. Has properties of a "dual superconductor".)



For electrostatics then, Neumann is not as physical as Dirichlet. It makes more sense, however, in fluid flow problems. Note the $\epsilon \rightarrow \infty, 0$ remarks above. ϵ is the "dielectric constant" to be introduced in Ch.4. I am saying that the $\epsilon \rightarrow \infty$ limit of a point charge near a dielectric gives you the correct Dirichlet Green's function for this geometry (except that the conductor will be uncharged) and the $\epsilon \rightarrow 0$ limit in the same situation gives the correct Neumann one. Note that the inside fields in these cases may or may not be physical.

An important aspect of Green functions is the symmetry property, $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$. We can show it as follows:

$$\int_V d^3x (\phi \nabla^2 \psi - \psi \nabla^2 \phi) = \oint_S da \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right]. \quad (1.68)$$

Choose (Dirichlet case)

$$\begin{aligned}
\phi &= G_D(\vec{x}'', \vec{x}) \quad , \quad \psi = G_D(\vec{x}', \vec{x}) \\
\Rightarrow \int d^3x & \left[G(\vec{x}'', \vec{x}) \nabla^2 G(\vec{x}', \vec{x}) - G(\vec{x}', \vec{x}) \nabla^2 G(\vec{x}'', \vec{x}) \right] \\
&= \oint da \left[G(\vec{x}'', \vec{x}) \frac{\partial}{\partial n} G(\vec{x}', \vec{x}) - G(\vec{x}', \vec{x}) \frac{\partial}{\partial n} G(\vec{x}'', \vec{x}) \right]. \quad (1.69)
\end{aligned}$$

But of course since $G_D(\vec{x}', \vec{x}) = 0$ for $\vec{x}$ on S , we have

$$G_D(\vec{x}'', \vec{x}') = G_D(\vec{x}', \vec{x}''). \quad (1.70)$$

If we do the same thing for Neumann, the RHS does not automatically vanish, and we get:

$$(G_N(\vec{x}'', \vec{x}') - G_N(\vec{x}', \vec{x}'')) = \left\langle G_N(\vec{x}'', \vec{x}) - G_N(\vec{x}', \vec{x}) \right\rangle_{s(\vec{x})} \quad (1.71)$$

However, each of these terms vanishes separately if S includes the surface at infinity since

$$\left\langle G_N(\vec{x}', \vec{x}) \right\rangle_{s(\vec{x})} = \frac{1}{S} \oint_S da \, G_N(\vec{x}', \vec{x}) \xrightarrow{R \rightarrow \infty} 0 \quad (1.72)$$

$$\begin{aligned}
G_N(\vec{x}', \vec{x}) &\sim \frac{1}{R} \quad (|\vec{x}| \gg |\vec{x}'|) \\
da &\sim R^2 d\Omega \\
S &= 4\pi R^2
\end{aligned}$$

(If we choose $G_N = \text{const.}$ on the surface at infinity, the RHS of (1.71) still vanishes.) Therefore, $G_N(\vec{x}'', \vec{x}') = G_N(\vec{x}', \vec{x}'')$ also for the exterior Neumann problem. Is not necessarily true for interior Neumann problems. Let's look at this in some more detail.

Have shown ($G \equiv G_N$; $\vec{x}' \equiv \vec{x}'$)

$$G(\vec{x}'', \vec{x}') - G(\vec{x}', \vec{x}'') = \left\langle G(\vec{x}'', \vec{x}) - G(\vec{x}', \vec{x}) \right\rangle_{s(\vec{x})} \quad (1.73)$$

$$\begin{aligned}
\Rightarrow \langle G(x'', x') - G(x', x'') \rangle_{s(x')} &= \langle G(x'', x) \rangle_{s(x)} - \langle G(x', x) \rangle_{s(x), s(x')} \\
&\quad \begin{array}{cc} \text{source} \downarrow & \downarrow \text{field} \end{array} \\
\Rightarrow \langle G(x', x'') \rangle_{s(x')} &\equiv \text{const.} \quad (1.74)
\end{aligned}$$

Note that (1.74) implies

$$\frac{\partial}{\partial n'} \langle G(x, x') \rangle_{s(x)} = \langle \frac{\partial}{\partial n'} G(x, x') \rangle_{s(x)} = 0. \quad (1.75)$$

But we also have that

$$\frac{\partial}{\partial n'} G(x, x') = \frac{-4\pi}{S}, \quad (1.76)$$

for x' on S . Therefore, it would seem from (1.76) that

$$\langle \frac{\partial}{\partial n'} G(x, x') \rangle_{s(x)} = \frac{-4\pi}{S}, \quad (1.77)$$

contradicting (1.75) above. We can clear this up with the hypothesis that there is a surface delta function lurking in (1.76). The hypothesis is that

$$\frac{\partial}{\partial n'} G(x, x') = \frac{-4\pi}{S} + 4\pi\delta^{(s)}(x-x'). \quad (1.78)$$

[What the last statement really means is

$$\lim_{x' \rightarrow S} \frac{\partial}{\partial n'} G(x, x') \Big|_{x' \text{ on } S} = \frac{-4\pi}{S} + 4\pi\delta^{(s)}(x-x'). \quad]$$

This is now consistent with (1.75) since

$$\frac{\partial}{\partial n'} \langle G(x, x') \rangle_{s(x)} = \langle \frac{\partial}{\partial n'} G(x, x') \rangle_{s(x)} = \frac{-4\pi}{S} + \frac{4\pi}{S} = 0. \quad (1.79)$$

