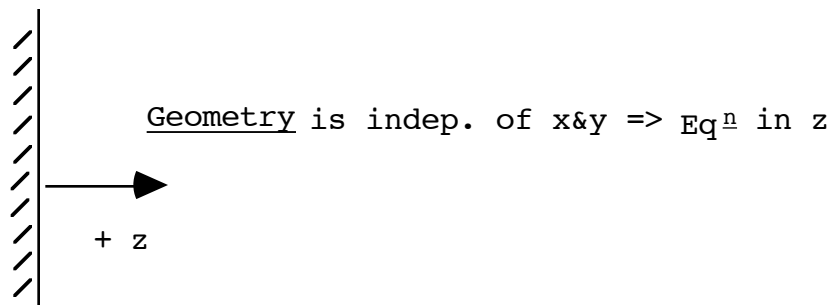


Chapter 3: Electrostatics: Cylindrical and Spherical Coordinates

3.1 Reduced Green function technique in cylindrical coordinates and Bessel functions

Basic problem in extending reduced G.f. method to cylindrical & spherically symmetric geometries: have to be able to separate the variables in order to reduce it to a 1-D problem. Techniques in this chapter largely based on methods from Schwinger. Let's go back to the wall problem. Reminder:



Set

$$G_D(\vec{x}, \vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')_{\perp}} g(z, z', k) \quad (3.1)$$

based upon

$$4\pi \delta(\vec{x} - \vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')_{\perp}} \delta(z - z'). \quad (3.2)$$

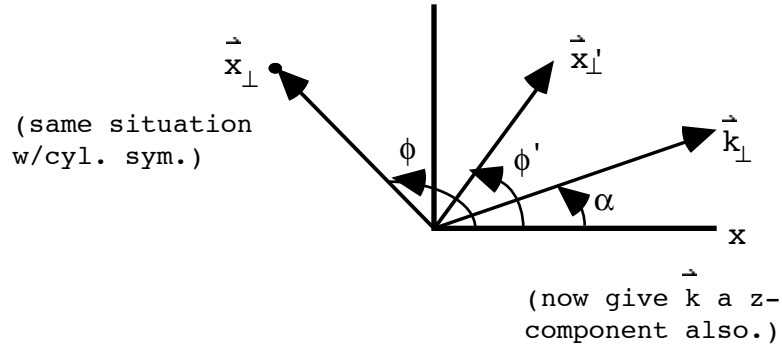
Since

$$-\nabla^2 G_D(\vec{x}, \vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')_{\perp}} \left(k^2 - \frac{\partial^2}{\partial z^2}\right) g(z, z', k), \quad (3.3)$$

$$\Rightarrow \left(k - \frac{\partial^2}{\partial z^2}\right) g = \delta(z - z'). \quad (3.4)$$

3.2

Now try it for cylindrical coordinates:



Geometry is indep. of ϕ & $z \Rightarrow$ eqⁿ in ρ .

$$\vec{k} \cdot (\vec{x} - \vec{x}') = k_z(z-z') + k_{\perp}\rho\cos(\phi - \alpha) - k_{\perp}\rho'\cos(\phi' - \alpha), \quad (3.5)$$

$$(\rho = |\vec{x}_{\perp}|, \rho' = |\vec{x}'_{\perp}|, k_{\perp} = |\vec{k}_{\perp}|)$$

$$\delta(\vec{x} - \vec{x}') = \int \frac{dk_z}{(2\pi)} e^{ik_z(z-z')} \int \frac{k_{\perp} dk_{\perp} d\alpha}{(2\pi)^2} e^{ik_{\perp}\rho\cos(\phi-\alpha)} e^{-ik_{\perp}\rho'\cos(\phi'-\alpha)}. \quad (3.6)$$

It is obvious that

$$e^{ik_{\perp}\rho\cos(\phi-\alpha)} \neq f(k\rho)g(\phi). \quad (3.7)$$

The reduced G.f. method depends upon a separation of variables. Brute force it:

$$k_{\perp}\rho \equiv t, \quad (3.8)$$

$$e^{i\phi} \equiv \frac{1}{i} u \quad (\text{a generic } \phi). \quad (3.9)$$

Then

$$e^{ik_{\perp}\rho\cos\phi} = e^{t/2(u-1/u)}. \quad (3.10)$$

Imagine expanding in powers of u . It must be of the form

$$e^{t/2(u-1/u)} = \sum_{m=-\infty}^{\infty} u^m J_m(t), \quad (3.11)$$

where the $J_m(t)$ are simply the coefficients of the expressions. These coefficients, which are functions of the t variable, turn out to be Bessel functions of integer order. LHS above is called a generating function. We have now separated; the price paid is an infinite sum. From the invariance of the gen. funct. under

$$u \rightarrow -\frac{1}{u}, \quad (3.12)$$

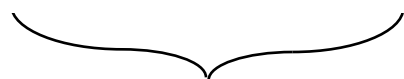
we learn that

$$\sum_{m=-\infty}^{\infty} u^m J_m(t) = \sum_{m=-\infty}^{\infty} (-1)^m u^{-m} J_m(t) = \sum_{m=-\infty}^{\infty} (-1)^m u^m J_{-m}(t), \quad (3.13)$$

$$\Rightarrow J_{-m}(t) = (-1)^m J_m(t). \quad (3.14)$$

Therefore $(\cos\phi = (u - \frac{1}{u})/2i)$

$$e^{it\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(t) = J_0(t) + 2 \sum_{m=1}^{\infty} i^m \cos m\phi J_m(t). \quad (3.15)$$



Can solve for $J_m(t)$
by using: $\int_0^{2\pi} \frac{d\phi}{2\pi} e^{-im'\phi} e^{im\phi} = \delta_{mm'}.$ (3.16)

(Can do the integral directly to see this.) This gives

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(t\cos\phi - m\phi)}. \quad (3.17)$$

Called an integral representation. From either the generating or integral eq^{ns}, can now show that $(m \geq 0)$

$$J_m(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(t/2)^{m+2n}}{n!(m+n)!}. \quad (3.18)$$

We can develop the diff. eqⁿ that $J_m(t)$ satisfies as follows.
Consider

$$\frac{d}{dt} J_m(t) = \frac{1}{i^m} \int_0^{2\pi} \frac{d\phi}{2\pi} (i \cos \phi) e^{i(t \cos \phi - m\phi)}. \quad (3.19)$$

But

$$\begin{aligned} J_{m-1}(t) &= \frac{1}{i^{m-1}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(t \cos \phi - (m-1)\phi)}, \\ " &= \frac{1}{i^m} \int_0^{2\pi} \frac{d\phi}{2\pi} (i e^{i\phi}) e^{i(t \cos \phi - m\phi)}, \end{aligned} \quad (3.20)$$

and

$$J_{m+1}(t) = \frac{1}{i^{m+1}} \int_0^{2\pi} \frac{d\phi}{2\pi} (-i e^{-i\phi}) e^{i(t \cos \phi - m\phi)}. \quad (3.21)$$

Therefore

$$\begin{aligned} J_{m-1}(t) - J_{m+1}(t) &= \frac{1}{i^m} \int_0^{2\pi} \frac{d\phi}{2\pi} i (e^{i\phi} + e^{-i\phi}) e^{i(t \cos \phi - m\phi)} \\ &= 2 \frac{d}{dt} J_m(t). \end{aligned} \quad (3.22)$$

Likewise

$$\begin{aligned} J_{m-1}(t) + J_{m+1}(t) &= \frac{1}{i^m} \int_0^{2\pi} \frac{d\phi}{2\pi} (-2 \sin \phi) e^{i(t \cos \phi - m\phi)} \\ &= \frac{2}{i^m} \frac{1}{it} \int_0^{2\pi} \frac{d\phi}{2\pi} \left(\frac{d}{d\phi} e^{it \cos \phi} \right) e^{-im\phi} \\ &\text{by parts } 0 - \underbrace{\frac{2}{i^m} \left\{ \frac{1}{it} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{it \cos \phi} \frac{d}{d\phi} e^{-im\phi} \right\}}_{-ime^{-im\phi}} \end{aligned}$$

$$= 2 \frac{m}{t} J_m(t). \quad (3.23)$$

These are called recurrence relations. So, we have

$$2 \frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t), \quad (3.24)$$

$$2 \frac{m}{t} J_m(t) = J_{m-1}(t) + J_{m+1}(t), \quad (3.25)$$

$$\Rightarrow J_{m+1}(t) = \left(\frac{m}{t} - \frac{d}{dt} \right) J_m(t), \quad (3.26)$$

$$J_{m-1}(t) = \left(\frac{m}{t} + \frac{d}{dt} \right) J_m(t). \quad (3.27)$$

Therefore

$$\underbrace{\left(\frac{m-1}{t} - \frac{d}{dt} \right) \left(\frac{d}{dt} + \frac{m}{t} \right) J_m(t)}_{J_{m-1}(t)} = J_m(t). \quad (3.28)$$

In more standard form, this gives

$$\left[\frac{d^2}{dt^2} + \frac{1}{t} \frac{d}{dt} - \frac{m^2}{t^2} + 1 \right] J_m(t) = 0. \quad (3.29)$$

[This is the dimensionless form of the Bessel function. Re-instating $t=k_\perp \rho$ gives us the more physical dimensionful form (what we get when we separate variables in Laplace's eqⁿ in cylindrical coordinates): ($k_\perp \rightarrow k$ here)]

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2 \right] J_m(k\rho) = 0. \quad]$$

3.2 Completeness of Bessel functions

What have we shown? We know

$$\delta(\vec{x} - \vec{x}') = \int \frac{dk_z}{2\pi} e^{ik_z(z-z')} \int \frac{dk_{\perp} k_{\perp} d\alpha}{(2\pi)^2} e^{ik_{\perp} \rho \cos(\phi-\alpha)} e^{-ik_{\perp} \rho' \cos(\phi'-\alpha)}. \quad (3.30)$$

Expand

$$e^{ik_{\perp} \rho \cos(\phi-\alpha)} = \sum_{-\infty}^{\infty} i^m e^{im(\phi-\alpha)} J_m(k_{\perp} \rho), \quad (3.31)$$

$$e^{-ik_{\perp} \rho' \cos(\phi'-\alpha)} = \sum_{-\infty}^{\infty} (-i)^{m'} e^{-im'(\phi'-\alpha)} J_{m'}(k_{\perp} \rho'). \quad (3.32)$$

