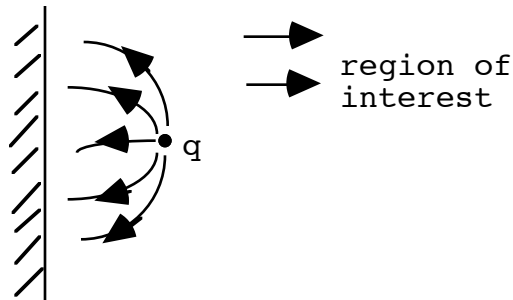


Chapter 2: Boundary Value Problems in Electrostatics

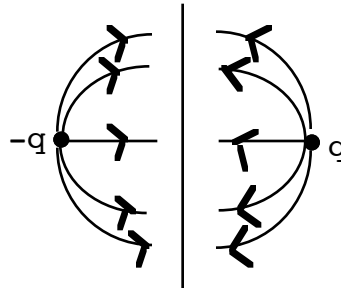
2.1 Method of images: flat conducting surface

Going to solve some specific problems in this chapter using: **1)** reduced Green function technique, **2)** method of images, **3)** expansion in orthogonal functions, and **4)** conformal mapping.

First problem: (Methods 1 & 2 illustrated)



is equivalent to



This says the Dirichlet Green function is just

$$G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}''|}, \quad (2.1)$$

($\vec{x}'$ = source pt.; $\vec{x}$ = field pt.) where

$$\vec{x}' = x'\hat{i} + y'\hat{j} + z'\hat{k} \quad (z' > 0)$$

2.2

locates the +unit charge and

$$\vec{x}'' = x' \hat{i} + y' \hat{j} - z' \hat{k}$$

locates the image charge. Notice that $G_D(\vec{x}, \vec{x}') \Big|_{z=0} = 0$ and that $G_D(\vec{x}, \vec{x}') = G_D(\vec{x}', \vec{x})$. We can write down the solution by inspection, but let us confirm it from a differential equation point of view. Want to solve

$$\nabla^2 G_D(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}'), \quad (2.2)$$

subject to

$$G_D(\vec{x}, \vec{x}') \Big|_{z=0} = 0. \quad (2.3)$$

First, however, just to get some experience, let us solve the easier problem of a point charge in free space from the diff. eqⁿ point of view. Obviously, the solution to $\nabla^2 G_f(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}')$ in free space is

$$G_f(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|}. \quad (2.4)$$

This can be derived from the diff. eqⁿ as follows using the representations,

$$\delta(\vec{x} - \vec{x}') = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}, \quad (2.5)$$

$$G_f(\vec{x}, \vec{x}') = \int \frac{d^3k}{(2\pi)^3} G(k) e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}. \quad (2.6)$$

($k = |\vec{k}|$) It is easy to see that $G(k) = \frac{4\pi}{k^2}$; the fact that

$G(k)$ depends only on $k = |\vec{k}|$ is due to the fact that the geometry is translationally invariant in x, y & z directions. Let's now do the integrals:

$$G_f = \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2} e^{ik \cdot (\vec{x} - \vec{x}') - k\varepsilon} \quad (\varepsilon > 0) \quad (2.7)$$

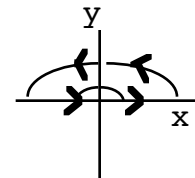
↑ convergence factor
(sometimes suppressed)

$$G_f = \frac{4\pi}{(2\pi)^3} \int d\phi \, d\cos\theta \frac{k^2 dk}{k^2} e^{ik|\vec{x} - \vec{x}'| \cos\theta} \quad (2.8)$$

$$G_f = \frac{8\pi^2}{(2\pi)^3} \int dk \int_{-1}^1 d\cos\theta e^{ik|\vec{x} - \vec{x}'| \cos\theta} \quad (2.9)$$

$$\underbrace{\int_{-1}^1 d\cos\theta e^{ik|\vec{x} - \vec{x}'| \cos\theta}}_{\frac{2}{k|\vec{x} - \vec{x}'|} \sin k|\vec{x} - \vec{x}'|}$$

$$\Rightarrow G_f = \frac{8\pi^2}{(2\pi)^3} \frac{2}{|\vec{x} - \vec{x}'|} \int_0^\infty dk \frac{\sin k|\vec{x} - \vec{x}'|}{k}. \quad (2.10)$$



↓ consider $\int dz \frac{e^{iz}}{z}$

$$\int_0^\infty dx \frac{\sin x}{x} = \frac{\pi}{2} \quad (\text{complex variables})$$

$$\Rightarrow G_f(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|}.$$

[Many other ways of evaluating this, including inserting a convergence factor, $e^{-k\varepsilon}$, in (2.10), doing the integral, and then taking $\varepsilon \rightarrow 0$ limit. Can't resist an alternate evaluation, which also uses a convergence factor earlier in the evaluation and inverts the order of the k and angular integrals:

$$G_f = \frac{4\pi}{(2\pi)^3} \int d\Omega \underbrace{\int_0^\infty dk e^{ik|\vec{x} - \vec{x}'| \cos\theta - k\varepsilon}}$$

(below)

$$\frac{1}{i|\vec{x} - \vec{x}'| \cos\theta - \varepsilon} e^{k|\vec{x} - \vec{x}'| \cos\theta - k\varepsilon} \Big|_0^\infty = \frac{-1}{i|\vec{x} - \vec{x}'| \cos\theta - \varepsilon}$$

$$\Rightarrow G_f = \frac{-4\pi}{(2\pi)^3} 2\pi \frac{1}{i|\vec{x} - \vec{x}'|} \int_{-1}^1 d\cos\theta \frac{1}{\cos\theta + i\varepsilon}$$

(ε has been re-scaled, but is still positive)

$$G_f = \frac{-1}{i\pi|\vec{x} - \vec{x}'|} \int_{-1}^1 d\cos\theta \frac{\cos\theta - i\varepsilon}{\cos^2\theta + \varepsilon^2}$$

$$\text{"Magic":} \quad \delta(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{x^2 + \varepsilon^2}$$

$$\Rightarrow G_f = \frac{1}{|\vec{x} - \vec{x}'|} \int_{-1}^1 dx \delta(x) \quad (x = \cos\theta)$$

$$= \frac{1}{|\vec{x} - \vec{x}'|} \cdot]$$

2.2 Reduced Green function technique applied to flat conductor

Now go back to our half-plane geometry. The geometry here is translationally invariant also, but only in the x and y directions. Therefore, assume

$$G_D(\vec{x}, \vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} g(z, z', k), \quad (2.11)$$

where

$$d^2k = dk_x dk_y,$$

$$\vec{k} \cdot (\vec{x} - \vec{x}')_{\perp} = k_x(x - x') + k_y(y - y'). \quad [\vec{k} = k_x \hat{i} + k_y \hat{j}].$$

Let's call $g(z, z', k)$ the "reduced Green function". We then have

$$\begin{aligned} -\nabla^2 G(\vec{x}, \vec{x}') &= 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')_{\perp}} [k^2 - \frac{\partial^2}{\partial z^2}] g(z, z', k), \\ &= 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')_{\perp}} \delta(z - z'), \end{aligned} \quad (2.12)$$

which implies

$$[k^2 - \frac{\partial^2}{\partial z^2}] g(z, z', k) = \delta(z - z'). \quad (2.13)$$

The BC on $G_D(\vec{x}, \vec{x}')$ now read

$$g(z, z', k) \Big|_{z=0} = 0. \quad (2.14)$$

Let's try to solve this simpler eqⁿ. For $z \neq z'$ we have

$$[k^2 - \frac{\partial^2}{\partial z^2}] g(z, z', k) = 0, \quad (2.15)$$

the solution for which is

$$g(z, z', k) = C(z') e^{\pm kz}. \quad (2.16)$$

Our solution must be bounded as $z \rightarrow \infty$, so

$$g(z, z', k) = \begin{cases} C_1 e^{-kz}, & z > z' \\ C_2 e^{kz} + C_3 e^{-kz}, & z < z'. \end{cases} \quad (2.17)$$

The above BC now gives

$$C_2 = -C_3. \quad (2.18)$$

The diff. eqⁿ implies a discontinuity in $g(z, z', k)$ at $z = z'$:

$$\int_{z'-}^{z'+} dz \left[k^2 - \frac{\partial^2}{\partial z^2} \right] g(z, z', k) = \int_{z'-}^{z'+} \delta(z-z') dz. \quad (2.19)$$

Since $g(z, z', k)$ is continuous, $g(z, z', k) \big|_{z'-}^{z'+} = 0$

as well as $-\frac{\partial}{\partial z} g(z, z', k) \big|_{z'-}^{z'+} = 1$ from the above. These imply

$$C_1 e^{-kz'} = C_2 (e^{kz'} - e^{-kz'}), \quad (\text{continuity of } g) \quad (2.20)$$

$$C_1 k e^{-kz'} + C_2 k (e^{kz'} + e^{-kz'}) = 1. \quad (\text{discontinuity of } \frac{\partial g}{\partial z}) \quad (2.21)$$

The solution is

$$C_1 = \frac{1}{2k} (e^{kz'} - e^{-kz'}), \quad (2.22)$$

$$C_2 = \frac{1}{2k} e^{-kz'}. \quad (2.23)$$

This means that

$$g(z, z', k) = \frac{1}{2k} \times \begin{cases} e^{k(z'-z)} - e^{-k(z'+z)}, & z > z' \\ e^{k(z-z')} - e^{-k(z'+z)}, & z < z'. \end{cases} \quad (2.24)$$

which we can also write as

$$g(z, z', k) = \frac{1}{2k} (e^{-k|z-z'|} - e^{-k|z+z'|}), \quad \forall z', z. \quad (2.25)$$

Therefore

$$G_D(\vec{x}, \vec{x}') = 4\pi \int \frac{d^2 k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \frac{1}{2k} (e^{-k|z-z'|} - e^{-k|z+z'|}). \quad (2.26)$$

We will actually do the integrals to verify this is a correct result. Brute force. Consider (first term above):

$$I \equiv \frac{4\pi}{2(2\pi)^2} \int_0^{2\pi} d\phi \int_0^\infty dk e^{ik\rho\cos\phi} e^{-k|z-z'|}$$

$$I = \frac{-1}{2\pi} \int_0^{2\pi} d\phi \frac{1}{i\rho\cos\phi - |z-z'|}$$

$$I = -\frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{-i\rho\cos\phi - |z-z'|}{\rho^2\cos^2\phi + (z-z')^2}$$

$$I = \frac{|z-z'|}{2\pi} \int_0^{2\pi} d\phi \frac{1}{\rho^2\cos^2\phi + (z-z')^2}$$

Look it up in a table of indefinite integrals (CRC, 22nd ed., #375, p.442). Get

$$I = \frac{|z-z'|}{2\pi} \left[\frac{1}{|z-z'| \sqrt{\rho^2 + (z-z')^2}} \tan^{-1} \left(\underbrace{\frac{|z-z'| \tan\phi}{\sqrt{\rho^2 + (z-z')^2}}}_{2\pi} \right) \right] \Big|_0^{2\pi}$$

$$= \frac{1}{\sqrt{\rho^2 + (z-z')^2}}.$$

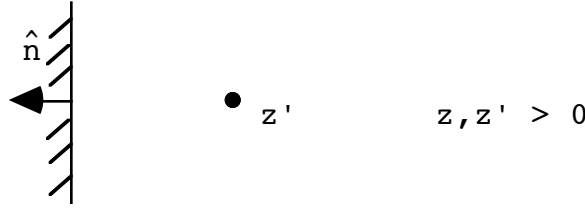
Similarly for the 2nd term. Replacing $\rho \rightarrow |(\vec{x}-\vec{x}')_\perp|$, we recover that