We can confirm this hypothesis from our earlier expression for Φ :

$$\Phi(\vec{x}) = \langle \Phi \rangle_s + \int d^3x' G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') + \frac{1}{4\pi} \oint_s da' G(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi}{\partial n'}, \quad (1.80)$$

which implies

$$\left. \frac{\partial \Phi}{\partial n} \right|_s = \int d^3x' \frac{\partial}{\partial n} G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') + \frac{1}{4\pi} \oint_s da' \frac{\partial}{\partial n} G(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi}{\partial n'}. \quad (1.81)$$

(Note an indescrption here: the normal derivatives are on the first, not second arguments of $G(\mathbf{x}, \mathbf{x}')$. We will clear this up in prob. 1.7.3.) This must reduce to an identity:

$$\begin{aligned} \left. \frac{\partial \Phi}{\partial n} \right|_s &= \int d^3x' \left\{ \frac{-4\pi}{S} + 4\pi \delta^{(s)}(\mathbf{x} - \mathbf{x}') \right\} \rho(\mathbf{x}') \\ &\quad + \frac{1}{4\pi} \oint_s da' \left\{ \frac{-4\pi}{S} + 4\pi \delta^{(s)}(\mathbf{x} - \mathbf{x}') \right\} \frac{\partial \Phi}{\partial n'}, \end{aligned} \quad (1.82)$$

$$\Rightarrow \left. \frac{\partial \Phi}{\partial n} \right|_s = \frac{-4\pi Q}{S} + 4\pi \int d^3x' \delta^{(s)}(\mathbf{x} - \mathbf{x}') \rho(\mathbf{x}') - \frac{1}{S} \oint_s da' \frac{\partial \Phi}{\partial n'} + \left. \frac{\partial \Phi}{\partial n} \right|_s. \quad (1.83)$$

But

$$\frac{-1}{S} \oint_s da' \frac{\partial \Phi}{\partial n'} = \frac{-1}{S} \int d^3x' \underbrace{\nabla'^2 \Phi(\mathbf{x}')}_{-4\pi \rho(\mathbf{x}')} = \frac{4\pi Q}{S}. \quad (1.84)$$

Therefore

$$\left. \frac{\partial \Phi}{\partial n} \right|_s = \left. \frac{\partial \Phi}{\partial n} \right|_s + 4\pi \int d^3x' \delta^{(s)}(\mathbf{x} - \mathbf{x}') \rho(\mathbf{x}'), \quad (1.85)$$

and consistency is maintained as long as we take

$$\int d^3x' \delta^{(s)}(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') = 0, \quad (1.86)$$

for any $f(\mathbf{x}')$. (This last fact can be seen from other, more mathematical, points of view as well. So the surface is not contained in the volume!)

Actually, surface delta functions appear using Dirichlet B.C. as well. Remember

$$\Phi(\mathbf{x}) = \int d^3\mathbf{x}' G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') - \frac{1}{4\pi} \oint_S d\mathbf{a}' \Phi(\mathbf{x}') \frac{\partial G}{\partial n'} . \quad (1.87)$$

Take $\vec{\mathbf{x}}$ on S . Then

$$\Phi \Big|_S = - \frac{1}{4\pi} \oint_S d\mathbf{a}' \Phi(\mathbf{x}') \frac{\partial G}{\partial n'} , \quad (1.88)$$

which means that

$$- \frac{1}{4\pi} \frac{\partial G}{\partial n'} \Big|_S = \delta^{(S)}(\mathbf{x} - \mathbf{x}') . \quad (1.89)$$

[This really means

$$\lim_{\mathbf{x} \rightarrow \mathbf{s}} - \frac{1}{4\pi} \frac{\partial G}{\partial n'} (\mathbf{x}, \mathbf{x}') \Big|_{\mathbf{x}' \text{ on } S} = \delta^{(S)}(\mathbf{x} - \mathbf{x}') .]$$

We will illustrate the presence of surface delta functions in some explicit problems later.

I claimed earlier that $G_N(\mathbf{x}', \mathbf{x})$ is not necessarily symmetric in it's arguments for interior BC's. Can see this as follows. First, we will show that, in general, the Dirichlet Green function is unique, while the Neumann one is not. Both cases at once:

$$\left. \begin{aligned} \nabla^2 G_1(\mathbf{x}', \mathbf{x}) &= -4\pi \delta(\mathbf{x} - \mathbf{x}') \\ \nabla^2 G_2(\mathbf{x}', \mathbf{x}) &= -4\pi \delta(\mathbf{x} - \mathbf{x}') \end{aligned} \right\} \begin{array}{l} \text{either Dirichlet} \\ \text{or Neumann} \end{array} \quad (1.90)$$

Form

$$U(\mathbf{x}', \mathbf{x}) = G_1(\mathbf{x}', \mathbf{x}) - G_2(\mathbf{x}', \mathbf{x}) , \quad (1.91)$$

$$\Rightarrow \nabla^2 U(x', x) = 0. \quad (1.92)$$

Then

$$U(x', x) \Big|_{s(x)} = 0 \quad \text{Dirichlet} \quad (1.93)$$

$$\frac{\partial}{\partial n} U(x', x) \Big|_{s(x)} = 0 \quad \text{Neumann} \quad (1.94)$$