Perform the α -integration (just showing the relevant part)

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} e^{-i\alpha(m-m')} = \delta_{mm'}, \quad (3.33)$$

so

$$\delta(\vec{x} - \vec{x}') = \int \frac{dk_z}{2\pi} e^{ik_z(z-z')} \int \frac{dk_{\perp} k_{\perp}}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} J_m(k_{\perp} \rho') J_m(k_{\perp} \rho) \quad (3.34)$$

$$\delta(\vec{x} - \vec{x}') = \delta(z-z') \frac{1}{\rho} \delta(\rho-\rho') \delta(\phi-\phi'). \quad (3.35)$$

(The form of the last line uses the UVW $\delta(u-u')\delta(v-v')\delta(w-w')$ form I talked about in Chapter 1.) Since we can easily identify the $\delta(z-z')$ part above, we must have

$$\frac{1}{\rho} \delta(\rho-\rho') \delta(\phi-\phi') = \int \frac{dk_{\perp} k_{\perp}}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} J_m(k_{\perp} \rho) J_m(k_{\perp} \rho'). \quad (3.36)$$

(In an equation like this where $k_{\perp}$ is just an integration variable that has lost its geometric meaning, I probably should just put $k_{\perp} \rightarrow k$.) Now mult. by $e^{-im'(\phi-\phi')}$ and integrate over ϕ : ($m' \rightarrow m$)

$$\int_0^{\infty} dk k J_m(k\rho) J_m(k\rho') = \frac{1}{\rho} \delta(\rho-\rho') \text{ ("completeness")}. \quad (3.37)$$

(Notice rhs is indep. of m ; true for each m .) It now follows that

$$\sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{im(\phi-\phi')} = \delta(\phi-\phi') \quad (\text{more completeness}). \quad (3.38)$$

Completeness in ρ implies ($t=k\rho$)

$$F(t) = \int_0^{\infty} du \, u J_m(ut) \mathfrak{F}(u), \quad (3.39)$$

$$\Rightarrow \int_0^{\infty} dt \, t F(t) J_m(tv) = \mathfrak{F}(v). \quad (3.40)$$

Similarly for ϕ . These results are useful only if the full range of ϕ or ρ values is allowed in some problem.

3.3 Zeros and orthogonality properties of Bessel functions

The dimensionful eqⁿ satisfied by $J_m(k\rho)$:

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2 \right] J_m(k\rho) = 0, \quad (3.41)$$

$$\Rightarrow \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dJ_m}{d\rho} \right) + \left(k^2 - \frac{m^2}{\rho^2} \right) J_m = 0. \quad (3.42)$$

We have that

$$- \int_0^a d\rho \rho J_m(k'\rho) \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dJ_m(k\rho)}{d\rho} \right) = \int_0^a d\rho \rho \left(k^2 - \frac{m^2}{\rho^2} \right) J_m(k'\rho) J_m(k\rho). \quad (3.43)$$

On the other hand ($k \Leftrightarrow k'$)

$$- \int_0^a d\rho \rho J_m(k\rho) \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dJ_m(k'\rho)}{d\rho} \right) = \int_0^a d\rho \rho \left(k'^2 - \frac{m^2}{\rho^2} \right) J_m(k\rho) J_m(k'\rho). \quad (3.44)$$

Integrate by parts on lhs of (3.44):

$$\int_0^a d\rho J_m(k\rho) \frac{d}{d\rho} \left(\rho \frac{dJ_m(k'\rho)}{d\rho} \right)$$

$$= \rho J_m(k\rho) \left. \frac{dJ_m(k'\rho)}{d\rho} \right|_0^a - \int_0^a d\rho \rho \frac{dJ_m(k\rho)}{d\rho} \frac{dJ_m(k'\rho)}{d\rho}. \quad (3.45)$$

The combination $\rho J_m(k\rho) \frac{dJ_m(k'\rho)}{d\rho}$ vanishes at $\rho=0$. Can see this from our earlier result:

$$J_m(k\rho) \xrightarrow{k\rho \ll 1} \frac{(k\rho)^m}{2^m m!}, \quad (3.46)$$

$$\Rightarrow \rho J_m(k\rho) \frac{dJ_m(k'\rho)}{d\rho} \xrightarrow{k'\rho \ll 1} \frac{m(kk')^m}{(2^m m!)^2} \rho^{2m}. \quad (3.47)$$

(Actually vanishes like ρ^2 for $m=0$ also) As for the other limit, we simply impose

$$J_m(ka) = 0. \quad (3.48)$$

Call $k_{mn}a \equiv x_{mn}$

$$\Rightarrow J_m(x_{mn}) = 0, \quad n = 1, 2, 3, \dots \quad (3.49)$$

There are an infinite number of such roots for each m value. Such a condition, we will see, is appropriate if we are building a function w/Dirichlet B.C. However, we can also impose

$$\left. \frac{d}{d\rho} J_m(k\rho) \right|_{\rho=a} = 0. \quad (3.50)$$

Define $k_{mn}a = y_{mn}$

$$\Rightarrow \left. \frac{d}{dx} J_m(x) \right|_{x=y_{mn}} = 0. \quad (3.51)$$

Therefore, (3.43) & (3.44) become (for either set of roots)

$$\left(\begin{matrix} 3.43 \\ 3.44 \end{matrix} \right) \Rightarrow \int_0^a d\rho \rho \frac{dJ_m(k_{mn}\rho)}{d\rho} \frac{dJ_m(k_{mn}'\rho)}{d\rho}$$

$$= \int_0^a d\rho \rho \left(\left(\frac{k_{mn}^2}{k_{mn'}^2} \right) - \frac{m^2}{\rho^2} \right) J_m(k_{mn}\rho) J_m(k_{mn'}\rho). \quad (3.52)$$

Subtracting yields

$$(k_{mn}^2 - k_{mn'}^2) \int_0^a d\rho \rho J_m(k_{mn}\rho) J_m(k_{mn'}\rho) = 0, \quad (3.53)$$

$$\Rightarrow \int_0^a d\rho \rho J_m(k_{mn}\rho) J_m(k_{mn'}\rho) = 0, \quad n \neq n'. \quad (3.54)$$

Orthogonality for $n \neq n'$ (for a given m). One can also show that

$$\int_0^a d\rho \rho [J_m(k_{mn}\rho)]^2 = \frac{a^2}{2} [J_{m+1}(x_{mn})]^2 \quad (3.55)$$

when we are dealing with zeros of Bessel functions, and

$$\int_0^a d\rho \rho (J_m(k_{mn}\rho))^2 = \frac{a^2}{2} \left(1 - \frac{m^2}{y_{mn}^2} \right) (J_m(y_{mn}))^2 \quad (3.56)$$

for zeros of derivatives of Bessel functions.

Material on how one obtains these integrals now follows. We are dealing here with the Dirichlet case, (3.55), only. Eq.(3.56) is derived in an analagous fashion.

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2 \right] J_m(k\rho) = 0, \quad (3.57)$$

$$(k_{mn} = \frac{x_{mn}}{a})$$

$$\Rightarrow \frac{\partial J_m''}{\partial a} + \frac{1}{\rho} \frac{\partial J_m'}{\partial a} + \left[-\frac{m^2}{\rho^2} + k^2 \right] \frac{\partial J_m}{\partial a} + \frac{\partial k^2}{\partial a} J_m = 0, \quad (3.58)$$

$$\Rightarrow \rho \frac{\partial J_m''}{\partial a} J_m + \frac{\partial J_m'}{\partial a} J_m + \left[-\frac{m^2}{\rho^2} + k^2 \right] \rho \frac{\partial J_m}{\partial a} J_m + \rho \frac{\partial k^2}{\partial a} J_m^2 = 0. \quad (3.59)$$

($J_m'(x) \equiv \frac{dJ_m(x)}{dx}$, etc.) Multiply (3.57) by $\rho \frac{\partial J_m}{\partial a}$:

$$\rho J''_m \frac{\partial J_m}{\partial a} + J'_m \frac{\partial J_m}{\partial a} + \left[-\frac{m^2}{\rho^2} + k^2\right] \rho J_m \frac{\partial J_m}{\partial a} = 0. \quad (3.60)$$

Subtract (3.59) - (3.60):

$$\rho \frac{\partial J''_m}{\partial a} J_m - \rho J''_m \frac{\partial J_m}{\partial a} + \frac{\partial J'_m}{\partial a} J_m - J'_m \frac{\partial J_m}{\partial a} = -\frac{\partial k^2}{\partial a} \rho J_m^2. \quad (3.61)$$

However

$$\frac{d}{d\rho} \left[\rho \frac{\partial J'_m}{\partial a} J_m - \rho J'_m \frac{\partial J_m}{\partial a} \right] = \text{LHS of (3.61)} = -\frac{\partial k^2}{\partial a} \rho J_m^2. \quad (3.62)$$

Can now integrate:

$$\rho \left[\frac{\partial J'_m}{\partial a} J_m - J'_m \frac{\partial J_m}{\partial a} \right] \Big|_0^a = -\frac{\partial k^2}{\partial a} \int_0^a d\rho \rho J_m^2. \quad (3.63)$$

As expressed before: (m=0 separate)

$$\rho J_m(k\rho) \frac{dJ_m(k\rho)}{d\rho} \xrightarrow{\rho \rightarrow 0} \frac{m(kk')^m}{(2^m m!)^2} \rho^{2m}. \quad (3.64)$$