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}''|},$$

where $\vec{x}''$ has the same meaning as before. Explicitly

$$G(\vec{x}, \vec{x}') = \frac{1}{[(x-x')^2 + (y-y')^2 + \underbrace{(z-z')^2}_{\uparrow *}]^{1/2}} - \frac{1}{[(x-x')^2 + (y-y')^2 + \underbrace{(z+z')^2}_{\uparrow *}]^{1/2}} \quad (2.27)$$

We can confirm the presence of the surface delta function mentioned earlier. We have



$$\frac{\partial G}{\partial n'} \Big|_s = \frac{-\partial G}{\partial z'} \Big|_{z'=0} = - \left\{ \frac{\frac{1}{2} \cdot 2(z-z')}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} - \frac{\frac{-1}{2} \cdot 2(z+z')}{[(x-x')^2 + (y-y')^2 + (z+z')^2]^{3/2}} \right\} \Big|_{z'=0} \quad (2.28)$$

$$\Rightarrow \frac{\partial G}{\partial n'} \Big|_s = \frac{-2z}{[\rho^2 + z^2]^{3/2}} \quad [\rho^2 = (x-x')^2 + (y-y')^2] \quad (2.29)$$

Notice

$$\rho \neq 0, \quad z = 0, \quad \frac{\partial G}{\partial n'} \Big|_{z'=0} = 0. \quad (2.30)$$

$$\rho = 0, \quad z \rightarrow 0^+, \quad \frac{\partial G}{\partial n'} \Big|_{z'=0} = -\infty. \quad (2.31)$$

Also

$$\frac{1}{4\pi} \lim_{z \rightarrow 0^+} \int_s \downarrow dx' dy' \frac{\partial G}{\partial n'} = \frac{1}{4\pi} \lim_{z \rightarrow 0^+} \iint \rho d\rho d\phi \frac{-2z}{[\rho^2 + z^2]^{3/2}}, \quad (2.32)$$

(take origin at $x=y=0$)

$$= \lim_{z \rightarrow 0^+} \left(-\frac{z}{2} \int_0^\infty \frac{2\rho d\rho}{[\rho^2 + z^2]^{3/2}} \right), \quad (2.33)$$

$$= \lim_{z \rightarrow 0^+} \left(-\frac{z}{2} \left(\frac{1}{-1/2} \right) \frac{1}{\sqrt{\rho^2 + z^2}} \Big|_0^\infty \right) = \lim_{z \rightarrow 0^+} \left(-\frac{z}{|z|} \right) = -1, \quad (2.34)$$

$$\Rightarrow \lim_{z \rightarrow 0^+} \frac{\partial G}{\partial n'} \Big|_{z'=0} = -4\pi \delta^{(s)}(\vec{x}_\perp - \vec{x}'_\perp). \quad (2.35)$$

(See Eq.(1.89) of the last Chapter.) We see that the surface delta function arises from the image charge which has been placed on the surface for $z'=0$.

[We can also see the surface delta function from

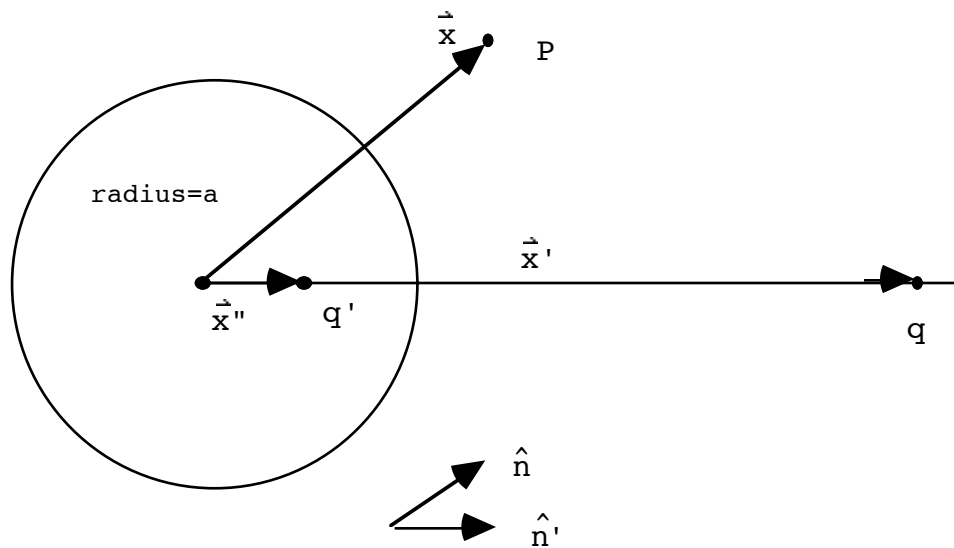
$$\begin{aligned}
 G_D(\vec{x}, \vec{x}') &= 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x}, \vec{x}')_{\perp}} \frac{1}{2k} (e^{-k|z-z'|} - e^{-k|z+z'|}) \\
 \Rightarrow \frac{\partial}{\partial z'} G_D(\vec{x}, \vec{x}') \Big|_{z'=0} &= 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x}, \vec{x}')_{\perp}} \frac{1}{2} (-e^{-kz} - e^{-kz}) \\
 \Rightarrow \lim_{z \rightarrow 0^+} \frac{1}{4\pi} \frac{\partial}{\partial n'} G_D(\vec{x}, \vec{x}') \Big|_{z'=0} &= - \underbrace{\int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')_{\perp}}}_{-\delta^{(s)}(\vec{x}_{\perp} - \vec{x}'_{\perp})} .]
 \end{aligned}$$

Of course, the total surface charge induced (when z' is not on the surface, say) is -1 because of the nature of our Green function. [Since $\sigma_{\text{surface}} = \frac{1}{4\pi} \frac{\partial G}{\partial n} \Big|_s$ we see that

$$\sigma_{\text{surface}} = \frac{-2z'}{4\pi[\rho^2 + z^2]^{3/2}} .]$$

2.3 Method of images: conducting sphere

Can do other problems with the method of images (usually need an infinite number of them, however). Point charge outside sphere:



Can find $q', \vec{x}$ as follows. Guess:

$$\Phi(\vec{x}) = \frac{q}{|\vec{x} - \vec{x}'|} + \frac{q'}{|\vec{x} - \vec{x}''|}, \quad (2.36)$$

or

$$\Phi(\vec{x}) = \frac{q}{|x\hat{n} - x'\hat{n}'|} + \frac{q'}{|x\hat{n} - x''\hat{n}'|}. \quad (2.37)$$

Now we want

$$\Phi(\vec{x}) \Big|_{x=a} = \frac{q}{a|\hat{n} - \frac{x'}{a}\hat{n}'|} + \frac{q'}{x''|\hat{n}' - \frac{a}{x''}\hat{n}|} = 0. \quad (2.38)$$

Notice

$$\frac{q}{a} = \frac{-q'}{x''}, \quad \frac{x'}{a} = \frac{a}{x''} \quad \text{works.}$$

$$\Rightarrow q' = \frac{-a}{x'}q, \quad x'' = \frac{a^2}{x'}, \quad (2.39)$$

$$\Rightarrow \Phi(\vec{x}) = \frac{q}{|\vec{x} - \vec{x}'|} - \frac{qa}{x'|\vec{x} - \frac{a^2}{x'^2}\vec{x}'|}. \quad (2.40)$$

Essentially the Green function for this geometry. More explicitly

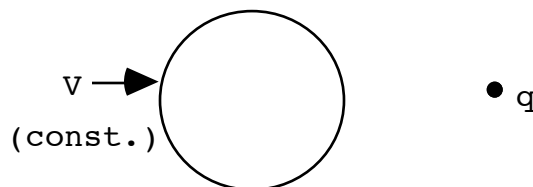
$$G_D(\vec{x}, \vec{x}') = \frac{1}{(x^2 + x'^2 - 2xx'\cos\gamma)^{1/2}} - \frac{1}{(\frac{x^2x'^2}{a^2} + a^2 - 2xx'\cos\gamma)^{1/2}}. \quad (2.41)$$

($\cos\gamma = \hat{n} \cdot \hat{n}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$) Can see explicitly that $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$ and can show that the gradient on the surface (inward normal to sphere) is given by

$$\left. \frac{\partial G}{\partial n'} \right|_{x'=a} = - \left. \frac{\partial G}{\partial x'} \right|_{x'=a} = - \frac{(x^2 - a^2)}{a(x^2 + a^2 - 2ax \cos\gamma)^{3/2}}. \quad (2.42)$$

Can also verify that as $x \rightarrow a^+$, $\left. \frac{\partial G}{\partial n'} \right|_s$ acts as a surface delta function. I am not going to do this now, but this problem can also be done using the reduced Green function approach. (Don't have the mathematical machinery yet - Ch.3!)

Now that we have the Green function, we can solve other problems for this geometry. In particular, consider:



The solution is

$$\Phi(\vec{x}) = \int d^3x' G_D(\vec{x}, \vec{x}') \underset{q\delta(\vec{x} - \vec{x}')}{\underset{\uparrow}{\rho(\vec{x}')}} \frac{-1}{4\pi} \oint_s da' \underset{\underset{\uparrow}{V}}{\Phi(\vec{x}') \frac{\partial G}{\partial n'}}, \quad (2.43)$$

so all we need is the surface integral,

$$\oint_s \left. \frac{\partial G}{\partial n'} \right|_{x'=a} da' = \frac{-a^2}{a} \int d\Omega' \frac{(x^2 - a^2)}{(x^2 + a^2 - 2ax \cos\gamma)^{3/2}}. \quad (2.44)$$

Take z' -axis along $\vec{x}$: ($d\Omega' = \sin\gamma d\gamma d\phi$)

$$\begin{aligned}
 \Rightarrow \oint_s \frac{\partial G}{\partial n'} \Big|_{x'=a} da' &= \frac{-a(x^2 - a^2)2\pi}{2ax} \int \frac{2ax \sin\gamma d\gamma}{(x^2 + a^2 - 2ax \cos\gamma)^{3/2}}, \\
 &= \frac{-(x^2 - a^2)}{2x} 2\pi \left(\frac{1}{-1/2} \right) \frac{1}{(x^2 + a^2 - 2ax \cos\gamma)^{1/2}} \Big|_0^\pi, \\
 &= (2\pi x^2 - a^2) \left\{ \frac{1}{|x+a|} - \frac{1}{|x-a|} \right\}, \\
 &\quad \uparrow \frac{1}{x-a} \text{ when } x > a, \\
 &= \frac{-4\pi a}{x} \quad (x = |\vec{x}|). \tag{2.45}
 \end{aligned}$$