As before, look at Green's first identity:

$$\begin{aligned} \int_V d^3x \left(U(x', x) \nabla^2 U(x', x) + [\vec{\nabla} U(x', x)]^2 \right) \\ = \oint_S da \, U(x', x) \frac{\partial}{\partial n} U(x', x), \end{aligned} \quad (1.95)$$

vanishes in either case

$$\Rightarrow \int_V d^3x \left(\vec{\nabla} U(x', x) \right)^2 = 0, \quad (1.96)$$

$$\Rightarrow \vec{\nabla} U(x', x) = 0 \text{ in } V. \quad (1.97)$$

In the Dirichlet case, we proved that $G_D(x', x'') = G_D(x'', x')$
Combined with $U \Big|_s = 0$, we have that

$$U(x', x) = 0, \quad (1.98)$$

and $G_D(x', x'')$ is unique. On the other hand, we have proven the symmetry $G_N(x', x'') = G_N(x'', x')$ only for the exterior problem (in which case $G_N(x', x'')$ is unique up to a constant.) Thus, in general for the interior problem we only have that $\vec{\nabla} U(x', x) = 0$ implies

$$U(x', x) = F(x'), \quad (1.99)$$

where $F(x')$ is some function of x' . It is also easy to see that the most general solution is then

$$G_N(x', x) = G_N^{\text{sym}}(x', x) + F(x') \quad (1.100)$$

where

$$G_N^{\text{sym}}(x', x) = G_N^{\text{sym}}(x, x'). \quad (1.101)$$

It is easy to see that $\nabla^2 F(x) = 0$ and that $G_N^{\text{sym}}(x', x)$ is unique up to a constant. We can get a physical interpretation of $F(x')$ from anti-symmetrizing (1.100). We can write

$$F(x') - F(x'') = G_N(x', x'') - G_N(x'', x'). \quad (1.102)$$

We now compare to our earlier (1.73):

$$\langle G_N(x', x) - G_N(x'', x) \rangle_{s(x)} = G_N(x', x'') - G_N(x'', x').$$

Averaging over the $s(x)$ surface there gives

$$F(x') = \langle G_N(x', x) \rangle_{s(x)} + \text{Const.} \quad (1.103)$$

Thus, the arbitrary function $F(x')$ represents the average potential of the surfaces (which vanishes when the surface at ∞ is included) up to a constant. The fact that this potential can take an arbitrary value for each source position is due to the fact that the system is isolated. The sensible thing to do would be to refer it to a constant outside potential (via a thin wire, say), which would then remove the arbitrariness. Thus, we can always require the Neumann Green function, in any case, to be symmetric in its arguments.

1.8 Electrostatic energy

Enough of this! Let's investigate some aspects of electrostatic energy. Imagine sources in infinite space. Work done on a test charge:

$$W_i = q_i \Phi(\vec{x}_i) \quad (\text{brought in from } \infty) \quad (1.104)$$

If Φ is due to point charges, [(n-1) of them]

$$\Phi(\vec{x}_i) = \sum_{j=1}^{n-1} \frac{q_j}{|\vec{x}_i - \vec{x}_j|}. \quad (1.105)$$

Total energy: (now n of them)

$$W = \sum_{i=1}^n \sum_{j<i} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}, \quad (1.106)$$

or

$$W = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}. \quad (1.107)$$

Continuous distributions (now has self-energy contributions):

$$W = \frac{1}{2} \iint \frac{\rho(\vec{x}) \rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x d^3x'. \quad (1.108)$$

In contrast to the discrete sum, this expression is in general finite (because of the d^3x , d^3x' elements) when self-energies are included as long as $\rho(\vec{x}')$ is not too singular.

Another expression:

$$W = \frac{1}{2} \int \rho(\vec{x}) \Phi(\vec{x}) d^3x. \quad (1.109)$$

But now

$$\rho(\vec{x}) = -\frac{1}{4\pi} \nabla^2 \Phi(\vec{x}), \quad (1.110)$$

$$\Rightarrow W = \frac{1}{8\pi} \int d^3x [\nabla^2 \Phi] \Phi, \quad (1.111)$$

$$= \frac{-1}{8\pi} \int d^3x \left[\vec{\nabla} \cdot (\Phi \vec{\nabla} \Phi) - \vec{\nabla} \Phi \cdot \vec{\nabla} \Phi \right], \quad (1.112)$$

$$= \frac{-1}{8\pi} \oint_S da \Phi \frac{\partial}{\partial n} \Phi + \frac{1}{8\pi} \int d^3x |\vec{E}|^2 \quad (1.113)$$

$$\begin{aligned} \text{because } da &\sim R^2 d\Omega, \quad \Phi \sim \frac{1}{R} \Rightarrow \frac{\partial}{\partial n} \Phi \sim \frac{1}{R^2} \\ \Rightarrow W &= \int d^3x w, \quad w = \frac{1}{8\pi} |\vec{E}|^2. \end{aligned} \quad (1.114)$$

"w" is energy density. What if we try to calculate self-energy of a stationary point charge:

$$|\vec{E}|^2 = \frac{q^2}{|\vec{x} - \vec{x}'|^4}. \quad (1.115)$$

Then (take $\vec{x}' = 0$; $x \equiv \frac{1}{r}$)

$$W_{\text{self}} = q^2 \int \frac{d^3x}{r^4} = 4\pi q^2 \int_0^\infty \frac{dr}{r^2} \sim 4\pi q^2 \int_0^\infty dx. \quad (1.116)$$

Divergence is at small distances, which means high energies. Called a linear divergence. Actually, divergent in the quantum theory also, but only logarithmically. This permits renormalization to take place. (Ask me what this means.)