(Derivatives wrt "a" don't change the power of ρ & therefore don't change $\rho \rightarrow 0$ limit.) Also (Dirichlet case)

$$J_m(\underbrace{k_{mn}a}_{x_{mn}}) = 0, \quad (3.65)$$

$$\Rightarrow \rho \frac{\partial J_m}{\partial \rho} \frac{\partial J_m}{\partial a} \Big|_{\rho=a} = \frac{\partial k^2}{\partial a} \int_0^a d\rho \rho J_m^2, \quad (3.66)$$

$$\frac{\partial J_m}{\partial \rho} = \frac{dJ_m(t)}{dt} \frac{\partial t}{\partial \rho} = \frac{dJ_m}{dt} \left(\frac{x}{a}\right), \quad (3.67)$$

$$(t = k\rho = \frac{x\rho}{a})$$

$$\frac{\partial J_m}{\partial a} \Big|_{\rho=a} = \frac{dJ_m(t)}{dt} \frac{\partial t}{\partial a} \Big|_{\rho=a} = -\frac{x\rho}{a^2} \frac{dJ_m}{dt} \Big|_{\rho=a} = -\frac{x}{a} \frac{dJ_m}{dt} \Big|_{t=x_{mn}}. \quad (3.68)$$

$$\frac{\partial k^2}{\partial a} = \frac{\partial}{\partial a} \left(\frac{x^2}{a^2} \right) = - \frac{2x^2}{a^3}, \quad (3.69)$$

$$\Rightarrow a \frac{dJ_m}{dt} \frac{x}{a} \left(\frac{-x}{a} \frac{dJ_m}{dt} \right) = - \frac{2x^2}{a^3} \int_0^a d\rho \rho J_m^2, \quad (3.70)$$

$$\Rightarrow \int_0^a d\rho \rho J_m^2(k_{mn}\rho) = \frac{a^2}{2} \left(\frac{d}{dt} J_m \right)^2 \Big|_{t=x_{mn}}. \quad (3.71)$$

However

$$2 \frac{d}{dt} J_m = J_{m-1} - J_{m+1} \quad (2 \frac{m}{t} J_m = J_{m-1} + J_{m+1}), \quad (3.72)$$

$$\Rightarrow \frac{d}{dt} J_m = - J_{m+1} \quad (\text{at } J_m = 0), \quad (3.73)$$

$$\Rightarrow \int_0^a d\rho \rho J_m^2(k_{mn}\rho) = \frac{a^2}{2} (J_{m+1}(x_{mn}))^2. \quad (3.74)$$

Assuming completeness, we may expand

$$f(\rho) = \sum_{n=1}^{\infty} A_{mn} J_m(k_{mn}\rho), \quad (3.75)$$

for any m . Using orthonormality and the above integral, we may now invert this for A_{mn} :

$$A_{mn} = \frac{2}{a^2 J_{m+1}^2(x_{mn})} \int_0^a d\rho' \rho' J_m\left(\frac{x_{mn}}{a} \rho'\right) f(\rho'). \quad (3.76)$$

Plugging A_{mn} in above, what this says is

$$f(\rho) = \int_0^a d\rho' \rho' f(\rho') \sum_{n=1}^{\infty} \frac{2}{a^2 J_{m+1}^2(x_{mn})} J_m(k_{mn}\rho) J_m(k_{mn}\rho'), \quad (3.77)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2}{a^2 J_{m+1}^2(x_{mn})} J_m\left(\frac{x_{mn}\rho'}{a}\right) J_m\left(\frac{x_{mn}\rho}{a}\right) = \frac{1}{\rho} \delta(\rho - \rho'). \quad (3.78)$$

Similarly, we find

$$\sum_{n=1}^{\infty} \frac{2}{a^2 \left(1 - \frac{m^2}{y_{mn}^2}\right)} J_m\left(\frac{y_{mn}\rho'}{a}\right) J_m\left(\frac{y_{mn}\rho}{a}\right) = \frac{1}{\rho} \delta(\rho - \rho'). \quad (3.79)$$

To save some wear and tear, define

$$J_{1m}(k_{mn}\rho) \equiv \frac{\sqrt{2}}{a} \frac{J_m(k_{mn}\rho)}{J_{m+1}(x_{mn})}, \quad (3.80)$$

$$J_{2m}(k_{mn}\rho) \equiv \frac{\sqrt{2}}{a \sqrt{1 - \frac{m^2}{y_{mn}^2}}} \frac{J_m(k_{mn}\rho)}{J_m(y_{mn})}. \quad (3.81)$$

Then the above reads (for each m)

$$\int_0^a d\rho \rho J_{1,2m}(k_{mn}\rho) J_{1,2m}(k_{mn}\rho') = \delta_{nn'}, \quad (3.82)$$

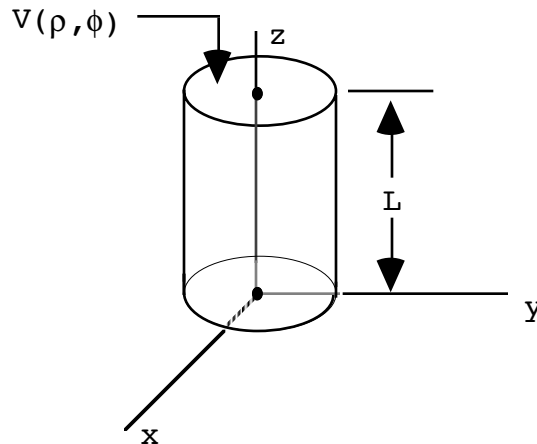
or

$$\sum_{n=1}^{\infty} J_{1m}(k_{mn}\rho') J_{1m}(k_{mn}\rho) = \frac{1}{\rho} \delta(\rho - \rho'), \quad (3.83)$$

$$\sum_{n=1}^{\infty} J_{2m}(k_{mn}\rho') J_{2m}(k_{mn}\rho) = \frac{1}{\rho} \delta(\rho - \rho'). \quad (3.84)$$

3.4 Reduced Green function for the conducting cylinder

First problem we will consider is the conducting cylinder:



Will do this interior problem as a Green function problem first, then as a B.V. problem.

We have a Dirichlet problem with restricted values of ρ, z . Therefore, choose

$$\begin{aligned} 4\pi\delta(\vec{x}-\vec{x}') &= \frac{4\pi}{\rho} \delta(\rho-\rho') \delta(\phi-\phi') \delta(z-z') \\ &= 4\pi \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} J_{1m}(k_{mn}\rho') J_{1m}(k_{mn}\rho) \frac{e^{im(\phi-\phi')}}{2\pi} \delta(z-z'). \end{aligned} \quad (3.85)$$

Similarly, assume

$$G_D(\vec{x}, \vec{x}') = 4\pi \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} J_{1m}(k_{mn}\rho') J_{1m}(k_{mn}\rho) \frac{e^{im(\phi-\phi')}}{2\pi} g_{mn}(z, z'), \quad (3.86)$$

Using (from Eq.(3.42)),

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dJ_m}{d\rho} \right) = \left(\frac{m^2}{\rho^2} - k^2 \right) J_m, \quad (3.87)$$

we have

$$\Rightarrow -\nabla^2 G_D(\vec{x}, \vec{x}') = 4\pi \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} J_{1m} J_{1m} \frac{e^{im(\phi-\phi')}}{2\pi} \left[\left(-\frac{m^2}{\rho^2} + k_{mn}^2 \right) + \frac{m^2}{\rho^2} - \frac{\partial^2}{\partial z^2} \right] g_{mn}(z, z'). \quad (3.88)$$

Therefore, we must solve

$$\left[\frac{k_{mn}^2}{a^2} - \frac{\partial^2}{\partial z^2} \right] g_{mn}(z, z') = \delta(z-z'), \quad (3.89)$$

subject to

$$g_{mn} \Big|_{z=0, L} = 0. \quad (3.90)$$

We have done exactly this problem before for the parallel plates ($k \rightarrow k_{mn}$, $a \rightarrow L$):

$$g_{mn}(z, z') = \frac{\sinh(k_{mn}z<) \sinh(k_{mn}(L-z>))}{k_{mn} \sinh(k_{mn}L)}. \quad (3.91)$$

Full solution:

$$G_D(\vec{x}, \vec{x}') = 2 \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} J_{1m}(k_{mn}\rho') J_{1m}(k_{mn}\rho) \times \frac{\sinh(k_{mn}z<) \sinh(k_{mn}(L-z>))}{k_{mn} \sinh(k_{mn}L)}. \quad (3.92)$$

Now get the solution of our BV problem:

$$\left. \frac{\partial}{\partial z'} g_{mn}(z, z') \right|_{z'=L} = - \frac{\sinh(k_{mn}z)}{\sinh(k_{mn}L)}. \quad (3.93)$$

Potential is

$$\Phi(\vec{x}) = \sum_{n,m} c_{mn} e^{im\phi} J_{1m}(k_{mn}\rho) \sinh(k_{mn}z), \quad (3.94)$$

where

$$c_{mn} = \frac{1}{2\pi} \frac{1}{\sinh(k_{mn}L)} \int_0^a d\rho' \rho' \int_0^{2\pi} d\phi' e^{-im\phi'} J_{1m}(k_{mn}\rho') V(\rho', \phi'). \quad (3.95)$$

3.5 The cylinder as a boundary value problem

Can also get the above result directly from BV point of view using separation of variables. (Did this before in 2-D.) Laplace eqⁿ:

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0. \quad (3.96)$$

Assume

$$\Phi(\rho, \phi, z) = P(\rho) Q(\phi) Z(z). \quad (3.97)$$

Leads to

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0, \quad (3.98)$$

$$\frac{d^2 Q}{d\phi^2} + v^2 Q = 0, \quad (3.99)$$

$$\frac{d^2 P}{d\rho^2} + \frac{1}{\rho} \frac{dP}{d\rho} + \left(k^2 - \frac{v^2}{\rho^2}\right) P = 0. \quad (3.100)$$

Solutions to the first two are

$$Z(z) \sim e^{\pm kz}, \quad (3.101)$$

$$Q(\phi) \sim e^{\pm iv\phi}. \quad (3.102)$$

If the full azimuthal range is allowed for $\phi \Rightarrow v = \text{integer}$.

$v \equiv m$. Other eqⁿ is just dimensionful form of Bessel's eqⁿ.

Solⁿs:

$$P(\rho) \sim J_m(k\rho), N_m(k\rho)$$

where (m integer; "Neumann function")

$$N_m(x) \equiv \lim_{v \rightarrow m} \left[\frac{J_v(x) \cos v\pi - J_{-v}(x)}{\sin v\pi} \right]. \quad (3.103)$$

(J_v and J_{-v} are linearly indep. if $v \neq m$. When $v = m$ we know $J_{-m} = (-1)^m J_m$; however, the above defⁿ of N_m does give a second linearly indep. solⁿ. Form is indeterminate if we just plug in. Must use l'Hospital's rule in order v . Explicit form of the $N_m(x)$ is somewhat sickening.)