Therefore

$$\Phi(\vec{x}) = q G_D(\vec{x}, \vec{x}') + \frac{Va}{|\vec{x}|}. \tag{2.46}$$

The extra term has a simple interpretation: due to surface charge on the sphere, uniformly distributed. The total charge on the surface is

$$Q = -q \frac{a}{|\vec{x}'|} + Va. \tag{2.47}$$

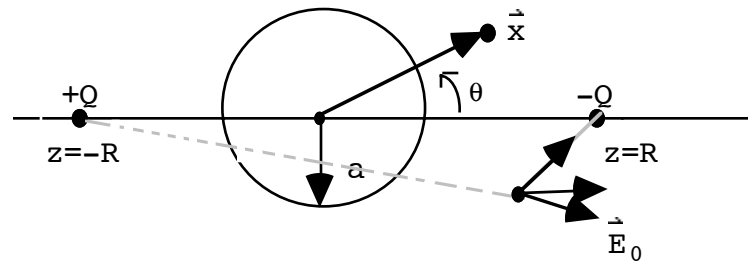
Q is determined because V is given. Turn this around. Specify $Q \Rightarrow V$ is determined.

$$\Rightarrow Va = Q + q \frac{a}{|\vec{x}'|}. \tag{2.48}$$

Gives solution with a given charge Q on the surface:

$$\Phi(\vec{x}) = q G_D(\vec{x}, \vec{x}') + \frac{\left(Q + q \frac{a}{|\vec{x}'|} \right)}{|\vec{x}|}. \tag{2.49}$$

Another situation:



Specify

$$\Phi|_s = 0, \quad (2.50)$$

$$\rho(\vec{x}) = Q \left[\delta(\vec{x} - (0, 0, -R)) - \delta(\vec{x} - (0, 0, R)) \right]. \quad (2.51)$$

Gives

$$\begin{aligned} \Phi(\vec{x}) = & \frac{Q}{(r^2 + R^2 + 2rR\cos\theta)^{1/2}} - \frac{aQ}{R(r^2 + a^4/R^2 + 2a^2r/R\cos\theta)^{1/2}} \\ & - \frac{Q}{(r^2 + R^2 - 2rR\cos\theta)^{1/2}} + \frac{aQ}{R(r^2 + a^4/R^2 - 2a^2r/R\cos\theta)^{1/2}}. \end{aligned} \quad (2.52)$$

Now

$$\frac{1}{(R^2 + r^2 \pm 2rR\cos\theta)^{1/2}} \approx_{R \gg r} \frac{1}{R} \left(1 \mp \frac{r}{R} \cos\theta \right), \quad (2.53)$$

$$\frac{1}{(r^2 + a^4/R^2 \pm 2a^2r/R\cos\theta)^{1/2}} \approx_{R \gg r} \frac{1}{r} \left(1 \mp \frac{a^2}{Rr} \cos\theta \right), \quad (2.54)$$

so

$$\Phi(\vec{x}) \approx -2Q/R^2 \left(r - \frac{a^3}{r^2} \right) \cos\theta. \quad (2.55)$$

Identify: $2Q/R^2 \rightarrow E_0$.

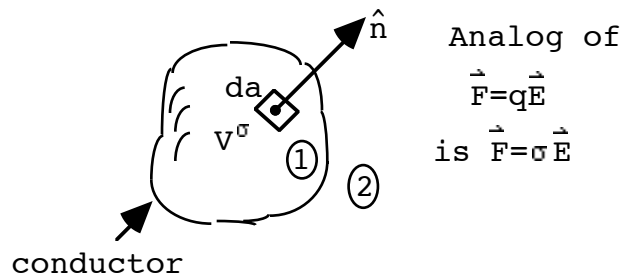
$$\Rightarrow \Phi(\vec{x}) = -E_0 \left(r - \frac{a^3}{r^2} \right) \cos\theta = -E_0 z + \frac{\vec{d} \cdot \vec{r}}{r^3},$$

$$\vec{d} = a^3 \vec{E}_0. \quad (2.56)$$

With this interpretation, result is exact. Of course, if we were to specify a potential on the surface, we would just get a term $\frac{Va}{|\vec{x}|}$ as before.

2.4 Normal force on a charged surface

We need the force/area on conductor surfaces.



("F" is now force/area). In this case

$$\vec{\mathcal{F}} = \sigma (\vec{E}_2^{\text{total}} - \vec{E}_2^{\text{disk}}). \quad (2.57)$$

Now we know that

$$\begin{aligned} (\vec{E}_2^{\text{total}} - \vec{E}_1^{\text{total}}) \cdot \hat{n} &= 4\pi\sigma, \\ \Rightarrow \vec{E}_{2n}^{\text{total}} \cdot \hat{n} &= 4\pi\sigma. \end{aligned} \quad (2.58)$$

Remember

$$E_{2n}^{\text{disk}} = - \int_{\substack{\vec{x} \\ (\text{flat})}} \sigma(\vec{x}) d\Omega, \quad (2.59)$$

$$\Rightarrow E_{2n}^{\text{disk}} = 2\pi\sigma \text{ (outside surface, } d\Omega < 0), \quad (2.60)$$

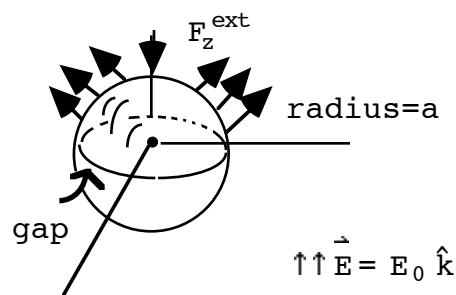
$$\Rightarrow \vec{F} \cdot \hat{n} = \sigma(4\pi\sigma - 2\pi\sigma) = 2\pi\sigma^2. \quad (2.61)$$

Notice $F_n > 0$, so the force is directed outward (likes repel). What has happened is just:

$$\text{cancel } \left\{ \begin{array}{l} \text{(remainder)} \\ \text{(disk)} \end{array} \right. \begin{array}{l} 2\pi\sigma \rightarrow \\ -2\pi\sigma \leftarrow \end{array} \left. \begin{array}{l} \nearrow \\ \searrow \end{array} \right) \begin{array}{l} \rightarrow 2\pi\sigma \text{ (remainder)} \\ \rightarrow 2\pi\sigma \text{ (disk)} \end{array} \right\} \text{add}$$

2.5 Charged conducting sphere force example

Now can do a problem using our result (2.56) above.
Picture:



We just found that

$$\Phi = -E_0 \left(r - \frac{a^3}{r^2} \right) \cos\theta. \quad (2.62)$$

Therefore the induced surface charge density is

$$\sigma = - \left. \frac{1}{4\pi} \frac{\partial \Phi}{\partial r} \right|_a = \frac{3}{4\pi} E_0 \cos\theta \quad (\text{total charge}=0). \quad (2.63)$$

Get total upward force by integrating over the upper surface:

$$F_z = 2\pi \int_{\text{upper}} \sigma^2 \cos\theta da = 2\pi \int_0^{\pi/2} \int_0^{2\pi} \frac{9}{16\pi^2} E_0^2 \cos^3\theta \sin\theta a^2 d\theta d\phi, \quad (2.64)$$

↑ by symmetry ↑!!

$$F_z = \frac{9(2\pi)^2}{16\pi^2} E_0^2 a^2 \int_0^{\pi/2} \cos^3 \theta \sin \theta d\theta, \quad (2.65)$$

$$\underbrace{\quad}_{-\frac{\cos^4 \theta}{4} \int_0^{\pi/2} = \frac{1}{4}}$$

$$\Downarrow \text{upper sphere} \\ F_z = \frac{9E_0^2 a^2}{16} \text{ (this is the force we must counteract).} \quad (2.66)$$

Now, imagine some total charge Q placed on the sphere before it is cut into two. Then, the external forces on the two pieces will no longer be equal in magnitude and opposite in direction. Now, the potential is

$$\Phi = \Phi_{\text{uncharged}} + \frac{Q}{r}, \quad (2.67)$$

since the charge Q will distribute itself uniformly over a spherical equipotential surface. This gives (magnitude only)

$$\sigma = \frac{3}{4\pi} E_0 \cos \theta + \frac{1}{4\pi} \frac{Q}{a^2}, \quad (2.68)$$

add in $[0, \pi/2]$

$$F_z^{\text{upper}} = (2\pi)^2 a^2 \int_0^{\pi/2} d\theta \sin \theta \cos \theta \left(\frac{3}{4\pi} E_0 \cos \theta + \frac{1}{4\pi} \frac{Q}{a^2} \right)^2, \quad (2.69)$$

$$F_z^{\text{lower}} = (2\pi)^2 a^2 \int_{\pi/2}^{\pi} d\theta \sin \theta \cos \theta \left(\frac{3}{4\pi} E_0 \cos \theta + \frac{1}{4\pi} \frac{Q}{a^2} \right)^2. \quad (2.70)$$

$\underbrace{\quad}_{\text{subtract in } [\pi/2, \pi]}$

(the "add in..." and "subtract in..." comments are true if E_0 and Q have the same sign.) Get

$$F_z^U = \frac{9E_0^2 a^2}{16} + \frac{1}{2} E_0 Q + \frac{Q^2}{8a^2}, \quad (2.71)$$

$$F_z^D = -\frac{9E_0^2 a^2}{16} + \frac{1}{2} E_0 Q - \frac{Q^2}{8a^2}. \quad (2.72)$$

Notice

$$F_z^U + F_z^D = E_0 Q \quad (\text{as it should}), \quad (2.73)$$

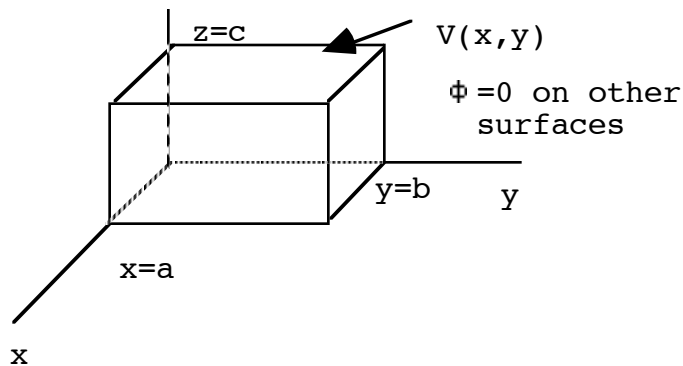
$$\frac{F_z^U - F_z^D}{2} = \frac{9E_0^2 a^2}{16} + \frac{Q^2}{8a^2}. \quad (2.74)$$

↑

This is the force that must be exerted on each sphere to keep the spheres together, but they still accelerate in the $\vec{E}$ field.