The equation $W_i = q_i \Phi(\vec{x}_i)$ also allows a physical interpretation of the fact that we may always choose $G(x', x'') = G(x'', x')$ for Green functions. We learned that $G(x', x'')$ represents the potential of a (+) unit charge in given BC's. Because of this connection between interaction energy and potential, we see that the Dirichlet or Neumann Green function can also be considered the interaction energy of two positive (or two negative) point charges (at x', x'') with given BC's. Since the two charges are equivalent, the Green function must be symmetric in its arguments. This argument fails in the case of interior Neumann boundary conditions,

again because one can imagine defining the zero of potential for each new source position. This is not possible in the Dirichlet case since the potentials are specified by the BC's.

1.9 Capacitance

Let's talk about capacitance. To emphasize the fact that the Green function contains all the information to solve an electrostatic problem in a given geometry, let's connect this to the Green function. I learned the following formulas from Schwinger. Adopting Dirichlet BC's, the potential is

$$\Phi(\vec{x}) = \int d^3x' G_D(\vec{x}, \vec{x}') \rho(\vec{x}') - \frac{1}{4\pi} \oint_S da' \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}. \quad (1.117)$$

Let us apply this to a volume, V , bounded by a number of conducting surfaces. Let's let

$$\rho(\vec{x}') = 0 \quad (1.118)$$

in V , and

$$\Phi(\vec{x}') \Big|_{S_i} = V_i \quad (\text{const.}) \quad (1.119)$$

on the i^{th} surface. Then, using superposition, the potential in the volume V is

$$\Phi(\vec{x}) = \frac{-1}{4\pi} \sum_j V_j \oint_{S_j} da_j' \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n_j'}. \quad (1.120)$$

Therefore, the surface charge density, given by the discontinuity of the normal electric field, is ($\hat{n}_i$ is an outward normal to V on the i^{th} conductor, not necessarily to the conductor volume)

$$\sigma_i(\vec{x}) = \frac{1}{4\pi} \hat{n}_i \cdot \vec{\nabla} \Phi(\vec{x}) = \frac{1}{4\pi} \frac{\partial}{\partial n_i} \Phi, \quad (1.121)$$

$$= -\frac{1}{16\pi^2} \sum_j v_j \oint_{s_j} da'_j \frac{\partial}{\partial n_i} \frac{\partial}{\partial n'_j} G_D(\vec{x}, \vec{x}'). \quad (1.122)$$

The total charge on the i^{th} surface is therefore

$$Q_i = \oint_{s_i} da_i \sigma_i = \frac{-1}{16\pi^2} \sum_j v_j \oint_{s_i, s_j} da_i da'_j \frac{\partial}{\partial n_i} \frac{\partial}{\partial n'_j} G_D(\vec{x}, \vec{x}'). \quad (1.123)$$

We can write this more simply as

$$Q_i = \sum_j C_{ij} v_j, \quad (1.124)$$

where the capacitances (C_{ii}) or coefficients of induction ($C_{ij}, i \neq j$) are given by

$$C_{ij} = -\frac{1}{16\pi^2} \oint_{s_i, s_j} da_i da'_j \frac{\partial}{\partial n_i} \frac{\partial}{\partial n'_j} G_D(\vec{x}, \vec{x}'). \quad (1.125)$$

The capacitance of a conductor is therefore the charge on the conductor when it is maintained at unit potential, all other conductors being grounded. A similar statement holds for the coefficients of capacitance, except the total charge is measured on a conductor other than the one being held at unit potential. The coefficients are symmetric in their indices,

$$C_{ij} = C_{ji}. \quad (1.126)$$

because $G_D(\vec{x}, \vec{x}') = G_D(\vec{x}', \vec{x})$. Given the energy expression,

$$W = \frac{1}{2} \int d^3x d^3x' \frac{\rho(\vec{x}) \rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \Rightarrow \frac{1}{2} \int da da' \frac{\sigma(\vec{x}) \sigma(\vec{x}')}{|\vec{x} - \vec{x}'|}, \quad (1.127)$$

$$\Rightarrow W = \frac{1}{2} \int da \Phi(\vec{x}) \sigma(\vec{x}), \quad (1.128)$$

we have, in this case

$$W = \frac{1}{2} \sum_i V_i \oint_{S_i} da_i \sigma_i(\vec{x}) = \frac{1}{2} \sum_i V_i Q_i \quad (1.129)$$

or

$$W = \frac{1}{2} \sum_{ij} C_{ij} V_i V_j \quad (\text{gives diff. perspective on the } C_{ij}\text{'s.}) \quad (1.130)$$

We have to be careful when the surface at ∞ is included in our analysis since it's capacitance and coeff.'s of induction can only be defined in a limiting sense in a given problem. With this understanding then, we can now show that

$$\sum_i C_{ij} = 0, \quad (1.131)$$

for an interior or "exterior" problem as follows:

$$\sum_i C_{ij} = \frac{-1}{16\pi^2} \oint_{S_j} da'_j \frac{\partial}{\partial n'_j} \left(\sum_i \oint_{S_i} da_i \frac{\partial}{\partial n_i} G_D(\vec{x}', \vec{x}) \right). \quad (1.132)$$

But

$$\frac{1}{4\pi} \sum_i \oint_{S_i} da_i \frac{\partial}{\partial n_i} G_D(\vec{x}', \vec{x}) = \frac{1}{4\pi} \int d^3x \nabla^2 G_D(\vec{x}', \vec{x}) = -1. \quad (1.133)$$