Remind ourselves of the BC's:

$$\Phi \Big|_{\rho=a} = 0 \Rightarrow J_m(ka) = 0 \Rightarrow k_{mn} = \frac{x_{mn}}{a}, \quad (3.104)$$

$$\Phi \Big|_{z=0} = 0 \Rightarrow Z(0) = 0 \Rightarrow Z(z) \sim \sinh(k_{mn}z). \quad (3.105)$$

Most general form:

$$\Phi(\rho, \phi, z) = \sum_{n,m} c_{mn} e^{im\phi} J_{1m}(k_{mn}\rho) \sinh(k_{mn}z). \quad (3.106)$$

3.16

Last BC:

$$V(\rho, \phi) = \sum_{n,m} c_{mn} e^{im\phi} J_{1m}(k_{mn}\rho) \sinh(k_{mn}L). \quad (3.107)$$

Use our orthogonality formulas

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi(m-m')} = \delta_{mm'}, \quad (3.108)$$

$$\int_0^a d\rho \rho J_{1m}(k_{mn}\rho) J_{1m}(k_{nn'}\rho) = \delta_{nn'}. \quad (3.109)$$

(Notice the last statement requires a single m value.)

$$\Rightarrow c_{mn} = \frac{1}{2\pi \sinh(k_{mn}L)} \int_0^a d\rho' \rho' \int_0^{2\pi} d\phi e^{-im\phi} J_{1m}(k_{mn}\rho') V(\rho', \phi'), \quad (3.110)$$

as before. This way is quicker, but less general.

3.6 Bessel functions of imaginary argument

Bessel function eqⁿ again: (v arbitrary; dimensionful form)

$$\left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \right) - \frac{v^2}{\rho^2} + k^2 \right] \begin{pmatrix} J_v(k\rho) \\ N_v(k\rho) \end{pmatrix} = 0. \quad (3.111)$$

↑ oscillating at large ρ

This is if the separation constant, k, is real. If it is imaginary, we get

$$\left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \right) - \frac{v^2}{\rho^2} - k^2 \right] \begin{pmatrix} I_v(k\rho) \\ K_v(k\rho) \end{pmatrix} = 0. \quad (3.112)$$

↑ Exponential at large ρ

"Bessel functions of imaginary argument":

$$\begin{cases} I_\nu(x) \equiv i^{-\nu} J_\nu(ix), \\ K_\nu(x) \equiv \frac{\pi}{2i^{\nu+1}} H_\nu^{(1)}(ix). \end{cases} \quad (3.113)$$

$$(H_\nu^{(1)}(x) \equiv J_\nu(x) + iN_\nu(x))$$

Situation is similar to

$$\begin{aligned} \frac{d^2x}{dt^2} + \omega_0^2 x &= 0, \\ \omega_0^2 > 0, \quad x &\sim \sin t, \cos t, \\ \omega_0^2 < 0, \quad x &\sim \sinh t, \cosh t. \end{aligned} \quad (3.114)$$

Unlike the J_m , N_m , the I_ν , K_ν are not complete for ρ in the range $[0^+, \infty]$ (just like $\sin t$, $\cos t$ being complete for $t \in [-\infty, \infty]$, but $\sinh t$, $\cosh t$ are not.) Also, can not build functions in a finite interval from them. In general, can use either form in a given problem, but in a different sense. Usually (solution in ϕ , $e^{im\phi}$, is oscillatory for $0 \leq \phi \leq 2\pi$):

<u>oscill.</u>	<u>non-oscill.</u> (reduced G.f.)
$J_m(k\rho), N_m(k\rho)$	$e^{\pm kz}$
$e^{\pm ikz}$	$K_m(k\rho), I_m(k\rho)$

The $K_m(k\rho)$ go to zero for $\rho \rightarrow \infty$, while $I_m(k\rho)$ is regular at the origin (except for $m=0$).

3.7 Free charge solution using Bessel functions

Let's do a simple problem to illustrate the use of these functions. We will also see the Wronskian here for the first time, which is a great labor saving technical device. Isolated charge in free space (use cyl. coord.'s). One way:

$$\nabla^2 G(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}'), \quad (3.115)$$

$$\begin{aligned}
4\pi\delta(\vec{x} - \vec{x}') &= \frac{4\pi}{\rho} \delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z'), \\
&= 2 \int_0^\infty dk k \sum_{-\infty}^\infty e^{im(\phi-\phi')} J_m(k\rho)J_m(k\rho')\delta(z - z'). \quad (3.116)
\end{aligned}$$

Assume

$$G(\vec{x}, \vec{x}') = 2 \int_0^\infty dk k \sum_{-\infty}^\infty e^{im(\phi-\phi')} J_m(k\rho)J_m(k'\rho)g_m(z, z'), \quad (3.117)$$

$$\begin{aligned}
\Rightarrow -\nabla^2 G(\vec{x}, \vec{x}') &= -2 \int_0^\infty dk k \sum_{-\infty}^\infty \left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] \\
&\quad \times e^{im(\phi-\phi')} J_m(k\rho)J_m(k'\rho)g_m(z, z'), \quad (3.118)
\end{aligned}$$

$$\Rightarrow \left[k^2 - \frac{\partial^2}{\partial z^2} \right] g_m(z, z') = \delta(z - z'). \quad (3.119)$$

Easy way to apply continuity:

$$g_m(z, z') = \begin{cases} f(z')\psi_1(z), & z \leq z' \\ g(z')\psi_2(z), & z \geq z'. \end{cases} \quad (3.120)$$

However,

$$g_m(z, z') = g_m(z', z), \quad (3.121)$$

$$\Rightarrow g_m(z, z') = \begin{cases} \psi_1(z)\psi_2(z'), & z \leq z', \\ \psi_2(z)\psi_1(z'), & z \geq z'. \end{cases} \quad (3.122)$$

$$\text{or } g_m(z, z') = \psi_1(z_{<})\psi_2(z_{>}). \quad (3.123)$$

We must now have that

$$-\frac{dg_m}{dz} \Big|_{z=z'+} + \frac{dg_m}{dz} \Big|_{z=z'-} = 1, \quad (3.124)$$

$$\Rightarrow -\psi_2'(z')\psi_1(z') + \psi_1'(z')\psi_2(z') = 1. \quad (3.125)$$

Define: $W[\psi_1, \psi_2] \equiv \psi_1\psi_2' - \psi_2\psi_1'$ "Wronskian".

$$(3.126)$$

Explicitly, here

$$\psi_1(z') = C_1 e^{kz} \quad (\text{finite as } z \rightarrow -\infty), \quad (3.127)$$

$$\psi_2(z') = C_2 e^{-kz} \quad (\text{finite as } z \rightarrow +\infty), \quad (3.128)$$

$$\Rightarrow 2kC_1C_2 = 1, \quad C_1C_2 = \frac{1}{2k}, \quad (3.129)$$

$$\Rightarrow g_m(z, z') = \frac{1}{2k} e^{-k(z_> - z_<)}. \quad (3.130)$$

Put the pieces together (prob. 3.14(b)):

$$\frac{1}{|\vec{x} - \vec{x}'|} = \int_0^\infty dk \sum_{-\infty}^\infty e^{im(\phi - \phi')} J_m(k\rho) J_m(k\rho') e^{-k(z_> - z_<)}. \quad (3.131)$$

$\Downarrow$ non-oscill.

$\Uparrow$ oscillatory

Another way: (sect. 3.11)

$$\delta(z - z') = \frac{1}{2\pi} \int_{-\infty}^\infty dk e^{ik(z - z')} = \frac{1}{\pi} \int_0^\infty dk \cos[k(z - z')], \quad (3.132)$$

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{-\infty}^\infty e^{im(\phi - \phi')}, \quad (3.133)$$

$$4\pi\delta(\vec{x} - \vec{x}') = \frac{4\pi}{\rho} \delta(\rho - \rho') \delta(z - z') \delta(\phi - \phi'), \quad (3.134)$$

$$4\pi\delta(\vec{x} - \vec{x}') = \frac{1}{2\pi^2} \int_0^\infty dk \sum_{-\infty}^\infty \cos[k(z - z')] e^{im(\phi - \phi')} \frac{4\pi}{\rho} \delta(\rho - \rho'). \quad (3.135)$$

Assume

$$G(\vec{x}, \vec{x}') = \frac{1}{2\pi^2} \int_0^\infty dk \sum_{-\infty}^\infty \cos[k(z - z')] e^{im(\phi - \phi')} g_m(\rho, \rho'), \quad (3.136)$$

$$\Rightarrow \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dg_m}{d\rho} \right) - (k^2 + \frac{m^2}{\rho^2}) g_m = - \frac{4\pi}{\rho} \delta(\rho - \rho'). \quad (3.137)$$

Solutions: $I_m(k\rho)$, $K_m(k\rho)$

$$g_m(\rho, \rho') = \psi_1(k\rho_<) \psi_2(k\rho_>). \quad (3.138)$$

BC's:

$$g_m(\rho, \rho') \Big|_{\rho=0} = \text{finite} \Rightarrow \psi_1(k\rho) \sim I_m(k\rho), \quad (3.139)$$

$$g_m(\rho, \rho') \Big|_{\rho \rightarrow \infty} = 0 \Rightarrow \psi_2(k\rho) \sim K_m(k\rho), \quad (3.140)$$

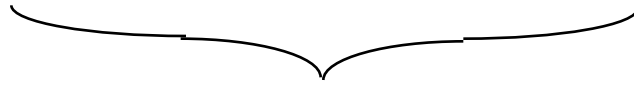
$$\Rightarrow g_m(\rho, \rho') = A I_m(k\rho_{<}) K_m(k\rho_{>}). \quad (3.141)$$

$\Uparrow$ unknown

Build in discontinuity:

$$- \frac{dg_m}{d\rho} \Big|_+ + \frac{dg_m}{d\rho} \Big|_- = \frac{4\pi}{\rho'}. \quad (3.142)$$

$$\Rightarrow A [-K'_m(k\rho') I_m(k\rho') + K_m(k\rho') I'_m(k\rho')] = \frac{4\pi}{\rho'}. \quad (3.143)$$



$$-kW[I_m(k\rho'), K_m(k\rho')]$$

Can show (from known forms for I_m, K_m for $x \ll 1$ or $x \gg 1$):

$$W[I_m(x), K_m(x)] = -\frac{1}{x}, \quad (3.144)$$

$$\Rightarrow A = 4\pi. \quad (3.145)$$

$$\Rightarrow \frac{1}{|\vec{x} - \vec{x}'|} = \frac{2}{\pi} \int_0^\infty dk \sum_{-\infty}^{\infty} e^{im(\phi - \phi')} I_m(k\rho_{<}) K_m(k\rho_{>}) \cos[k(z - z')]. \quad (3.146)$$

With the use of the Wronskian, continuity is automatic, and one can compute the necessary scale factor as in (3.141) by comparison of the discontinuity condition with the explicitly computed Wronskian. In summary, if one has

$$g_m(z, z') = C \psi_1(z_{<}) \psi_2(z_{>}),$$

for $\psi_1(z)$ and $\psi_2(z)$ known solutions, one rewrites the boundary condition as

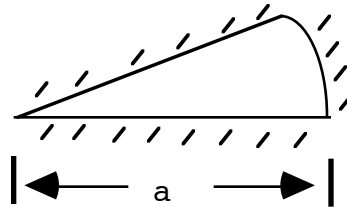
$$C(\psi_1(z')\psi_2'(z') - \psi_1'(z')\psi_2(z')) = D(z'),$$

$$\Rightarrow C = \frac{D(z')}{W[\psi_1, \psi_2]}.$$

Explicitly evaluating the Wronskian then gives the constant C.