2.6 Expansion in orthogonal series: conducting box

Next problem (new technique):



Find the inside solution. Since there is no charge in V , need to solve

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (2.75)$$

Assume

$$\phi(x,y,z) = X(x) Y(y) Z(z). \quad (2.76)$$

Substituting above yields (divide through by $\phi=XYZ$)

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0. \quad (2.77)$$

(Notice total derivatives have replaced partials). Must be that

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\alpha^2, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = -\beta^2, \quad \frac{1}{Z} \frac{d^2 Z}{dz^2} = \alpha^2 + \beta^2. \quad (2.78)$$

Nothing to determine these constants yet. Keep our minds open as to their values (α, β can be real or imag. at this point). Solution will be unique $\Rightarrow$ BC's will tell us their values. Assuming $\alpha, \beta \neq 0$:

$$X(x) \sim C_1 \sin \alpha x + C_2 \cos \alpha x, \quad (2.79)$$

$$Y(y) \sim C_3 \sin \beta y + C_4 \cos \beta y, \quad (2.80)$$

$$Z(z) \sim C_5 \sinh \gamma z + C_6 \cosh \gamma z, \quad (2.81)$$

$$\gamma \equiv \sqrt{\alpha^2 + \beta^2}.$$

BC's:

$$\phi \Big|_{x=0} = 0 \Rightarrow C_2 = 0, \quad (2.82)$$

$$\phi \Big|_{x=a} = 0 \Rightarrow \alpha = \frac{n\pi}{a}, \quad n = 1, 2, 3, \dots \quad (2.83)$$

$\Downarrow$

$$\phi \Big|_{y=0} = 0 \Rightarrow C_4 = 0, \quad \gamma = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \quad (2.84)$$

$\Uparrow$

$$\phi \Big|_{y=b} = 0 \Rightarrow \beta = \frac{m\pi}{b}, \quad m = 1, 2, 3, \dots \quad (2.85)$$

Must also have

$$\phi \Big|_{z=0} = 0 \Rightarrow C_6 = 0. \quad (2.86)$$

Most general solution:

$$\phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{mn} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{mn} z). \quad (2.87)$$

Last BC:

$$V(x,y) = \sum_{n,m} A_{mn} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{mn} c). \quad (2.88)$$

Use orthogonality to invert this ($n, n' \geq 1$)

$$\int_{-a}^a dx \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) = a\delta_{nn'}, \quad (2.89)$$

$$\int_{-a}^a dx \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n'\pi x}{a}\right) = a\delta_{nn'}. \quad (2.90)$$

[Can get these by using

$$\sin x \sin y = -\frac{1}{2} (\cos(x+y) - \cos(x-y))$$

$$\cos x \cos y = \frac{1}{2} (\cos(x+y) + \cos(x-y)).]$$

$$\Rightarrow A_{mn} = \frac{4}{ab \sinh(\gamma_{mn} c)} \int_0^a dx \int_0^b dy V(x,y) \sin(\alpha_n x) \sin(\beta_m y). \quad (2.91)$$

2.7 Fourier series problems: conducting box

Another way to solve this problem: reduced G.F. method. However, we have to proceed differently from before since we no longer have translational invariance. It really combines the image, reduced G.F. and Fourier series techniques. Start with fundamental theorem for Fourier series ($f(x)$ piecewise continuous in $-a \leq x \leq a$):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{a}\right) + b_n \sin\left(\frac{n\pi x}{a}\right) \right), \quad (2.92)$$

where (from orthogonality)

$$a_n = \frac{1}{a} \int_{-a}^a f(x) \cos\left(\frac{\pi n x}{a}\right) dx, \quad (2.93)$$

$$b_n = \frac{1}{a} \int_{-a}^a f(x) \sin\left(\frac{\pi n x}{a}\right) dx. \quad (2.94)$$

Now consider

$$Q(x, x') \equiv \frac{1}{a} \sum_{n=1}^{\infty} \left[\sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n x'}{a}\right) \right]. \quad (2.95)$$

We have that

$$\begin{aligned} \int_{-a}^a dx f(x) Q(x, x') &= \int_{-a}^a dx \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{a}\right) Q(x, x') \\ &= \frac{1}{a} \sum_{n, n'} b_n \sin\left(\frac{\pi n' x'}{a}\right) \int_{-a}^a dx \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n' x}{a}\right) \\ &= \sum_n b_n \sin\left(\frac{\pi n x'}{a}\right) = \frac{f(x) - f(-x)}{2}. \end{aligned} \quad (2.96)$$

↑ Since this is the odd part of $f(x)$

Since this holds for any $f(x)$

$$\Rightarrow Q(x, x') = \frac{1}{2} [\delta(x-x') - \delta(x+x')] , \quad -a \leq x, x' \leq a. \quad (2.97)$$

Likewise, for

$$P(x, x') = \frac{1}{a} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi x'}{a}\right) \right], \quad (2.98)$$

one can show

$$\int_{-a}^a f(x) P(x, x') dx = \frac{f(x) + f(-x)}{2}, \quad (2.99)$$

$$\Rightarrow P(x, x') = \frac{1}{2} [\delta(x-x') + \delta(x+x')] , \quad -a \leq x, x' \leq a. \quad (2.100)$$

Explicit BC on $G(\vec{x}, \vec{x}')$:

$$G_D(\vec{x}, \vec{x}') \Big|_{x=0,a} = 0, \quad (2.101)$$

$$G_D(\vec{x}, \vec{x}') \Big|_{y=0,b} = 0, \quad (2.102)$$

$$G_D(\vec{x}, \vec{x}') \Big|_{z=0,c} = 0. \quad (2.103)$$

My claim: Solving

$$\nabla^2 G_D(\vec{x}, \vec{x}') = -4\pi[\delta(x-x') - \delta(x+x')] [\delta(y-y') - \delta(y+y')] \delta(z-z') \quad (2.104)$$

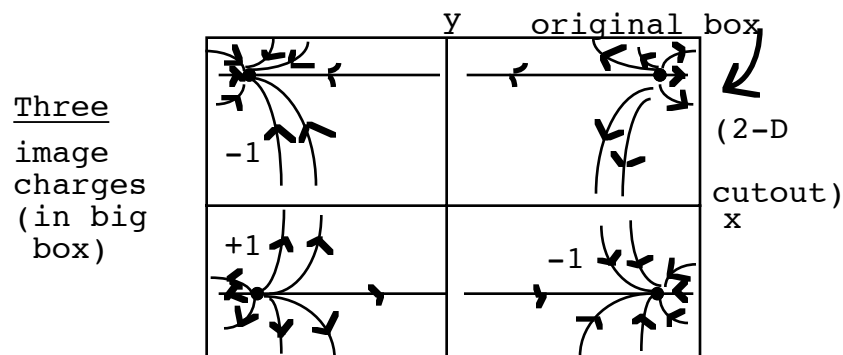
$$\text{for } G_D \Big|_{\text{big box}} = 0 \text{ in } \begin{aligned} &-a \leq x, x' \leq a \\ &-b \leq y, y' \leq b \\ &0 \leq z, z' \leq c \end{aligned}$$

gives the same solution in the small box as

$$\nabla^2 G_D(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}'). \quad (2.105)$$

$$\text{for } G_D \Big|_{\text{small box}} = 0 \text{ in } \begin{aligned} &0 \leq x, x' \leq a \\ &0 \leq y, y' \leq b \\ &0 \leq z, z' \leq c \end{aligned}$$

Point: Fourier series can only represent periodic functions.



By the way, this means we can formally write down the form of G_D here as (an infinite number of them, however)

$$G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}''|} + \frac{1}{|\vec{x} - \vec{x}'''|} - \frac{1}{|\vec{x} - \vec{x}'^{IV}|} + \dots \quad (2.106)$$

where

↑
rest of periodic images
in x, y, and z.

$$\begin{aligned}\vec{x}' &= (x', y', z'), \\ \vec{x}'' &= (x', -y', z'), \\ \vec{x}''' &= (-x', -y', z'), \\ \vec{x}'^{IV} &= (-x', y', z'), \text{ etc.}\end{aligned}$$

Since $G_D(\vec{x}, \vec{x}')$ is now an odd function in x' and y' , let's expand (also $G_D(\vec{x}, \vec{x}') = G_D(\vec{x}', \vec{x})$)

$$G_D(\vec{x}, \vec{x}') = \frac{16\pi}{ab} \sum_{n,m=1} g_{nm}(z, z') \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \times \sin\left(\frac{m\pi y'}{b}\right). \quad (2.107)$$

Then

$$\begin{aligned}-\nabla^2 G_D(\vec{x}, \vec{x}') &= \frac{16\pi}{ab} \sum_{n,m=1} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right) \\ &\quad \times \left[\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} - \frac{\partial^2}{\partial z^2} \right] g_{nm}(z, z'). \quad (2.108)\end{aligned}$$

On the other hand

$$\begin{aligned}4\pi[\delta(x-x') - \delta(x+x')] [\delta(y-y') - \delta(y+y')] \delta(z-z') \\ = \frac{16\pi}{ab} \sum_{n',m'=1} \sin\left(\frac{n'\pi x}{a}\right) \sin\left(\frac{n'\pi x'}{a}\right) \sin\left(\frac{m'\pi y}{b}\right) \sin\left(\frac{m'\pi y'}{b}\right) \delta(z-z'). \quad (2.109)\end{aligned}$$

Must be that

$$\left[\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} - \frac{\partial^2}{\partial z^2} \right] g_{nm}(z, z') = \delta(z-z'). \quad (2.110)$$

As before, we can solve for $g_{nm}(z, z')$. Solutions:

$$g_{nm}(z, z') = \begin{cases} C_1(z') \sinh \gamma_{nm} z, & z \leq z' \\ C_2(z') \sinh \gamma_{nm} z + C_3(z') \cosh \gamma_{nm} z, & z \geq z' \end{cases} \quad (2.111)$$

where $\gamma_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$. The above diff. eqⁿ and BC's require

$$g_{nm}(z, z') \Big|_{z'=c}^{z'^{+}} = 0, \quad (2.112)$$

$$-\frac{\partial}{\partial z} g_{nm}(z, z') \Big|_{z'=c}^{z'^{+}} = 1. \quad (2.113)$$

$$g_{nm}(z, z') \Big|_{z=c} = 0. \quad (2.114)$$

Solution is

$$g_{nm}(z, z') = \frac{\sinh(\gamma_{nm} z_{<}) \sinh[\gamma_{nm}(c - z_{>})]}{\gamma \sinh \gamma c}, \quad (2.115)$$

where $z_{<}$ is the lesser of z, z' ; $z_{>}$ is the greater of z, z' .

Full Green function for the box:

$$G_D(\vec{x}, \vec{x}') = \frac{16\pi}{ab} \sum_{n,m} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right) \\ \times \left[\frac{\sinh(\gamma_{nm} z_{<}) \sinh[\gamma_{nm}(c - z_{>})]}{\gamma \sinh \gamma c} \right]. \quad (2.115)$$

Now recover result for $\phi \Big|_{z=c} = V(x, y)$:

$$\phi(\vec{x}) = -\frac{1}{4\pi} \oint_S da' \phi(\vec{x}') \frac{\partial G}{\partial n'} \\ = -\frac{1}{4\pi} \int dx' dy' V(x', y') \frac{\partial G}{\partial n'} \Big|_{z'=c}. \quad (2.116)$$

For $z' > z$:

$$\frac{\partial g_{nm}(z, z')}{\partial z'} \Big|_{z'=c} = -\frac{\sinh \gamma_{nm} z}{\sinh \gamma_{nm} c}, \quad (2.118)$$

so, as before

$$\phi(\vec{x}) = \sum_{n,m=1}^{\infty} \frac{4}{ab} \frac{\sinh \gamma_{nm} z}{\sinh \gamma_{nm} c} \int_0^a dx' \int_0^b dy' V(x', y') \sin\left(\frac{n\pi x'}{a}\right) \times \sin\left(\frac{m\pi y'}{b}\right) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right). \quad (2.119)$$

2.8 Separation of variables in cylindrical coordinates

Many, many ways of solving 2-D electrostatics problems with simple geometry.