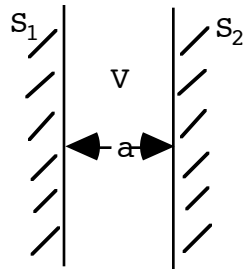
Since $\frac{\partial}{\partial n'_j} (\text{const.}) = 0$, the top line says $\sum_i C_{ij} = 0$. This means that

$$Q \equiv \sum_i Q_i = \sum_{ij} C_{ij} V_j = 0, \quad (1.134)$$

as it must since we are imagining a closed system with no volume charge. (Can you see why this line of reasoning fails in the exterior problem, when the surface at infinity is not included?) In addition, only relative values of the potential have significance here. We can see this from

$$\sum_j C_{ij} (V_j + \text{const.}) = \sum_j C_{ij} V_j. \quad (1.135)$$

Simple example: parallel conductor geometry



Since the C_{ij} are symmetric in i & j and must sum to zero on one index, this means

$$C_{11} = -C_{21} = -C_{12} = C_{22} \equiv C. \quad (1.136)$$

Also $Q_1 = -Q_2 = C(V_1 - V_2) = C\Delta V. \quad (1.137)$

The energy is

$$E = \frac{1}{2} \sum_{i,j} C_{ij} V_i V_j = \frac{1}{2} C \Delta V^2. \quad (1.138)$$

These are the usual expressions for a parallel plate capacitor. C can be shown from Gauss' law to be,

$$C \approx \frac{A}{4\pi a}. \quad (\text{finite surface, } A. \text{ Can be directly evaluated; see prob. 2.4.}) \quad (1.139)$$

Note: Usually it is difficult to get exact solutions for the Green function for a given geometry, so approximation methods must be used in computing the C_{ij} .

Problems

1.1.1 (a) A vector field is defined by

$$\vec{E} = -\vec{\nabla}\Phi,$$

with

$$\Phi = \vec{d} \cdot \frac{\vec{r}}{r^3} \quad (\vec{d} = \text{constant vector})$$

Derive an expression for $\vec{E}$.

(b) Prove that $\vec{E}$ in part (a) can be written as follows:

$$\vec{E} = \vec{\nabla} \times \vec{A},$$

with

$$\vec{A} = \vec{d} \times \frac{\vec{r}}{r^3}.$$

(c) Calculate $\vec{\nabla} \cdot \vec{E}$ and $\vec{\nabla} \times \vec{E}$.

[Note: This problem neglects certain subtleties that occur at $\vec{r} = 0$. Adapted from DiBartolo's "Classical Theory of Electromagnetism".]

1.2.1 Given the scale factor definitions,

$$U^{-1} \equiv \left| \frac{\partial \vec{x}}{\partial u} \right|, \quad V^{-1} \equiv \left| \frac{\partial \vec{x}}{\partial v} \right|, \quad W^{-1} \equiv \left| \frac{\partial \vec{x}}{\partial w} \right|,$$

and

$$\delta(\vec{x} - \vec{x}') = \delta(u - u') \delta(v - v') \delta(w - w') UVW,$$

find the form of the delta function for:

1.32

(a) cylindrical coordinates

$$\begin{aligned}x &= \rho \cos \phi, \\y &= \rho \sin \phi,\end{aligned}$$

where $u=\rho$, $v=\phi$, $w=z$;

(b) spherical coordinates

$$\begin{aligned}x &= r \sin \theta \cos \phi, \\y &= r \sin \theta \sin \phi, \\z &= r \cos \theta,\end{aligned}$$

where $u=r$, $v=\cos \theta$, $w=\phi$.

1.2.2 Find the form of the Dirac delta function in oblate spheroidal coordinates. Use

$$\begin{aligned}(x) \quad x_1 &= R\sqrt{Z^2+1}\sin\theta\cos\phi, \\(y) \quad x_2 &= R\sqrt{Z^2+1}\sin\theta\sin\phi, \\(z) \quad x_3 &= RZ \cos\theta.\end{aligned}$$

Take $u=Z$, $v=\cos\theta$, $w=\phi$ as the new coordinates. (The θ and ϕ coordinates correspond to the usual polar and azimuthal angles, respectively, in spherical coordinates.)

1.3.1 (a) Given in two spatial dimensions the form

$$\vec{E}(\vec{x}) = \int d\vec{x}' \sigma(\vec{x}') \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^2},$$

for the electric field, argue as in the notes that one has

$$\oint_C d\vec{\ell} \cdot \hat{n} \cdot \vec{E} = \sum_{j \text{ in } S} 2\pi q_j,$$

for enclosed charge within an surface area S bounded by C , and

$$\vec{\nabla} \cdot \vec{E} = 2\pi\sigma(\vec{x}),$$

for the electric field due to charge density $\sigma(\vec{x})$.

(b) For a positive point charge in two dimensions, show that (a) gives

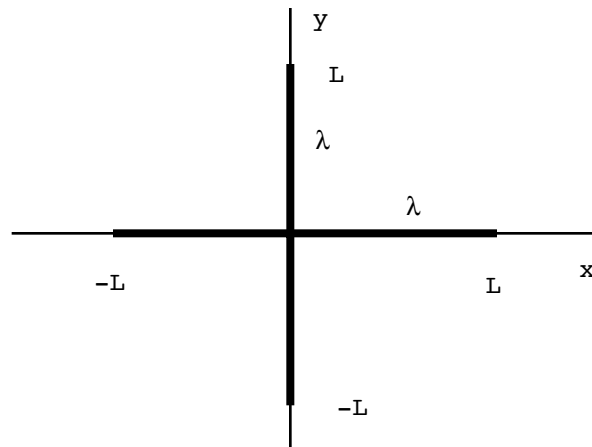
$$\nabla^2 \ln \frac{|\vec{x} - \vec{x}'|}{K} = 2\pi \delta^{(2)}(\vec{x} - \vec{x}'),$$

where K is an arbitrary positive constant.