3.8 Reduced Green function/image method: conducting wedge

Not all cyl. sym. problems need involve Bessel functions. Wedge problem (for $a \rightarrow \infty$ becomes Green function for wedge problem in Ch.2):



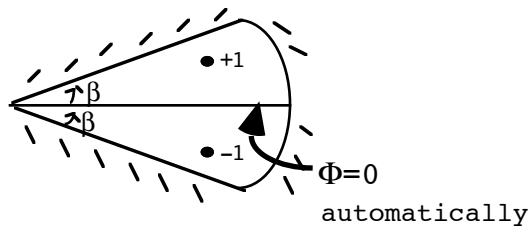
A 2D problem. Need to solve:

$$\nabla^2 G(\vec{x}, \vec{x}') = -4\pi \frac{1}{\rho} \delta(\rho - \rho') \delta(\phi - \phi'). \quad (3.147)$$

Can not assume

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{im(\phi - \phi')}, \quad (3.148)$$

because ϕ, ϕ' have a restricted range. Solution is to imagine:



A combined image/reduced G.f. method again (like box problem), except here we are working in ϕ rather than x . So solve

$$\nabla^2 G(\vec{x}, \vec{x}') = -4\pi \frac{1}{\rho} \delta(\rho - \rho') [\delta(\phi - \phi') - \delta(\phi + \phi')], \quad (3.149)$$

in the larger region. What happens, however, when $\beta > \pi$? We are forcing a periodicity on the problem which can not be realized physically. This is OK, as long as we realize $0 < \phi < \beta$ for physics. Think of the image extension as a sort of Riemann sheet, the "cut" for which is at $\phi = 0$. The sheets may lay on top of one another (if $\beta > \pi$), that is, we are allowing the potential to become a mult. valued function (in non-physical regions) in order to force a symmetry on the problem.

What all this means is that we take ($\alpha \rightarrow \beta$ in earlier expression)

$$\frac{1}{2} [\delta(\phi - \phi') - \delta(\phi + \phi')] = \frac{1}{\beta} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\phi}{\beta}\right) \sin\left(\frac{n\pi\phi'}{\beta}\right), \quad (3.150)$$

for $-\beta < \phi < \beta$ instead of the original range for ϕ, ϕ' . Also assume

$$G(\vec{x}, \vec{x}') = \frac{8\pi}{\beta} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\phi}{\beta}\right) \sin\left(\frac{n\pi\phi'}{\beta}\right) g_n(\rho, \rho'), \quad (3.151)$$

$$\begin{aligned} \Rightarrow -\nabla^2 G(\vec{x}, \vec{x}') &= \frac{8\pi}{\beta} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\phi}{\beta}\right) \sin\left(\frac{n\pi\phi'}{\beta}\right) \\ &\quad \times \left[-\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \right) + \frac{n^2 \pi^2}{\rho^2 \beta^2} \right] g_n(\rho, \rho'), \end{aligned} \quad (3.152)$$

$$\Rightarrow \left[\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \right) - \frac{n^2 \pi^2}{\rho^2 \beta^2} \right] g_n = -\frac{1}{\rho} \delta(\rho - \rho'). \quad (3.153)$$

I guess this can still be considered Bessel's eqⁿ but for $k=0$. We know that (see for example Jackson, Section 3.7)

$$\lim_{k \rightarrow 0} J_\gamma(k\rho) \sim (k\rho)^\gamma, \quad (3.154)$$

$$\lim_{k \rightarrow 0} N_\gamma(k\rho) \sim (k\rho)^{-\gamma} \quad (\gamma \neq 0), \quad (3.155)$$

which suggests: $(\gamma \equiv \frac{n\pi}{\beta})$

$$g_n \sim \rho^\gamma, \rho^{-\gamma}. \quad (3.156)$$

We let

$$\psi_1(\rho') = \rho'^\gamma, \quad (3.157)$$

$$\psi_2(\rho') = \left(\rho'^\gamma - \frac{a^{2\gamma}}{\rho'^\gamma} \right). \quad (3.158)$$

Then

$$g_n(\rho, \rho') = C \psi_1(\rho_<) \psi_2(\rho_>), \quad (3.159)$$

One has that

$$-\left. \frac{dg_n}{d\rho} \right|_+ + \left. \frac{dg_n}{d\rho} \right|_- = \frac{1}{\rho'}, \quad (3.160)$$

gives

$$-C W[\psi_1(\rho'), \psi_2(\rho')] = \frac{1}{\rho'}. \quad (3.161)$$

Explicitly

$$\begin{aligned} W[\psi_1(\rho'), \psi_2(\rho')] \\ = \gamma \rho'^\gamma (\rho'^{\gamma-1} + \frac{a^{2\gamma}}{\rho'^{\gamma+1}}) - \gamma (\rho'^\gamma - \frac{a^{2\gamma}}{\rho'^\gamma}) \rho'^{\gamma-1} = \frac{2\gamma a^{2\gamma}}{\rho'}, \end{aligned} \quad (3.162)$$

which means that

$$C = -\frac{1}{2\gamma a^{2\gamma}}, \quad (3.163)$$

$$\Rightarrow g_n(\rho, \rho') = \frac{1}{2\gamma} \rho^{\gamma_<} \left(-\frac{\rho^{\gamma_>}}{a^{2\gamma}} + \frac{1}{\rho^{\gamma_>}} \right), \quad (3.164)$$

$$\Rightarrow G(\vec{x}, \vec{x}') = \sum_{n=1}^{\infty} \frac{4}{n} \rho^{\gamma_{<}} \left(\frac{1}{\rho^{\gamma_{>}}} - \frac{\rho^{\gamma_{>}}}{a^{2\gamma}} \right) \sin\left(\frac{n\pi\phi}{\beta}\right) \sin\left(\frac{n\pi\phi'}{\beta}\right). \quad (3.165)$$

Of course, we have that

$$G(\vec{x}, \vec{x}') = \sum_{n=1}^{\infty} \frac{4}{n} \left(\frac{\rho_{<}}{\rho_{>}} \right)^{\gamma} \sin\left(\frac{n\pi\phi}{\beta}\right) \sin\left(\frac{n\pi\phi'}{\beta}\right), \quad (3.166)$$

in the $a \rightarrow \infty$ limit, which gives the Green function for the wedge problem of Ch.2. Using this form near the origin, we have

$$G(\vec{x}, \vec{x}') \sim 4 \left(\frac{\rho}{\rho'} \right)^{\pi/\beta} \sin\left(\frac{\pi\phi}{\beta}\right) \sin\left(\frac{\pi\phi'}{\beta}\right). \quad (3.167)$$

This is of the same form as in Ch.2:

$$\Phi(\rho, \phi) = V + a_1 \rho^{\pi/\beta} \sin\left(\frac{\pi\phi}{\beta}\right). \quad (3.168)$$

↑ can evaluate by comparison

3.9 Construction of spherical harmonics: Schwinger technique

We've beat cylindrical symmetry to death. Next: spherical symmetry. Start off like cylindrical case. Got everything from

$$e^{i\vec{k} \cdot \vec{x}} = e^{ik_z z} e^{ik_{\perp} \rho \cos(\phi - \alpha)}. \quad (3.169)$$

Deceptively simple structure. Try the same thing here. Write

$$e^{i\vec{k} \cdot \vec{r}} = e^{i(\vec{a} \cdot \vec{r} + \vec{b} \cdot \vec{r})}, \quad (3.170)$$

(notice the $\vec{r}$ instead of $\vec{x}$ here) where

$$\vec{k} = \vec{a} + \vec{b}, \quad (3.171)$$

$$\Rightarrow k^2 = a^2 + b^2 + 2\vec{a} \cdot \vec{b}. \quad (3.172)$$

However, require

$$\vec{a}^2 = \vec{b}^2 = 0. \quad (3.173)$$

$\vec{a}$, $\vec{b}$ now necessarily complex. Also require

$$\vec{b} = \vec{a}^*, \quad (3.174)$$

$$\Rightarrow 2\vec{a} \cdot \vec{b} = 2|\vec{a}|^2 = \vec{k}^2. \quad (3.175)$$

Can we really do this? Count parameters:

$$\vec{a} + \vec{b} = 2\text{Re}(\vec{a}) = \vec{k}, \quad (3.176)$$

$$\left. \begin{aligned} a_1 &= \frac{1}{2} k_x + iC_1 \\ a_2 &= \frac{1}{2} k_y + iC_2 \\ a_3 &= \frac{1}{2} k_z + iC_3 \end{aligned} \right\} C_{1,2,3} \text{ real} \quad (3.177)$$