First, some background: Separation of variables for ρ, ϕ . Want to solve Laplace eqⁿ. In polar coordinates it reads

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0. \quad (2.120)$$

Assume

$$\Phi(\rho, \phi) = R(\rho)\psi(\phi). \quad (2.121)$$

Substitute & multiply by $\frac{\rho^2}{\Phi}$:

$$\frac{\rho}{R} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + \frac{1}{\psi} \frac{d^2 \psi}{d\phi^2} = 0. \quad (2.122)$$

Must have

$$\frac{\rho}{R} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) = v^2, \quad (2.123)$$

$$\frac{1}{\psi} \frac{d^2 \psi}{d\phi^2} = -v^2. \quad (2.124)$$

$$v \neq 0 \Rightarrow R(\rho) = a\rho^v + b\rho^{-v}, \quad \psi(\phi) = A\cos v\phi + B\sin v\phi, \quad (2.125)$$

$$v = 0 \Rightarrow R(\rho) = a_0 + b_0 \ln \frac{\rho}{\rho_0}, \quad \psi(\phi) = A_0 + B_0\phi. \quad (2.126)$$

When solving for a problem where the full azimuthal range of ϕ is allowed, must have ($\phi = 0, 2\pi$ are the same)

$$v \neq 0: \sin(0) = \sin(2\pi v) \Rightarrow v = \pm 1, \pm 2, \dots, \quad (2.127)$$

$$v = 0: B_0 \cdot 0 = B_0 \cdot 2\pi \Rightarrow B_0 = 0. \quad (2.128)$$

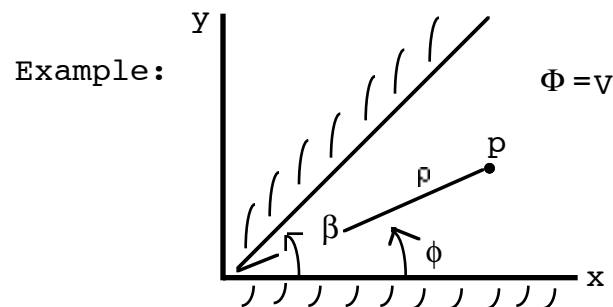
Full solution for this case

$$\begin{aligned} \Phi(\rho, \phi) = a_0 + b_0 \ln \frac{\rho}{\rho_0} + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) \\ + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n). \end{aligned} \quad (2.129)$$

Becomes a Fourier series problem.

2.9 Corner problem in cylindrical coordinates

Solve the corner problem:



$\rho=0$ is included $\Rightarrow b_n=0$. Want

$$\Phi \Big|_{\phi=0, \beta} = V \Rightarrow \psi \Big|_{\phi=0, \beta}^{v \neq 0} = 0, \quad \psi \Big|_{\phi=0, \beta}^{v=0} = V. \quad (2.130)$$

$$v \neq 0: A \cos(0) + B \sin(0) = 0, \quad (\phi=0) \quad (2.131)$$

$$\Rightarrow A = 0.$$

$$B \sin v \beta = 0, \quad (\phi = \beta)$$

$$\Rightarrow v = \frac{m\pi}{\beta}, \quad m=1, 2, 3, \dots \quad (2.132)$$

$$v=0: \quad a_0 (A_0 + B_0 \phi) = V, \\ \uparrow_{0,\beta}$$

$$\Rightarrow B_0 = 0, \quad a_0 A_0 = V \quad (2.133)$$

General solution is

$$\Phi(\rho, \phi) = V + \sum_{m=1}^{\infty} a_m \rho^{m\pi/\beta} \sin\left(\frac{m\pi\phi}{\beta}\right). \quad (2.134)$$

↓ undetermined

Notice that if the surfaces have $\Phi=V$ for $\rho \rightarrow \infty$, then the solution degenerates to $\Phi=V$ everywhere. Near $\rho=0$, the term with the lowest power will dominate. Assuming $a_1 \neq 0$,

$$\Phi(\rho, \phi) \approx V + a_1 \rho^{\pi/\beta} \sin\left(\frac{\pi\phi}{\beta}\right). \quad (2.135)$$

Electric fields are

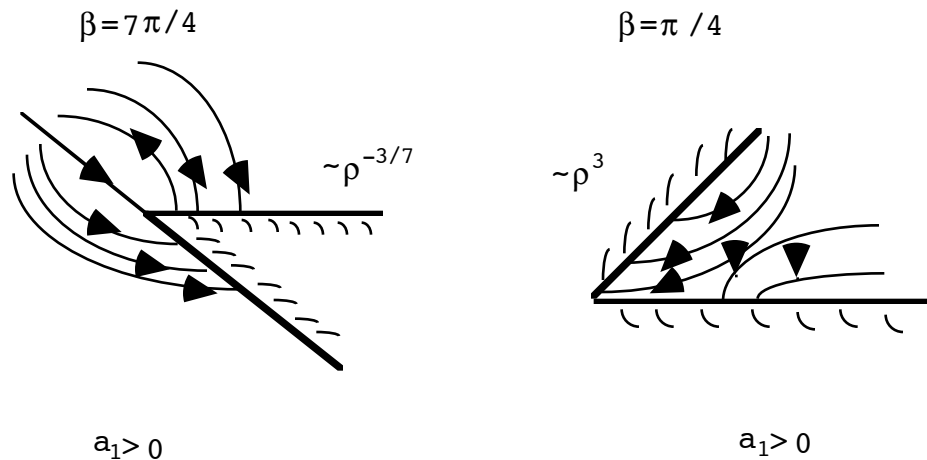
$$E_\rho = - \frac{\partial \Phi}{\partial \rho} \approx - \frac{\pi a_1}{\beta} \rho^{\pi/\beta-1} \sin\left(\frac{\pi\phi}{\beta}\right), \quad (2.136)$$

$$E_\phi = - \frac{1}{\rho} \frac{\partial \Phi}{\partial \phi} \approx - \frac{\pi a_1}{\beta} \rho^{\pi/\beta-1} \cos\left(\frac{\pi\phi}{\beta}\right). \quad (2.137)$$

Notice that

$$\sqrt{E_\rho^2 + E_\phi^2} \approx \frac{\pi |a_1|}{\beta} \rho^{\pi/\beta-1}. \quad (2.138)$$

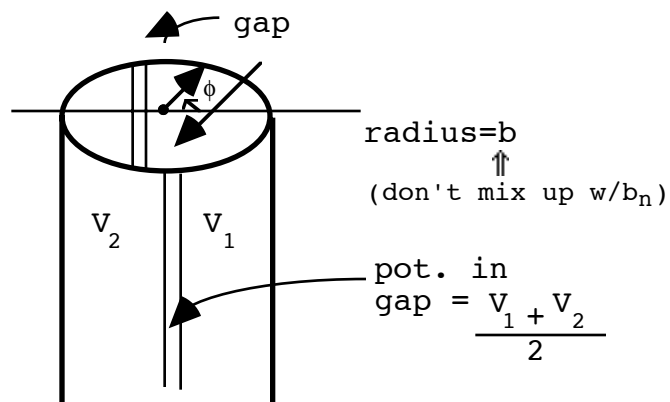
It looks like:



The Green functions for this geometry can be found, but I will resist this temptation until the next chapter.

2.10 Cylindrical halves at different potentials

Do this problem as an example:



This is a standard BV problem; however, the resulting Fourier series can be summed exactly. Start off:

$$\Phi = a_0 + b_0 \ln \rho / \rho_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n) \quad (2.139)$$

Origin is included $\Rightarrow b_n = 0$, all n . By integrating around the perimeter, we can immediately establish that a_0 represents the average value of the potential on the surface, so

$$a_0 = \frac{V_1 + V_2}{2}. \quad (2.140)$$

The BC are:

$$\Phi(b, \phi = -\frac{\pi}{2} < \phi < \frac{\pi}{2}) = V_1, \quad (2.141)$$

$$\Phi(b, \phi = \frac{\pi}{2} < \phi < \frac{3\pi}{2}) = V_2. \quad (2.142)$$

There are two sets of constants, the a_n and α_n , that we need to evaluate. The standard, long, way of doing this would be to

rewrite the $\sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n)$ term in (2.139) as

$$\sum_{n=1}^{\infty} (a_n \rho^n \sin(n\phi) + a'_n \rho^n \cos(n\phi))$$

and evaluate the two sets of constants, a_n and a'_n using (2.141) and (2.142). However, whenever we have a symmetry in the problem, we should exploit it to simplify the work. We note that the function $\left(\Phi - \frac{V_1 + V_2}{2}\right)$ on the inner surface is odd in the variable $\phi + \frac{\pi}{2}$ about $\phi = -\frac{\pi}{2}$. The symmetry of the problem implies that it is odd everywhere in the interior. Introducing $\phi' = \phi + \frac{\pi}{2}$, we have

$$\left(\Phi - \frac{V_1 + V_2}{2}\right) = \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi' - \frac{n\pi}{2} + \alpha_n). \quad (2.143)$$

This means

$$\alpha_n = \frac{n\pi}{2}. \quad (2.144)$$

Thus,

$$(2.141) \Rightarrow V_1 = \frac{V_1 + V_2}{2} + \sum_{n=1} a_n b^n \sin n(\phi + \frac{\pi}{2}), \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \quad (2.145)$$

This is a Fourier series problem for a_n :

$$\begin{aligned} \Rightarrow a_n b^n &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \left(\frac{V_1 - V_2}{2} \right) \sin n(\phi + \frac{\pi}{2}) d\phi, \\ \Rightarrow a_n b^n &= \frac{2}{\pi} \left[\frac{V_1 - V_2}{2} \right] \left[-\frac{\cos n\pi}{n} + \frac{1}{n} \right], \\ \Rightarrow a_n &= \begin{cases} 0, & n \text{ even} \\ b^{-n} \frac{2(V_1 - V_2)}{\pi n}, & n \text{ odd} \end{cases}. \end{aligned} \quad (2.146)$$

Check by using (2.142):

$$(2.142) \Rightarrow V_2 = \frac{V_1 + V_2}{2} + \sum_{n=1} a_n b^n \sin n(\phi + \frac{\pi}{2}), \quad \frac{\pi}{2} < \phi < \frac{3\pi}{2}. \quad (2.147)$$

$$\Rightarrow a_n b^n = -\frac{2}{\pi} \int_{\pi/2}^{3\pi/2} \left(\frac{V_1 - V_2}{2} \right) \sin n(\phi + \frac{\pi}{2}) d\phi, \quad (2.148)$$

$$\Rightarrow a_n = \begin{cases} 0, & n \text{ even} \\ b^{-n} \frac{2(V_1 - V_2)}{\pi n}, & n \text{ odd} \end{cases}. \quad (2.149)$$

(Same as before)

$$\Rightarrow \Phi = \frac{V_1 + V_2}{2} + \frac{2(V_1 - V_2)}{\pi} \sum_{n \text{ odd}} \left(\frac{\rho}{b} \right)^n \frac{\sin[n(\phi + \pi/2)]}{n}. \quad (2.150)$$