(c) Confirm this delta function representation by putting $\vec{x}' = 0$ and integrating both sides of (b) about an arbitrary area surrounding the origin. [Note: This identifies the potential of a point charge in 2 dimensions as $-\ln \frac{|\vec{x} - \vec{x}'|}{K}$.

However, the potential of a *line* charge in 3 dimensions with unit line density of charge is $-2 \ln \frac{|\vec{x} - \vec{x}'|}{K}$. Do you know where the extra 2 comes from?]

1.3.2 (a) Find the potential everywhere for a straight uniform line charge density, λ , which makes a X-figure at the origin in the xy plane extending from L to $-L$ on both axes (no conductors present):

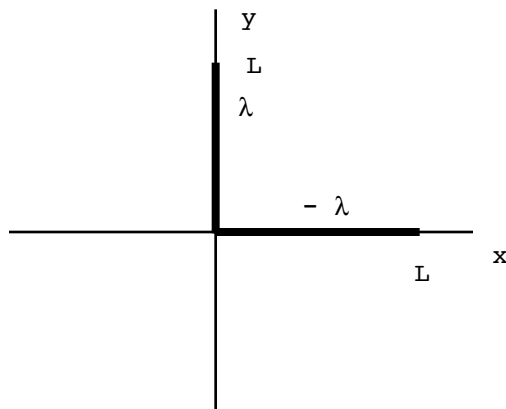


Get the exact $\Phi(x, y, z)$, then show for $L \gg |x|, |y|, |z|$ that

$$\Phi(x, y, z) \approx -\lambda \ln \left(\frac{(x^2 + z^2)(y^2 + z^2)}{16L^4} \right).$$

Why is this answer obvious? For the opposite extreme, $L \ll |x|, |y|, |z|$ show that you get the expected answer for an effective point charge.

(b)



The straight uniform line charges in the xy plane are now given by one line charge, λ , situated along the y -axis from the origin to L , and another, $-\lambda$, situated along the x -axis. Again, get the exact expression, then show for $L \gg |x|, |y|, |z|$ that

$$\Phi(x, y, z) \approx -\lambda \ln\left(\frac{r-y}{r-x}\right),$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

1.4.1 (a) For a potential of the form ($\delta \ll 1$)

$$\phi = \frac{q}{(1+\delta)r_{12}^{1+\delta}},$$

where r_{12} is the distance between source and field points, show that the potential difference between two concentric spheres with uniform surface charge (radii a and b with $b < a$) is given by

$$\phi(r) = \phi_{\text{in}}(r) + \phi_{\text{out}}(r)$$

where

$$\phi_{\text{in}}(r) = \frac{q_a}{1-\delta^2} \frac{1}{2ar} [(a+r)^{1-\delta} - (a-r)^{1-\delta}],$$

$$\phi_{\text{out}}(r) = \frac{q_b}{1-\delta^2} \frac{1}{2br} [(r+b)^{1-\delta} - (r-b)^{1-\delta}].$$

(b) After the spheres are connected by a wire, show that the charge on the inner sphere is given by

$$q_b \approx \frac{q_a \delta}{2(b-a)} \left[a \ln \left(\frac{a-b}{a+b} \right) + b \ln \left(\frac{4a^2}{a^2-b^2} \right) \right].$$

1.5.1 In deriving Eq.(1.42) of the text, consider two close surfaces, 1 and 2, with associated surface charge densities $\sigma_1(\vec{x}')$ and $\sigma_2(\vec{x}'')$, respectively; see the figure on p. 1.12. The separation between the surfaces is such that the points at which the charge densities are equal and opposite are separated a uniform distance, d , along the common perpendicular; ie, $\sigma_1(\vec{x}' - \hat{n}_{21}d) = -\sigma_2(\vec{x}'')$. We may characterize the potential in the limit of close separation of the two surfaces as,

$$\Phi(\vec{x}) = \int_{S_2} \frac{\sigma_2(\vec{x}'')}{|\vec{x} - \vec{x}''|} da'' - \int_{S_1} \frac{\sigma_1(\vec{x}')}{|\vec{x} - \vec{x}'|} da'.$$

With the connection

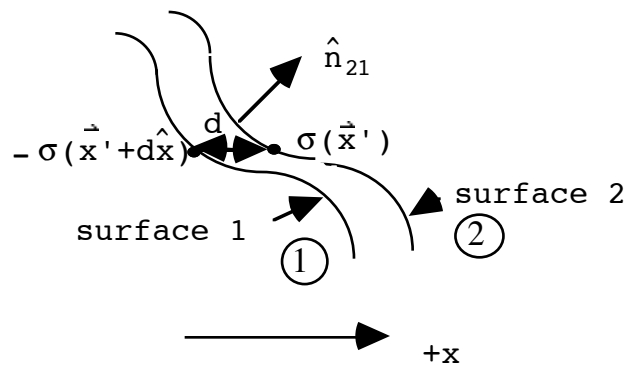
$$\vec{x}'' = \vec{x}' - \hat{n}_{21}d,$$

and the expansion, to lowest order in $\vec{a}$,

$$\frac{1}{|\vec{x} - \vec{a}|} = \frac{1}{|\vec{x}|} + \vec{a} \cdot \vec{\nabla} \left(\frac{1}{|\vec{x}|} \right),$$

one may derive Eq.(1.42) of the text.