There are 3 real parameters, but $\vec{a}^2 = 0$ represents 2 conditions. $\Rightarrow$ 1 arbitrary parameter allowed. Still leaves one free to choose. Let

$$C_3 = 0. \quad (3.178)$$

Gives an explicit realization of $\vec{a}$: (verify yourself)

$$\vec{a} = \left(\frac{1}{2}(k_x - i\frac{k}{k_\perp} k_y), \frac{1}{2}(k_y + i\frac{k}{k_\perp} k_x), \frac{k_z}{2} \right), \quad (3.179)$$

where $k_\perp^2 = k_x^2 + k_y^2$, $k^2 = k_z^2 + k_\perp^2$. Now notice that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) e^{i\vec{a} \cdot \vec{r}} = -\vec{a}^2 e^{i\vec{a} \cdot \vec{r}} = 0, \quad (3.180)$$

$\Rightarrow e^{i\vec{a} \cdot \vec{r}}$ is a solⁿ of Laplace's eqⁿ. Since

$$e^{i\vec{a}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} \frac{(i\vec{a}\cdot\vec{r})^{\ell}}{\ell!} \quad (3.181)$$

each monomial $(\vec{a}\cdot\vec{r})^{\ell}$ is also a solⁿ to this eqⁿ. Let's let

$$a_1 + ia_2 \equiv 2\xi_-^2, \quad (3.182)$$

$$a_1 - ia_2 \equiv -2\xi_+^2, \quad (3.183)$$

$$a_3 = 2\xi_+\xi_- . \quad (3.184)$$

$\xi_{+,-}$ two arbitrary complex nos. 6 equations but there are 2 conditions on $\vec{a}$ from $\vec{a}^2 = 0$. Indeed

$$\begin{aligned} \vec{a}^2 &= (a_1 + ia_2)(a_1 - ia_2) + a_3^2 \\ &= -4\xi_-^2\xi_+^2 + 4\xi_+^2\xi_-^2 = 0, \end{aligned} \quad (3.185)$$

is automatic. Concentrate on $(\vec{a}\cdot\vec{r})^{\ell}$:

$$\begin{aligned} \frac{\vec{a}\cdot\vec{r}}{r} &= \frac{a_1x}{r} + \frac{a_2y}{r} + \frac{a_3z}{r} \\ &= a_1\sin\theta\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right) + a_2\sin\theta\left(\frac{e^{i\phi}-e^{-i\phi}}{2i}\right) + a_3\cos\theta \\ &= \frac{(a_1 - ia_2)}{2} \sin\theta e^{i\phi} + \frac{(a_1 + ia_2)}{2} \sin\theta e^{-i\phi} + a_3\cos\theta \\ &= -\xi_+^2\sin\theta e^{i\phi} + \xi_-^2\sin\theta e^{-i\phi} + 2\xi_+\xi_-\cos\theta. \end{aligned} \quad (3.186)$$

Therefore

$$\begin{aligned} (\vec{a}\cdot\vec{r})^{\ell} &= r^{\ell} \left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^{\ell} \\ &= r^{\ell} \left(\frac{\xi_+^2 e^{i\phi}}{\sin\theta}\right)^{\ell} \left[\left(\frac{\xi_-}{\xi_+} \sin\theta e^{-i\phi}\right)^2 + 2\frac{\xi_-}{\xi_+} \sin\theta e^{-i\phi} \cos\theta - \sin^2\theta \right]^{\ell} \\ &= r^{\ell} \left(\frac{\xi_+^2 e^{i\phi}}{\sin\theta}\right)^{\ell} \left[\left(\frac{\xi_-}{\xi_+} \sin\theta e^{-i\phi} + \cos\theta\right)^2 - 1 \right]^{\ell}. \end{aligned} \quad (3.187)$$

Do a Taylor series expansion in $\xi \equiv \frac{\xi_-}{\xi_+}$. Reminder:

$$f(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \left(\frac{d}{d\xi} \right)^n f(\xi) \Big|_{\xi=0}. \quad (3.188)$$

In this case, the highest power = 2ℓ . So

$$\begin{aligned} & [(\xi \sin\theta e^{-i\phi} + \cos\theta)^2 - 1]^\ell \\ &= \sum_{n=0}^{2\ell} \frac{\xi^n}{n!} \left(\frac{d}{d\xi} \right)^n [(\xi \sin\theta e^{-i\phi} + \cos\theta)^2 - 1]^\ell \Big|_{\xi=0}. \end{aligned} \quad (3.189)$$

Define

$$\cos\tilde{\theta} = \xi \sin\theta e^{-i\phi} + \cos\theta, \quad (3.190)$$

$$\Rightarrow \xi = \frac{e^{i\phi}}{\sin\theta} (\cos\tilde{\theta} - \cos\theta), \quad (3.191)$$

$$\Rightarrow d\xi = \frac{e^{i\phi}}{\sin\theta} d\cos\tilde{\theta}. \quad (3.192)$$

Therefore

$$\begin{aligned} & \left(\frac{d}{d\xi} \right)^n [(\xi \sin\theta e^{-i\phi} + \cos\theta)^2 - 1]^\ell \Big|_{\xi=0} \\ &= \left(\frac{\sin\theta}{e^{i\phi}} \frac{d}{d\cos\tilde{\theta}} \right)^n [\cos^2\tilde{\theta} - 1]^\ell \Big|_{\xi=0} \\ &= (\sin\theta e^{-i\phi})^n \left(\frac{d}{d\cos\tilde{\theta}} \right)^n (\cos^2\tilde{\theta} - 1)^\ell. \end{aligned} \quad (3.193)$$

So, we have

$$\begin{aligned} & [(\xi \sin\theta e^{-i\phi} + \cos\theta)^2 - 1]^\ell = \sum_{n=0}^{2\ell} \frac{(\xi \sin\theta e^{-i\phi})^n}{n!} \left(\frac{d}{d\cos\tilde{\theta}} \right)^n (\cos^2\tilde{\theta} - 1)^\ell. \\ & \hspace{25em} (3.194) \end{aligned}$$

Now define $m \equiv \ell - n$, $-\ell \leq m \leq \ell$ ($2\ell + 1$ values) and put it all together:

$$(\vec{a} \cdot \vec{r})^\ell = r^\ell \left(\frac{\xi_+^2 e^{i\phi}}{\sin\theta} \right)^\ell \sum_{m=-\ell}^{\ell} \frac{((\xi_-/\xi_+) \sin\theta e^{-i\phi})^{\ell-m}}{(\ell-m)!} \left(\frac{d}{d\cos\theta} \right)^{\ell-m} (\cos^2\theta - 1)^\ell \quad (3.195)$$

or

$$(\vec{a} \cdot \vec{r})^\ell = r^\ell 2^\ell \ell! \sum_{m=-\ell}^{\ell} \frac{\xi_+^{\ell+m} \xi_-^{\ell-m}}{\sqrt{(\ell+m)! (\ell-m)!}} \sqrt{\frac{(\ell+m)!}{(\ell-m)!}} (\sin\theta)^{-m} e^{im\phi} \times \left(\frac{d}{d\cos\theta} \right)^{\ell-m} \frac{(\cos^2\theta - 1)^\ell}{2^\ell \ell!}. \quad (3.196)$$

Define spherical harmonics:

$$Y_{\ell m}(\theta, \phi) \equiv \sqrt{\frac{(2\ell+1)(\ell+m)!}{4\pi(\ell-m)!}} e^{im\phi} (\sin\theta)^{-m} \left(\frac{d}{d\cos\theta} \right)^{\ell-m} \frac{(\cos^2\theta - 1)^\ell}{2^\ell \ell!}. \quad (3.197)$$

so

$$\begin{aligned} (\vec{a} \cdot \vec{r})^\ell &= r^\ell 2^\ell \ell! \sum_{m=-\ell}^{\ell} \frac{\xi_+^{\ell+m} \xi_-^{\ell-m}}{\sqrt{(\ell+m)! (\ell-m)!}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\theta, \phi), \\ \Rightarrow e^{i\vec{a} \cdot \vec{r}} &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell r^\ell 2^\ell \frac{\xi_+^{\ell+m} \xi_-^{\ell-m}}{\sqrt{(\ell+m)! (\ell-m)!}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}. \end{aligned} \quad (3.198)$$

Since $e^{i\vec{a} \cdot \vec{r}}$ satisfies Laplace eqⁿ: (coeff's of $\xi_+^{\ell+m} \xi_-^{\ell-m}$ vanish)

$$\Rightarrow \nabla^2 [r^\ell Y_{\ell m}] = 0, \quad (3.199)$$

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2}. \quad (3.200)$$

Now

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial r^\ell}{\partial r} \right) = \frac{\ell}{r^2} \frac{\partial}{\partial r} r^{\ell+1} = \ell(\ell+1)r^{\ell-2}, \quad (3.201)$$

$$\Rightarrow \left[\ell(\ell+1)r^{\ell-2} Y_{\ell m} + r^{\ell-2} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y_{\ell m}}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} Y_{\ell m} \right) \right] = 0. \quad (3.202)$$

$$\Rightarrow \left[\left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y_{\ell m}}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y_{\ell m}}{\partial\phi^2} \right) \right] = -\ell(\ell+1)Y_{\ell m}. \quad (3.203)$$

Notice also $\frac{\partial^2}{\partial\phi^2} Y_{\ell m} = -m^2 Y_{\ell m},$

$$\Rightarrow \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y_{\ell m}}{\partial\theta} \right) - \frac{m^2}{\sin^2\theta} Y_{\ell m} \right] = -\ell(\ell+1)Y_{\ell m}. \quad (3.204)$$

$Y_{\ell m}$ is the solution of the separated Laplace eqⁿ in spherical coordinates. There is an alternate form for $Y_{\ell m}$. Remember

$$(\vec{a} \cdot \vec{r})^\ell = r^\ell [-\xi_+^2 \sin\theta e^{i\phi} + \xi_-^2 \sin\theta e^{-i\phi} + 2\xi_+ \xi_- \cos\theta]^\ell. \quad (3.205)$$

It is unchanged under:

$$\xi_+ \leftrightarrow \xi_-, \quad \theta \rightarrow -\theta, \quad \phi \rightarrow -\phi.$$

Implies that

$$\begin{aligned} \sum_{m=-\ell}^{\ell} \frac{\xi_+^{\ell+m} \xi_-^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\theta, \phi) \\ = \sum_{m'=-\ell}^{\ell} \frac{\xi_-^{\ell+m'} \xi_+^{\ell-m'}}{\sqrt{(\ell+m')!(\ell-m')!}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m'}(-\theta, -\phi). \end{aligned} \quad (3.206)$$

Coeff's of like powers must be the same:

$$\Rightarrow Y_{\ell m}(\theta, \phi) = Y_{\ell -m}(-\theta, -\phi). \quad (3.207)$$

So

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} e^{im\phi} (-\sin\theta)^m \left(\frac{d}{d\cos\theta}\right)^{\ell+m} \frac{(\cos^2\theta-1)^\ell}{2^\ell \ell!}. \quad (3.208)$$

Usual defⁿ for "Legendre polyⁿs":

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell,$$

$$\Rightarrow Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} e^{im\phi} (-\sin\theta)^m \left(\frac{d}{d\cos\theta}\right)^m P_\ell(\cos\theta) \quad (m \geq 0) \quad (3.209)$$

$$\Rightarrow Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell+m)!}{(\ell-m)!}} e^{im\phi} (\sin\theta)^{-m} \left(\frac{d}{d\cos\theta}\right)^{-m} P_\ell(\cos\theta) \quad (m \leq 0) \quad (3.210)$$

Can only use the top form when $m \geq 0$ and bottom when $m \leq 0$.