Now show it can be summed. Let

$$Z = \frac{\rho}{b} e^{i(\phi + \pi/2)}, \quad (2.151)$$

$$\text{Im } Z = \frac{\rho}{b} \sin(\phi + \pi/2) = \frac{\rho}{b} \cos \phi, \quad (2.152)$$

$$\Phi = \frac{V_1 + V_2}{2} + \frac{2(V_1 - V_2)}{\pi} \text{Im} \sum_{n \text{ odd}} \frac{Z^n}{n}. \quad (2.153)$$

By comparing infinite series representations and using properties of the log,

$$\sum_{n \text{ odd}} \frac{z^n}{n} = \frac{1}{2} \ln \left(\frac{1+z}{1-z} \right), \quad (2.154)$$

$$\Rightarrow \Phi = \frac{V_1 + V_2}{2} + \frac{2(V_1 - V_2)}{\pi} \frac{1}{2} \operatorname{Im} \left[\ln \left(\frac{1+z}{1-z} \right) \right]. \quad (2.155)$$

However

$$\operatorname{Im} [\ln(z')] = \tan^{-1} \left(\frac{\operatorname{Im} z'}{\operatorname{Re} z'} \right), \quad (2.156)$$

and

$$\left(\frac{1+z}{1-z} \right) = \frac{1 - |z|^2 + 2i \operatorname{Im} z}{|1 - z|^2}. \quad (2.157)$$

$$\Rightarrow \operatorname{Re} \left(\frac{1+z}{1-z} \right) = \frac{1 - |z|^2}{|1 - z|^2}, \quad \operatorname{Im} \left(\frac{1+z}{1-z} \right) = \frac{2 \operatorname{Im} z}{|1 - z|^2}. \quad (2.158)$$

$$\Rightarrow \Phi(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1} \left(\frac{2\rho/b \cos \phi}{1 - (\rho/b)^2} \right), \quad (2.159)$$

$$\Rightarrow \sigma = \frac{1}{4\pi} \frac{\partial \Phi}{\partial \rho} \Big|_{\rho=b} = \frac{V_1 - V_2}{4\pi^2} \frac{1}{b \cos \phi}. \quad (2.160)$$

2.11 Conformal mapping techniques

Another important technique is conformal mapping. Conformal mapping techniques are good for (essentially) 2-D problems with Dirichlet or Neumann BC's.

$$z = x + iy, \quad (2.161)$$

$$f(z) = u(x, y) + iv(x, y). \quad (2.162)$$

Cauchy:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad (2.163)$$

$$\Rightarrow \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} \right) = 0. \quad (2.164)$$

$$\nabla^2 v = 0 \text{ also.} \quad (2.165)$$

Also notice, if

$$\vec{E} = -\vec{\nabla} u, \quad (2.166)$$

$$\vec{\nabla} u \cdot \vec{\nabla} v = \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = -\frac{\partial u \partial u}{\partial x \partial y} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} = 0. \quad (2.167)$$

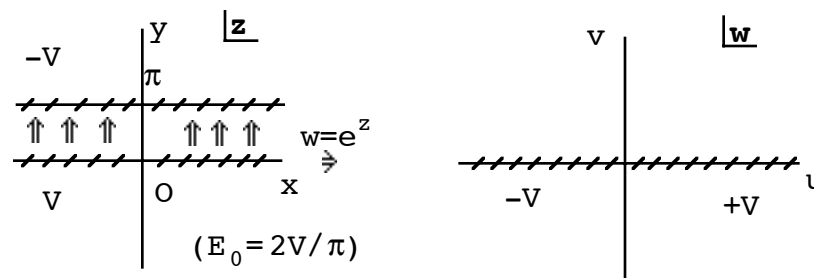
so lines of $v = \text{const.}$ lay along electric field lines. Basic theorem (Churchill, "Complex Variables and Applications", 3rd Ed., p.206):

Suppose that the analytic function $f(z) = u(x,y) + iv(x,y)$ maps an arc C in the z plane onto an arc Γ in the w plane. Let $f(z)$ be conformal ($f(z)$ analytic and $f'(z) \neq 0$ at z .) on C and let a function $h(u,v)$ be differentiable on Γ . If the function $h(u,v)$ satisfies either of the conditions

$$h = c \text{ or } \frac{dh}{dn} = 0 \quad (\text{not } \frac{dh}{dn} = c)$$

along Γ , then $H(x,y) = h[u(x,y), v(x,y)]$ satisfies the corresponding condition along C .

Do an example. Here is a mapping:



$$f(z) = iE_0 z + V, \quad E_0, V > 0. \quad (2.168)$$

We see that

$$u(x,y) = -E_0 y + V, \quad (2.169)$$

$$\Rightarrow E_x = 0, E_y = -\nabla_y u = E_0. \quad (2.170)$$

on the other hand

$$\begin{aligned} f[z(w)] &= iE_0 \ln w + V, \\ &= iE_0 \ln(u+iv) + V, \\ &= iE_0 \left[\ln \sqrt{u^2+v^2} + i \tan^{-1}(v/u) \right] + V. \end{aligned} \quad (2.171)$$

$\underbrace{\hspace{10em}}$
 use principle
 branch, if you
 wish

Identify

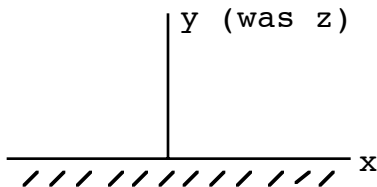
$$\Phi(u,v) = -E_0 \tan^{-1}(v/u) + V. \quad (2.172)$$

$$\Rightarrow E_u = -\frac{\partial \Phi}{\partial u} = -\frac{E_0 v}{u^2+v^2}, \quad (2.173)$$

$$E_v = -\frac{\partial \Phi}{\partial v} = \frac{E_0 u}{u^2+v^2}. \quad (2.174)$$

Notice
$$E_u^2 + E_v^2 = \frac{E_0^2}{\rho^2}, \rho^2 = u^2+v^2. \quad (2.175)$$

There are many ways of confirming this result. One way is to use our Green function for the geometry:



That Green function was

$$G_D(\vec{x}, \vec{x}') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y+y')^2 + (z-z')^2}},$$

Then

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \oint_s da' \Phi(\vec{x}') - \frac{\partial G}{\partial n'}, \quad (2.176)$$

$$\frac{\partial G}{\partial n'} \Big|_s = -\frac{\partial G}{\partial y'} \Big|_{y'=0} = \frac{-2y}{\left((x-x')^2 + y^2 + (z-z')^2\right)^{3/2}}. \quad (2.177)$$

$$\begin{aligned} \Phi(\vec{x}) = & -\frac{yV}{2\pi} \left\{ \int_{-\infty}^0 dx' \int_{-\infty}^{\infty} dz' \frac{1}{\left((x-x')^2 + y^2 + (z-z')^2\right)^{3/2}} \right. \\ & \left. - \int_0^{\infty} dx' \int_{-\infty}^{\infty} dz' \frac{1}{\left((x-x')^2 + y^2 + (z-z')^2\right)^{3/2}} \right\} \end{aligned} \quad (2.178)$$

Now

$$\begin{aligned} \int_{-\infty}^{\infty} dz' \frac{1}{\left((x-x')^2 + y^2 + (z-z')^2\right)^{3/2}} &= \int_{-\infty}^{\infty} dz'' \frac{1}{\left((x-x')^2 + y^2 + z''^2\right)^{3/2}} \\ &= \left(\frac{z''}{(x-x')^2 + y^2} \right) \frac{1}{\sqrt{(x-x')^2 + y^2 + z''^2}} \Big|_{-\infty}^{\infty} = \left(\frac{2}{(x-x')^2 + y^2} \right). \end{aligned} \quad (2.179)$$

We also have

$$\begin{aligned} \int_{-\infty}^0 dx' \left(\frac{1}{(x-x')^2 + y^2} \right) &= -\int_{-\infty}^x dx'' \frac{1}{x''^2 + y^2} \\ &= -\frac{1}{y} \tan^{-1} \frac{x}{y} \Big|_{-\infty}^x = -\frac{1}{y} \left(\tan^{-1} \left(\frac{x}{y} \right) - \pi/2 \right). \end{aligned} \quad (2.180)$$

Likewise

$$\int_0^{\infty} dx' \left(\frac{1}{(x-x')^2 + y^2} \right) = -\frac{1}{y} \left(-\pi/2 - \tan^{-1} \left(\frac{x}{y} \right) \right), \quad (2.181)$$

$$\begin{aligned} \Rightarrow \Phi(\vec{x}) &= \frac{2V}{\pi} \tan^{-1} \left(\frac{x}{y} \right) = \frac{2V}{\pi} \left(\pi/2 - \tan^{-1} \left(\frac{y}{x} \right) \right) \\ &= V - \frac{2V}{\pi} \tan^{-1} \left(\frac{y}{x} \right). \end{aligned} \quad (2.182)$$

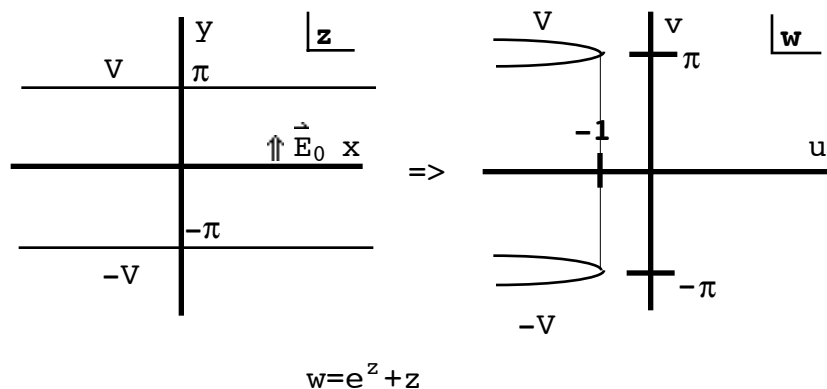
Same result as before if we realize

$$y \Leftrightarrow v, \quad (2.183)$$

$$x \Leftrightarrow u, \quad (2.184)$$

$$E_0 \Leftrightarrow \frac{2V}{\pi}. \quad (2.185)$$

Conformal methods can also give us some very nontrivial answers. Example:



$$f(z) = i \frac{V}{\pi} z, \quad \text{Re} (f(z)) = - \frac{V}{\pi} y. \quad (2.186)$$

Complicated in general, but let's just consider the electric field along $v=0$ ($v=0 \Leftrightarrow y=0$ here)

$$u = e^x \cos y + x, \quad (2.187)$$

$$v = e^x \sin y + y. \quad (2.188)$$

Must try to express $x(u,v)$ and $y(u,v)$. Now

$$E_v \Big|_{v=0} = - \frac{\partial}{\partial v} \left(- \frac{V}{\pi} y(u,v) \right) \Big|_{v=0} = \frac{V}{\pi} \frac{\partial y}{\partial v} \Big|_{v=0}. \quad (2.189)$$

$$(2.187) \Rightarrow 0 = e^x \cos y \frac{\partial x}{\partial v} - e^x \sin y \frac{\partial y}{\partial v} + \frac{\partial x}{\partial v}, \quad (2.190)$$

$$(2.188) \Rightarrow 1 = e^x \sin y \frac{\partial x}{\partial v} + e^x \cos y \frac{\partial y}{\partial v} + \frac{\partial y}{\partial v}. \quad (2.191)$$