Now consider the similar situation:



Two regions in free space (3 dimensions) are separated by a type of dipole surface. The dipole surface is constructed by placing equal and opposite surface charge densities on complementary surfaces which are then brought together such that a small constant separation, d , measured along one axis, remains (these are open surfaces).

(a) Show that the potential can be written as

$$\Phi^d(\vec{x}) = d\hat{x} \cdot \vec{E}^2(\vec{x}),$$

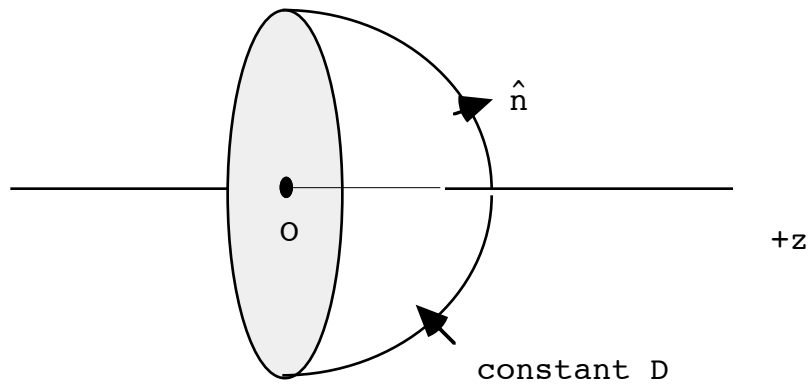
where $\hat{x}$ is a unit vector in the separation direction and $\vec{E}^2$ is the electric field due to surface 2.

(b) Show that the discontinuity of the potential in the *normal* direction (to surface 2) across the surfaces is given by

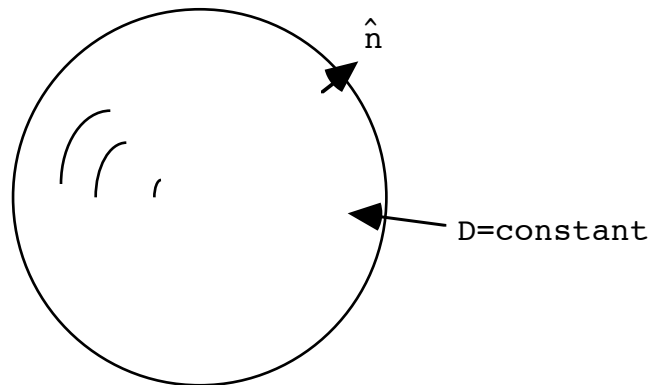
$$\Phi_2^d - \Phi_1^d = 4\pi D(\vec{x}) \hat{x} \cdot \hat{n}_{21},$$

where $D(\vec{x}) \equiv \sigma(\vec{x})d$.

1.5.2 (a) A hemisphere of radius " a " has a constant dipole surface density, D . D is defined relative to the unit vector, $\hat{n}$, pointing "out" as shown. Using the coordinates shown with the origin O at the hemisphere's center, find the $\vec{E}$ field everywhere along the z -axis ($-\infty \leq z \leq \infty$). Then, using spherical coordinates, find the electric field components, E_r and E_θ , far from the region of the sphere, $r \gg R$.

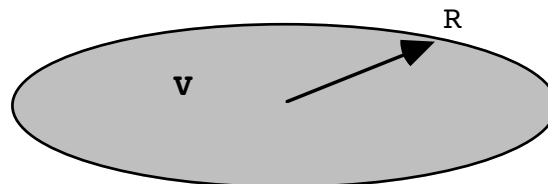


(b) A *sphere* of radius " a " is given a constant dipole surface charge density, D . D is defined relative to the sphere's outer normal, $\hat{n}$.



Find the electric field both inside and outside the sphere.

1.5.3 A thin isolated metallic disk of radius R is charged until it has acquired a potential V .



The solution for the electrostatic potential is found most easily in oblate spheroidal coordinates, Z and ξ , given by

$$z = R Z\xi,$$

$$\rho = R[(1+Z^2)(1-\xi^2)]^{1/2},$$

($\rho^2 = x^2 + y^2$ where x , y and z are the usual Cartesian coordinates) and is given by

$$\Phi = A \tan^{-1}\left(\frac{1}{Z}\right),$$

where A is an unknown constant. (The third coordinate is the usual azimuthal angle ϕ ; take your the origin of coordinates at the center of the disk and the z axis $\perp$ to the plane of the disk.)

(a) Change variables to show that the potential may be written as ($r^2 = \rho^2 + z^2$)

$$\Phi = A \tan^{-1}\left(\frac{\sqrt{2}R}{\sqrt{(r^2 - R^2) + \sqrt{(r^2 - R^2)^2 + 4R^2z^2}}}\right)$$

(b) By considering the limit $z \rightarrow 0$ when $r < R$, argue from consistency that

$$A = \frac{2V}{\pi}.$$

(c) Show that the surface charge density on the disk, $\sigma(\rho)$, is given by

$$\sigma(\rho) = \frac{V}{\pi^2} \frac{1}{\sqrt{R^2 - \rho^2}}.$$

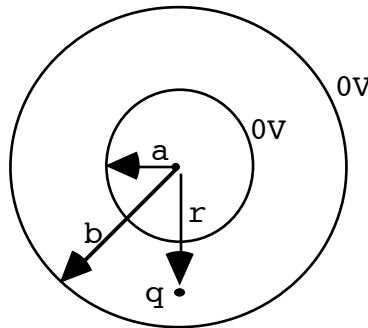
[Note: $\frac{d}{dx} \tan^{-1} y = \frac{1}{1+y^2} \frac{dy}{dx}$.]