Combine them:

$$Y_{\ell m}(\theta, \phi) = (-1)^{(m+|m|)/2} \sqrt{\frac{(2\ell+1)}{4\pi}} \sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}} e^{im\phi} \times (\sin\theta)^{|m|} \left(\frac{d}{d\cos\theta}\right)^{|m|} P_\ell(\cos\theta). \quad (3.211)$$

From the last form we can read off that

$$Y_{\ell m}(\theta, \phi) = (-1)^m Y_{\ell -m}(\theta, -\phi). \quad (3.212)$$

But since

$$Y_{\ell m}^*(\theta, \phi) = Y_{\ell m}(\theta, -\phi), \quad (3.213)$$

this gives

$$Y_{\ell m}^*(\theta, \phi) = (-1)^m Y_{\ell -m}(\theta, \phi). \quad (3.214)$$

We also have that (get diff. eqⁿ for $P_\ell(\cos\theta)$ by setting $m=0$ in $Y_{\ell m}$ eqⁿ)

$$Y_{\ell 0}(\theta, \phi) = \sqrt{\frac{(2\ell+1)}{4\pi}} P_\ell(\cos\theta). \quad (3.215)$$

3.10 Orthogonality properties of spherical harmonics

All this started with an exponential. Go back to it:

$$\int d\Omega e^{i\vec{k} \cdot \vec{r}} = \int d\Omega e^{i\vec{a}^* \cdot \vec{r} + \vec{a} \cdot \vec{r}} = \sum_{\ell_1, \ell_2=0}^{\infty} i^{\ell_1} i^{\ell_2} \int d\Omega \frac{(\vec{a}^* \cdot \vec{r})^{\ell_1}}{\ell_1!} \frac{(\vec{a} \cdot \vec{r})^{\ell_2}}{\ell_2!}. \quad (3.216)$$

Use our expansion:

$$\begin{aligned} \int d\Omega e^{i\vec{k} \cdot \vec{r}} &= \sum_{\ell_1, \ell_2=0}^{\infty} i^{\ell_1} i^{\ell_2} \sum_{m_1=-\ell_1}^{\ell_1} \sum_{m_2=-\ell_2}^{\ell_2} r^{\ell_1} r^{\ell_2} 2^{\ell_1} 2^{\ell_2} \frac{\xi_+^{*\ell_1+m_1} \xi_-^{*\ell_1-m_1}}{\sqrt{(\ell_1+m_1)! (\ell_1-m_1)!}} \\ &\times \frac{\xi_+^{\ell_2+m_2} \xi_-^{\ell_2-m_2}}{\sqrt{(\ell_2+m_2)! (\ell_2-m_2)!}} \sqrt{\frac{4\pi}{2\ell_1+1}} \sqrt{\frac{4\pi}{2\ell_2+1}} \int d\Omega Y_{\ell_1 m_1}^*(\theta, \phi) Y_{\ell_2 m_2}(\theta, \phi). \end{aligned} \quad (3.217)$$

On the other hand

$$\int d\Omega e^{i\vec{k} \cdot \vec{r}} = 2\pi \int_0^\pi d\theta \sin\theta e^{ikr \cos\theta} = \frac{4\pi}{kr} \sin(kr). \quad (3.218)$$

We can write

$$\frac{4\pi \sin(kr)}{kr} = \frac{4\pi}{kr} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(kr)^{2\ell+1}}{(2\ell+1)!} = 4\pi \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(r^2 k^2)^\ell}{(2\ell+1)!}. \quad (3.219)$$

But $\vec{k}^2 = 2\vec{a}^* \cdot \vec{a}$,

$$\Rightarrow \int d\Omega e^{i\vec{k} \cdot \vec{r}} = 4\pi \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{2^{\ell} (r^2)^{\ell}}{(2\ell+1)!} (\vec{a}^* \cdot \vec{a})^{\ell}. \quad (3.232)$$

But again

$$\begin{aligned} (\vec{a}^* \cdot \vec{a})^{\ell} &= 2^{\ell} (\xi_+^* \xi_+ + \xi_-^* \xi_-)^{2\ell} \\ &= 2^{\ell} \sum_{m'=0}^{2\ell} \frac{2\ell!}{(2\ell-m')! m'!} (\xi_+^* \xi_+)^{2\ell-m'} (\xi_-^* \xi_-)^{m'}. \end{aligned} \quad (3.221)$$

Let $m=\ell-m'$

$$(\vec{a}^* \cdot \vec{a})^{\ell} = 2^{\ell} \sum_{m=-\ell}^{\ell} \frac{2\ell!}{(\ell+m)! (\ell-m)!} (\xi_+^* \xi_+)^{\ell+m} (\xi_-^* \xi_-)^{\ell-m}, \quad (3.222)$$

so

$$\int d\Omega e^{i\vec{k} \cdot \vec{r}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} (-1)^{\ell} (2^{\ell} r^{\ell})^2 \frac{(\xi_+^* \xi_+)^{\ell+m} (\xi_-^* \xi_-)^{\ell-m}}{(\ell+m)! (\ell-m)!}. \quad (3.223)$$

Comparing (3.217) & (3.223), we see that in (3.223) the coeff. of the various ξ 's gives us information about the integral in (3.217). Zero unless:

$$\left. \begin{aligned} \xi_+^* &: \ell_1 + m_1 = \ell + m \\ \xi_-^* &: \ell_1 - m_1 = \ell - m \end{aligned} \right\} \Rightarrow \ell_1 = \ell, m_1 = m \quad (3.224)$$

$$\left. \begin{aligned} \xi_+ &: \ell_2 + m_2 = \ell + m \\ \xi_- &: \ell_2 - m_2 = \ell - m \end{aligned} \right\} \Rightarrow \ell_2 = \ell, m_2 = m \quad (3.225)$$

$\Rightarrow$ only non-zero terms are when $\ell_1 = \ell_2 = \ell$, $m_1 = m_2 = m$. Therefore

$$\int d\Omega Y_{\ell_1 m_1}^*(\theta, \phi) Y_{\ell_2 m_2}(\theta, \phi) = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}. \quad (3.226)$$

Since we know that $Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}} P_{\ell}(\cos\theta)$, we also have
 $(m_1=m_2=0)$

$$\int_{-1}^1 d(\cos\theta) P_{\ell}(\cos\theta) P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1} \delta_{\ell\ell'}. \quad (3.227)$$

Now consider $(\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'))$

$$\begin{aligned} \frac{1}{|\vec{r}-\vec{r}'|} &= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \frac{1}{r \sqrt{1 + (\frac{r'}{r})^2 - 2(\frac{r'}{r})\cos\gamma}} \\ &= \frac{1}{r' \sqrt{1 + (\frac{r}{r'})^2 - 2(\frac{r}{r'})\cos\gamma}}. \end{aligned} \quad (3.228)$$

Do a Taylor series expansion in $x = \frac{r_{<}}{r_{>}}$:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_{<}^{\ell}}{r_{>^{\ell+1}}} \frac{1}{\ell!} \left(\frac{d}{dx}\right)^{\ell} \frac{1}{\sqrt{1+x^2-2x\cos\gamma}} \Big|_{x=0}. \quad (3.229)$$

Define

$$Q_{\ell}(\cos\gamma) \equiv \frac{1}{\ell!} \left(\frac{d}{dx}\right)^{\ell} \frac{1}{\sqrt{1+x^2-2x\cos\gamma}} \Big|_{x=0}. \quad (3.230)$$

Clear that $Q_{\ell}(\cos\gamma)$ is a polynomial in $\cos\gamma$ of degree ℓ . Can show $Q_{\ell}(\cos\gamma) = P_{\ell}(\cos\gamma)$!

3.11 The Coulomb expansion, completeness of spherical harmonics and the "Addition Theorem"

Cushing argument ("Applied Analytical Mathematics for Physical Scientists", p.172): $(|\mu|, |v| < 1)$

$$I(\mu, v) = \int_{-1}^1 \frac{dx}{\sqrt{1-2\mu x + \mu^2} \sqrt{1-2vx + v^2}} \quad (3.231)$$

Direct, but tedious (see for example Gradshteyn & Ryzhik, (2.261)):

$$\begin{aligned} I(\mu, \nu) &= \frac{1}{\sqrt{\mu\nu}} \ln \left(\frac{1 + \sqrt{\mu\nu}}{1 - \sqrt{\mu\nu}} \right) \\ &= 2 \sum_{\ell=0}^{\infty} \frac{(\mu\nu)^{\ell}}{(2\ell+1)}. \end{aligned} \quad (3.232)$$

On the other hand

$$\frac{1}{\sqrt{1 - 2\mu x + \mu^2}} = \sum_{\ell=0}^{\infty} \mu^{\ell} Q_{\ell}'(x), \quad (3.233)$$

$$\frac{1}{\sqrt{1 - 2\nu x + \nu^2}} = \sum_{\ell=0}^{\infty} \nu^{\ell} Q_{\ell}(x). \quad (3.234)$$

Comparison shows

$$\int_{-1}^1 dx Q_{\ell}(x) Q_{\ell}'(x) = \frac{2}{2\ell+1} \delta_{\ell\ell'}. \quad (3.235)$$