For $y=0$ the second eqⁿ becomes

$$1 = e^x \left. \frac{\partial y}{\partial v} \right|_{y=0} + \left. \frac{\partial y}{\partial v} \right|_{y=0}, \quad (2.192)$$

$$\Rightarrow \left. \frac{\partial y}{\partial v} \right|_{y=0} = \frac{1}{e^x + 1}. \quad (2.193)$$

From (2.187) at $y = 0$ we have

$$u = e^x + x, \quad (2.194)$$

so that

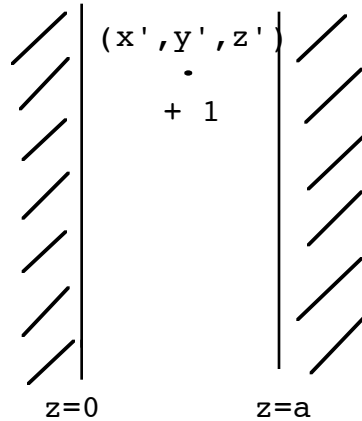
$$\frac{1}{e^x + 1} \rightarrow \begin{cases} 1/u, & u \rightarrow \infty \\ 1, & u \rightarrow -\infty \end{cases} \quad (2.195)$$

Tells us, for example, that $(E_0 = \frac{V}{\pi})$

$$E_v \rightarrow \begin{cases} \frac{V}{\pi} \frac{1}{u}, & u \rightarrow \infty \\ \frac{V}{\pi}, & u \rightarrow -\infty \end{cases} \quad (2.196)$$

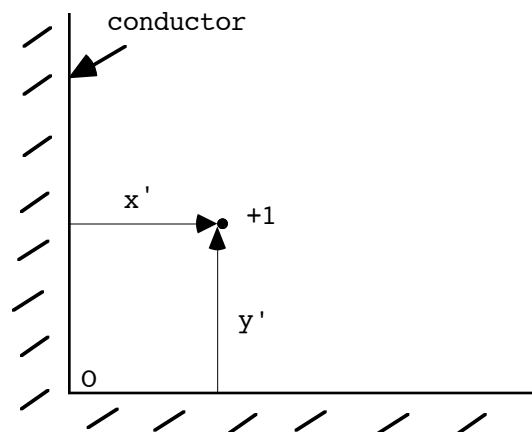
Problems

2.1.1 Consider the infinite parallel conducting plate interior problem.



Find the image charge locations and values of all the image charges needed to satisfy the BC's for the Dirichlet Green function. Write the Green function formally as a sum over these charges. Show that your expression converges absolutely.

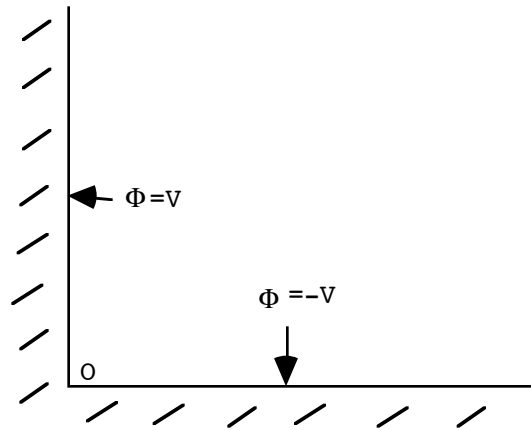
2.1.2 (a) Find the Dirichlet electrostatic Green function, $G(\vec{x}; \vec{x}')$, for the (3 dimensional) geometry shown in Cartesian coordinates. The walls are perfect conductors.



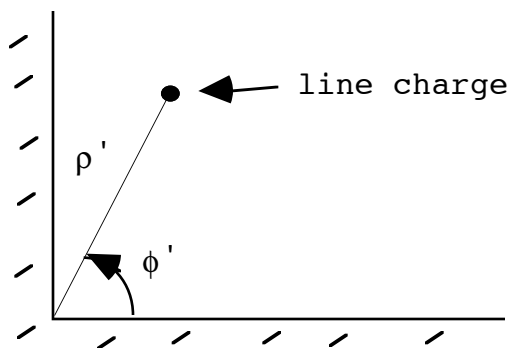
The angle between the infinitely long conducting planes is exactly 90° . (Take your coordinate origin at 0.)

(b) Find the work done, W , in removing the $+1$ charge to spatial infinity.

(c) Using the Green's function from the (a) part, find the potential for the situation where the vertical wall is at potential V and the horizontal wall is at $-V$, as shown.



2.1.3 An infinitely long unit line charge is located with cylindrical coordinates (ρ', ϕ') parallel to the line at which perpendicular conducting planes intersect. Using the image method, find the Dirichlet Green function.

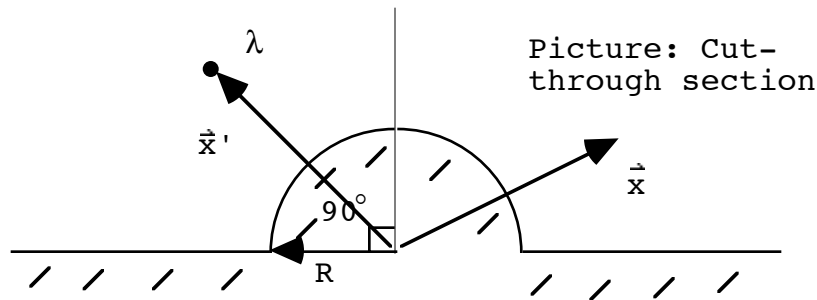


Express your answer in *cylindrical* coordinates. Show that

$$G(\vec{x}, \vec{x}') = \ln \left[\frac{\rho^4 + \rho'^4 - 2\rho'^2\rho^2 \cos(2(\phi + \phi'))}{\rho^4 + \rho'^4 - 2\rho'^2\rho^2 \cos(2(\phi - \phi'))} \right].$$

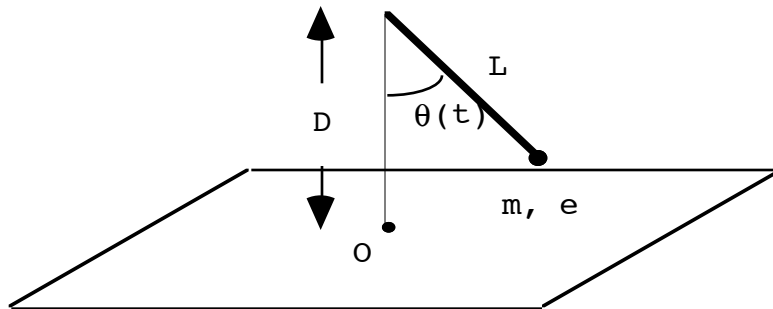
[Beware of the point raised in prob. 1.3.1.]

2.1.4 A half-cylinder of conducting material of radius R is formed on top of an infinite conducting plane. A long line charge, λ , is located in vacuum above the conductors.



Find the potential at an arbitrary position, $\vec{x}$, in vacuum. Show *explicitly* that all the boundary conditions are met.

2.1.5 A particle of charge e is attached to a massless string of length L and oscillates above a semi-infinite conducting plane that fills the entire region below the origin, O . The distance from the point of attachment to the plane is D . (Gravity is *not* operating in this problem and $D > L$.)



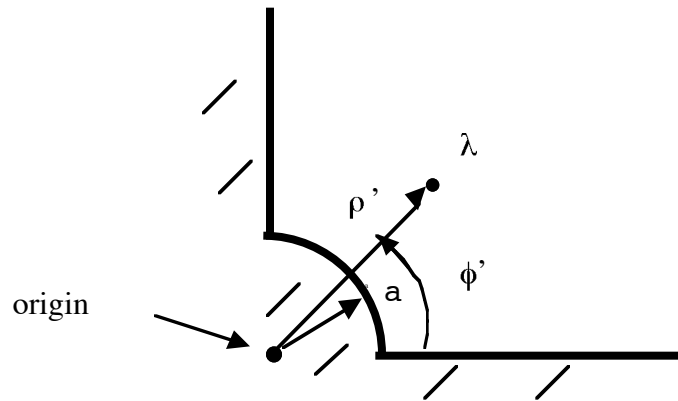
By computing the potential energy of the system to lowest order in the angle $\theta(t)$ (or other means), find the frequency of small oscillations of the charge.

2.1.6 By using the image method, find the Green function for a line charge of unit linear density inside a cylinder of radius b and parallel to it's axis. Assume two line charges

of equal and opposite density and require that the potential vanish on a circular cylinder containing the positive line charge. Given the radius b , the distance from the exterior image line charge to the cylinder's center as ρ'' , and the distance from the interior line charge to the cylinder's center as ρ' , one can show that $\rho'' = \frac{b^2}{\rho'}$. [Hint: the answer is

$$G(\rho, \phi; \rho', \phi') = \ln \left(\frac{b^4 + \rho^2 \rho'^2 - 2b^2 \rho \rho' \cos(\phi - \phi')}{b^2 (\rho^2 + \rho'^2 - 2\rho \rho' \cos(\phi - \phi'))} \right).$$

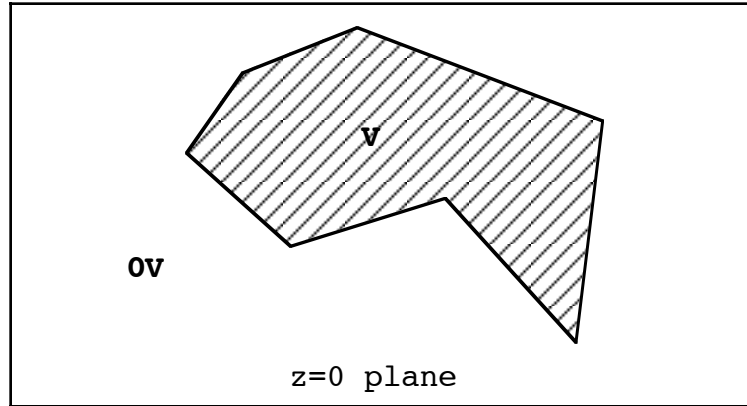
2.1.7 Consider a long line of charge, λ , located near a rounded conductor corner, as shown. The corner is $1/4$ of a circle of radius a .



Using cylindrical coordinates, ρ and ϕ , find the Dirichlet Green function ($\lambda=1$) by the method of images. Take your origin at the hypothetical corner, which is also the circle center, as shown.

2.2.1 An arbitrary patch of area on the $z = 0$ plane is raised to a constant potential V , as shown. The rest of the plane (extending to infinity) is specified to have $V=0$.