1.6.1 (a) Prove Green's reciprocity theorem: For two problems with the exact same geometry including conducting surfaces, one has

$$\int da \sigma \Phi' + \int d^3x \rho \Phi' = \int da \sigma' \Phi + \int d^3x \rho' \Phi,$$

where Φ , ρ and σ are the potential, volume charge density and surface charge density of problem 1, while Φ' , ρ' and σ' are similar quantities for problem 2.

(b) A point charge, q , is located somewhere between two grounded concentric spherical conducting shells ($a < r < b$) as shown.



Find the charge induced on the inner or outer conducting shells by use of the theorem of (a).

1.7.1 Given a certain geometry, show that the Neumann and Dirichlet Green functions are related by

$$G_N(\vec{x}'', \vec{x}) = G_D(\vec{x}, \vec{x}'') - \frac{1}{4\pi} \oint_S da' G_N(\vec{x}'', \vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}.$$

1.7.2 Show that a solution to the Poisson equation with both Φ and $\frac{\partial \Phi}{\partial n}$ specified arbitrarily on a closed boundary (Cauchy boundary conditions) does not exist. [Hint: Show that if one is known, the other is *determined*.]

1.7.3 The Neumann Green's function in 3 dimensions satisfies

$$\nabla^2 G_N(\vec{x}', \vec{x}) = -4\pi \delta(\vec{x} - \vec{x}').$$

Assume that the normal derivative is

$$\left. \frac{\partial G_N(\vec{x}', \vec{x})}{\partial n} \right|_{\vec{x} \text{ on } S} = - \frac{4\pi}{S}$$

where S is the total surface area. We saw that according to (1.100),

$$G_N(\vec{x}', \vec{x}) = G_{\text{sym}}(\vec{x}', \vec{x}) + F(\vec{x}').$$

Considering the interior problem, show that $F(\vec{x}')$ gives no contribution to the potential, $\Phi(\vec{x})$. When considering the normal derivative, $\left. \frac{\partial \Phi}{\partial n} \right|_S$, show that the general form for $G_N(\vec{x}', \vec{x})$ above and Eq.(1.78) remove the "indiscretion" in Eq.(1.81).

1.8.1 Prove a simplified Thompson's theorem for a single closed surface: If a surface is fixed in position and a given total charge is placed on it, then the electrostatic energy in the region bounded by the surface and infinity is an absolute minimum when the charges are placed so that the surface is an equipotential.

1.8.2 Prove the following theorem: If a number of conducting surfaces are fixed in position with a given total charge on each, the introduction of an uncharged, insulated conductor into the region bounded by the surfaces lowers the electrostatic energy.

1.8.3 Investigate the self-energy of a one-dimensional straight string of charge with linear charge density $\lambda = \frac{Q}{a}$ and length a . Is this quantity divergent, and if so, is it quadratically, linearly or logarithmically divergent?

1.8.4 Consider an infinitely thin circular disk of radius R with constant surface charge density σ . (This is not a conductor!) Give an argument that the self-energy is finite. [Hint: The origins of the primed and unprimed coordinate systems do not have to coincide. I used cylindrical coordinates and got an upper limit.]

1.9.1 Two conductors, A and B, form an isolated system. Show, based upon the positivity of the field energy W , that the coefficient of capacitance, $C_{AB} = C_{BA}$, and the two capacitances, C_{AA} and C_{BB} , obey the inequality,

$$C_{AB}^2 < C_{AA}C_{BB}.$$

1.9.2 (a) Use Green's reciprocity theorem (above) to prove

$$C_{ij} = C_{ji}.$$

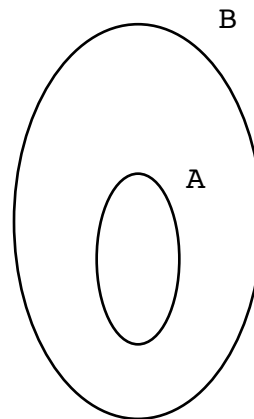
(b) Show the same result from (1.125) of the script.

(c) Show $C_{ii} > 0$ from first principles.

1.9.3 Using $Q_i = \sum_j C_{ij}V_j$ and the definition of the C_{ij} ,

find the six independent coefficients of capacitance for three co-centered spherical conducting shells having radii a, b and c ($a < b < c$). [Call these C_{aa} , C_{bb} , C_{cc} , C_{ab} , C_{ac} and C_{bc} .]

1.9.4 Given a system of two conductors of arbitrary shape, with conductor B being hollow and containing an interior conductor A, show that the capacitances satisfy



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$$C_{BB} \geq C_{AA}.$$

1.9.5 Consider two arbitrary, closed conductors, labelled "a" and "b" in free space, with total charge Q and $-Q$ present. Define,

$$C \equiv \frac{Q}{V_a - V_b},$$

where V_a and V_b are the potentials of conductors A and B. Show that

$$C = \frac{\det \begin{pmatrix} C_{aa} & C_{ab} \\ C_{ab} & C_{bb} \end{pmatrix}}{\sum_{i,j} C_{ij}}.$$

1.9.6 Consider an isolated conductor, a, with capacitance C_a^1 . Show that the introduction of a second conductor, b, raises the capacitance of the first conductor: $C_a^2 > C_a^1$.