2.40

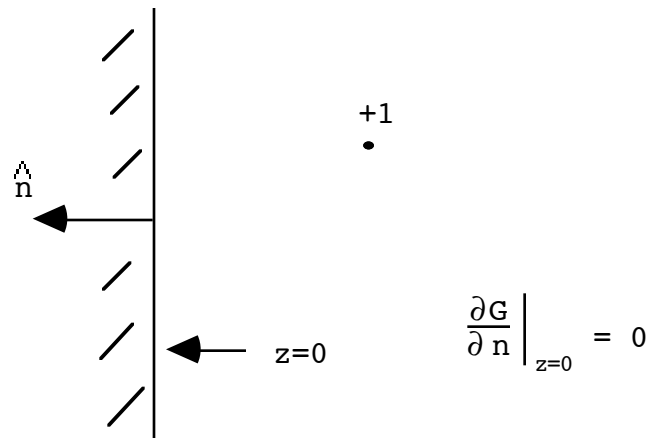


Show that the electric potential, Φ , at an arbitrary position above or below the plane can be written

$$\Phi(\vec{x}) = \frac{V}{2\pi} |\Omega(\vec{x})|,$$

where $\Omega(\vec{x})$ is the solid angle subtended by the patch at the observation position, $\vec{x}$. [Hint: Use the Green function for this geometry.]

2.2.2 Obtain the Neumann Green function for the half-infinite geometry:



Solve it:

- (a) By the method of images.
 (b) By the reduced Green function method [You can leave the result in integral form.]
 (c) Confirm the presence of a surface delta function in $\left. \frac{\partial G}{\partial n'} \right|_{z=0}$. That is, show

$$\rho \neq 0, \quad z' = 0, \quad \left. \frac{\partial G}{\partial n'} \right|_{z=0} = 0$$

$$\rho = 0, \quad z' \rightarrow 0^+, \quad \left. \frac{\partial G}{\partial n'} \right|_{z=0} = +\infty$$

$$(\rho^2 = (x-x')^2 + (y-y')^2) \quad \text{and that}$$

$$\frac{1}{4\pi} \lim_{z' \rightarrow 0^+} \int_S da \left. \frac{\partial G}{\partial n'} \right|_{z=0} = 1.$$

2.2.3 Consider again the Dirichlet Green function for infinite parallel conducting plates (see above).

- (a) Show that

$$G_0(\vec{x}, \vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')_{\perp}} g(z, z', k),$$

where

$$g(z, z', k) = \frac{\sinh(kz_{<}) \sinh(k(a-z_{>}))}{k \sinh(ka)},$$

and where $z_{<}(z_{>})$ is the lesser (greater) of z and z' .

- (b) Using the result of part (a), verify the result (which can be shown using the reciprocity theorem of Ch.1),

$$\left. \begin{aligned} Q_{z=0} &= -\left(1 - \frac{z'}{a}\right) \\ Q_{z=a} &= -\frac{z'}{a} \end{aligned} \right\} \quad z' \text{ is the source position}$$

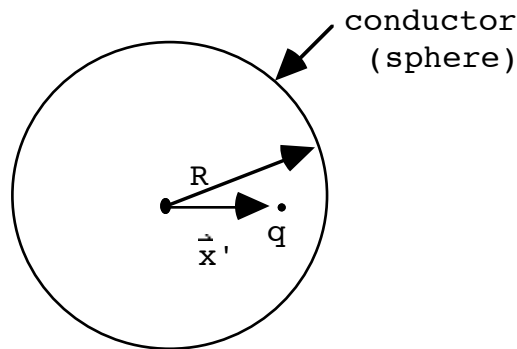
(c) Using the explicit expression for the capacitance (derived in Ch. 1) and the result of (b) to show that the capacitance is

$$C = \frac{A}{4\pi a} .$$

[Hint: Take one of the surface integrals to include only a finite area, A . We have used a sledgehammer to crack a nut.]

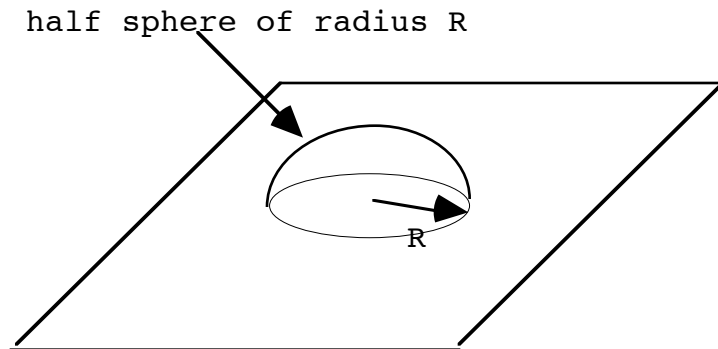
2.3.1 Find the Green's function solution for a unit point charge located *inside* a hollow, grounded, perfectly conducting spherical shell of radius " a ". Use the image method and find the potential both inside and outside the conducting shell.

2.3.2 Consider a point charge, q , located inside a thin spherical conducting sphere of radius R . In this case the conductor is neutral (no net charge on it).



Find the potential due to the charge q everywhere inside and outside the sphere. Can one tell where the charge is inside the sphere from the outside field? Generalize your conclusion to a point charge inside an arbitrarily shaped conducting neutral hollow object.

2.3.3 (a) A half-sphere of radius R is placed atop an infinite two dimensional plane. All surfaces are conducting.



Using a set of image charges, find the Dirichlet Green function at all points above the conducting surfaces.

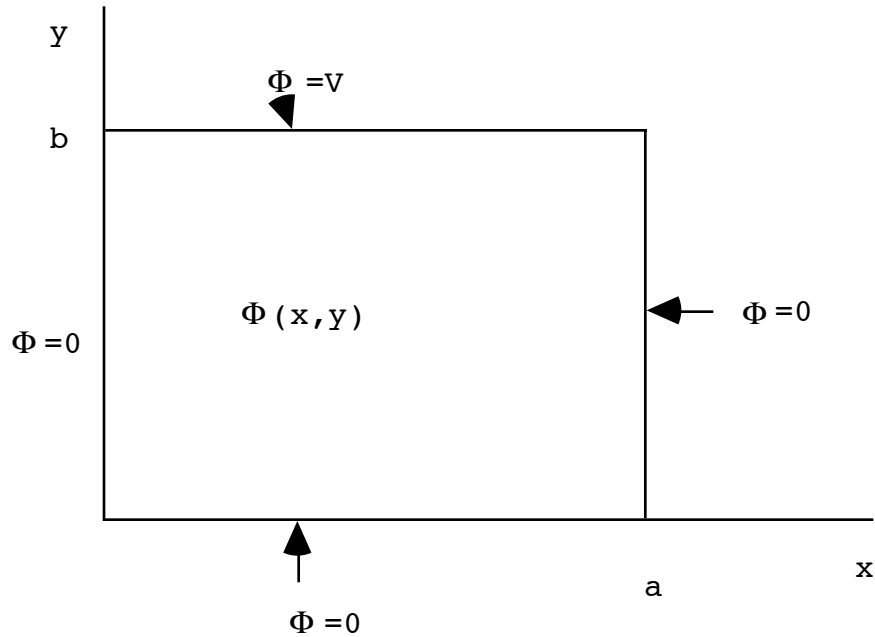
(b) Find the Dirichlet Green function for the vacuum region *inside* the hemispherical conducting half-sphere.

2.3.4 Will an image charge solution for the Neumann Green function for a sphere, either inside or outside, of radius R work? If so, find them; if not, explain why.

2.6.1 Solve the Laplace equation,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0,$$

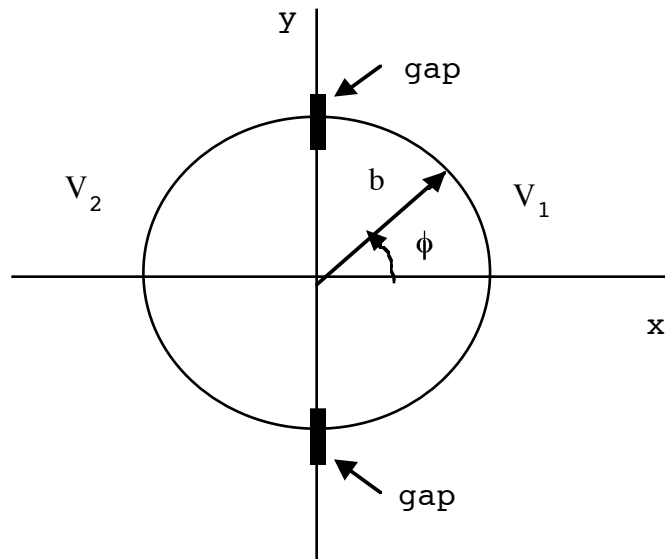
in two dimensions by separation of variables and find the potential, expressed as an infinite series, everywhere inside a two dimensional box which has $\Phi=0$ on it's borders except along $y=b$ where $\Phi=V$. (V is a constant).



$$\text{Ans.: } \Phi(x,y) = \frac{4V}{\pi} \sum_{n=1}^{\infty} \frac{\sinh(\frac{m\pi y}{a})}{m \sinh(\frac{m\pi b}{a})} \sin(\frac{m\pi x}{a}), \quad m=2n-1.$$

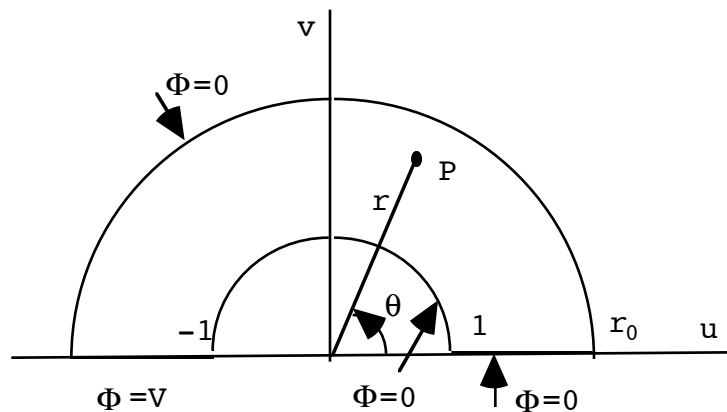
2.10.1 Using the Dirichlet Green function derived in prob. 2.1.6, rederive the expression (2.159) for the potential inside a long cylinder of radius b with potentials V_1 and V_2 as in the cylindrical halves section above. (Make sure you set up the angle ϕ relative to the potentials exactly as in the example.)

2.10.2 (a) Using Fourier series methods, find the potential, $\Phi(\rho, \phi)$, of an infinitely long cylinder (radius b) raised to two separate potentials, V_1 (for $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$) and V_2 (for $\frac{\pi}{2} < \phi < \pi$ and $-\pi < \phi < -\frac{\pi}{2}$) in the region **exterior** to the cylinder ($\rho > b$).



(b) By summing or other means produce a closed form solution. Compare with the interior solution and comment.

2.11.1 (Adapted from Churchill, Ch.9) The two dimensional Dirichlet problem for the geometry shown in prob. 2.6.1 is conformally mapped to the space between the semicircular regions shown, all surfaces having zero potential except the surface between $v=0$, $-r_0 < u < -1$, which again has potential V .



(a) Find the mapping, $w = f(z)$, which takes the geometry in prob. 2.6.1 to the above.

(b) Using this mapping, show that the potential everywhere inside the semicircular region is given by

$$\Phi(r, \theta) = \frac{4V}{\pi} \sum_{n=1}^{\infty} \frac{\sinh\left(\frac{m\pi\theta}{\ell_n(r_0)}\right)}{m \sinh\left(\frac{m\pi^2}{\ell_n(r_0)}\right)} \sin\left(m\pi \frac{\ell_n(r)}{\ell_n(r_0)}\right), \quad m=2n-1,$$

where θ and r are cylindrical coordinates which locate a given point P in the semicircular region.