Now

$$P_n(x) = \sum_{m=0}^n \alpha_{nm} Q_m(x), \quad (3.236)$$

since n, m are the order of the polynomials $P_n(x)$ and $Q_m(x)$.

$$\Rightarrow \int_{-1}^1 dx P_n(x) Q_r(x) = \frac{2}{2r+1} \alpha_{nr}, \quad r=0, 1, 2, \dots, n. \quad (3.237)$$

However

$$\begin{aligned} \int_{-1}^1 dx P_n(x) x^m &= \frac{1}{2^n n!} \int_{-1}^1 dx \frac{d^n [(-1+x^2)^n]}{dx^n} x^m \\ &\stackrel{\text{by parts}}{=} 0, \quad m < n. \end{aligned} \quad (3.238)$$

$$\Rightarrow \alpha_{nr} = 0 \quad \forall \quad r < n, \quad (3.239)$$

$$\Rightarrow \alpha_{nn} Q_n(x) = P_n(x). \quad (3.240)$$

Set $x=1$ on both sides:

$$P_\ell(1) = 1. \quad (3.241)$$

On the other hand

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r^{<\ell}}{r^{>\ell+1}} Q_\ell(\cos\gamma). \quad (3.242)$$

If $\cos\gamma = 1$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{|r-r'|}. \quad (3.243)$$

But

$$\frac{1}{|r-r'|} = \sum_{\ell=0}^{\infty} \frac{r^{<\ell}}{r^{>\ell+1}} \Rightarrow Q_\ell(1) = 1, \quad (3.244)$$

$$\Rightarrow P_\ell(x) = Q_\ell(x). \quad (3.245)$$

So

$$\frac{1}{|\vec{r}-\vec{r}'|} \stackrel{\text{"Coulomb expansion"}}{=} \sum_{\ell=0}^{\infty} \frac{r^{<\ell}}{r^{>\ell+1}} P_\ell(\cos\gamma). \quad (3.246)$$

Now we know that

$$\nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\delta(\vec{r}-\vec{r}'). \quad (3.247)$$

Substituting (3.246) in (3.247) and integrating across $r=r'$ gives

$$\left[-r^2 \frac{\partial}{\partial r} \sum_{\ell=0}^{\infty} \frac{r^{<\ell}}{r^{>\ell+1}} P_{\ell}(\cos\gamma) \right] \Big|_{r'-\varepsilon}^{r'+\varepsilon} = 4\pi\delta(\cos\theta-\cos\theta')\delta(\phi-\phi'), \quad (3.248)$$

which results in

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} P_{\ell}(\cos\gamma) = \delta(\cos\theta-\cos\theta')\delta(\phi-\phi'). \quad (3.249)$$

This is actually a statement of completeness of the $Y_{\ell m}(\theta, \phi)$.

To see why, consider the following. For $\vec{r} \neq \vec{r}'$, $\frac{1}{|\vec{r}-\vec{r}'|}$ is a

solⁿ to Laplace's eqⁿ. The only linearly indep. sol^{ns} to Laplace's eqⁿ of the form r^{ℓ} x something(θ, ϕ) are:

$$r^{\ell} Y_{\ell m}(\theta, \phi) \quad (2\ell+1 \text{ of them}).$$

Likewise, the only linearly indep. sol^{ns} of the form $r^{-\ell-1}$ x something(θ, ϕ) are:

$$r^{-\ell-1} Y_{\ell m}(\theta, \phi) \quad (2\ell+1 \text{ of them}).$$

Therefore, considered as functions of r, θ, ϕ we must have

$$r < r': \quad \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell, m} \frac{r^{\ell}}{r'^{-\ell-1}} Y_{\ell m}(\theta, \phi) A_{\ell m}, \quad (3.250)$$

$$r > r': \quad \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell, m} \frac{r'^{\ell}}{r^{-\ell-1}} Y_{\ell m}(\theta, \phi) B_{\ell m}. \quad (3.251)$$

Continuity across $r=r'$ demands $A_{\ell m} = B_{\ell m}$, giving the unified form

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell, m} \frac{r^{<\ell}}{r^{>\ell+1}} A_{\ell m} Y_{\ell m}(\theta, \phi). \quad (3.252)$$

Comparison with (3.246) just shows the $P_\ell(\cos\gamma)$, considered as a function of the r, θ, ϕ , must be a linear combination of the form,

$$P_\ell(\cos\gamma) = \sum_m A_{\ell m} Y_{\ell m}(\theta, \phi). \quad (3.253)$$

Substitution above in (3.249) gives

$$\sum_{\ell, m} \frac{(2\ell+1)}{4\pi} A_{\ell m} Y_{\ell m}(\theta, \phi) = \delta(\cos\theta - \cos\theta') \delta(\phi - \phi'). \quad (3.254)$$

Using orthonormality $\int d\Omega Y_{\ell m}(\theta, \phi) Y_{\ell' m'}^*(\theta, \phi) = \delta_{\ell\ell'} \delta_{mm'}$,

$$\Rightarrow A_{\ell m} = \frac{4\pi}{2\ell+1} Y_{\ell m}^*(\theta', \phi'). \quad (3.255)$$

Putting this in (3.254) gives us completeness for sph. harmonics:

$$\sum_{\ell, m} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi') = \delta(\cos\theta - \cos\theta') \delta(\phi - \phi'), \quad (3.256)$$

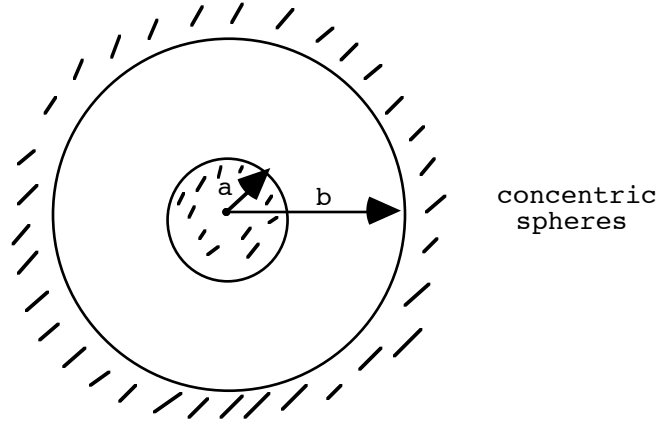
$$\Rightarrow \frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell, m} \frac{r < \ell}{r > \ell+1} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'). \quad (3.257)$$

In addition, (3.253) gives us the "addition theorem":

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'). \quad (3.258)$$

3.12 Green function for concentric spheres

Now do this problem:



Will get special results for $a \rightarrow 0$, $b \rightarrow \infty$ limits. We have

$$4\pi\delta(\vec{r}-\vec{r}') = 4\pi\sum_{\ell,m} Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi) \frac{1}{r^2}\delta(r-r'), \quad (3.259)$$

$$G(\vec{r},\vec{r}') = 4\pi\sum_{\ell,m} Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi) g_{\ell}(r,r'), \quad (3.260)$$

$$\Rightarrow -\vec{\nabla}^2 G(\vec{r},\vec{r}') = 4\pi\sum_{\ell,m} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi)$$

$$\left(-\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial g_{\ell}}{\partial r}) + \frac{\ell(\ell+1)}{r^2} g_{\ell} \right). \quad (3.261)$$

Eqn will be satisfied if

$$-\frac{\partial}{\partial r} (r^2 \frac{\partial g_{\ell}}{\partial r}) + \ell(\ell+1)g_{\ell} = \delta(r-r'). \quad (3.262)$$

When $r \neq r'$ solutions are

$$g_{\ell} \sim r^{\ell}, r^{-\ell-1}, \quad r \neq r'.$$

Therefore

$$g_{\ell} = C(r_{<}^{\ell} - \frac{a^{2\ell+1}}{r_{<}^{\ell+1}}) (r_{>}^{\ell} - \frac{b^{2\ell+1}}{r_{>}^{\ell+1}}). \quad (3.263)$$

Since

$$-r^2 \frac{\partial g_\ell}{\partial r} \Big|_{-}^{+} = 1, \quad (3.264)$$

we have

$$\begin{aligned} -C_{r',2} \left[\left(\ell r'^{\ell-1} + (\ell+1) \frac{b^{2\ell+1}}{r'^{\ell+2}} \right) \left(r'^\ell - \frac{a^{2\ell+1}}{r'^{\ell+1}} \right) \right. \\ \left. - \left(\ell r'^{\ell-1} + (\ell+1) \frac{a^{2\ell+1}}{r'^{\ell+2}} \right) \left(r'^\ell - \frac{b^{2\ell+1}}{r'^{\ell+1}} \right) \right] = 1, \\ \Rightarrow -C_{r',2} \left[\frac{a^{2\ell+1}}{r'^2} (-\ell - (\ell+1)) + \frac{b^{2\ell+1}}{r'^2} ((\ell+1) + \ell) \right] = 1, \quad (3.265) \end{aligned}$$

$$\Rightarrow C = \frac{1}{(2\ell+1) [a^{2\ell+1} - b^{2\ell+1}]}. \quad (3.266)$$

$$\Rightarrow g_\ell(r, r') = \frac{1}{(2\ell+1) \left[1 - \left(\frac{a}{b} \right)^{2\ell+1} \right]} \left(r_{<}^\ell - \frac{a^{2\ell+1}}{r_{<}^{\ell+1}} \right) \left(\frac{1}{r_{>}^{\ell+1}} - \frac{r_{>}^\ell}{b^{2\ell+1}} \right). \quad (3.267)$$

Examine some special cases:

1. $a \rightarrow 0$, $b \rightarrow \infty$. (Free Green's function)

$$-\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell, m} \frac{r_{<}^\ell}{r_{>}^{\ell+1}} \frac{4\pi}{2\ell+1} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi). \quad (3.268)$$

Compare this with

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell} \frac{r_{<}^\ell}{r_{>}^{\ell+1}} P_{\ell}(\cos \gamma), \quad (3.269)$$

get ("addition theorem" again)

$$P_\ell(\cos \gamma) = \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi). \quad (3.270)$$

2. $a \rightarrow 0$, b finite (Interior sphere)

$$G(\vec{r}, \vec{r}') \rightarrow \sum_{\ell, m} \left(\frac{r_{<}^\ell}{r_{>}^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}} \right) \frac{4\pi}{2\ell+1} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi). \quad (3.271)$$

Notice that

$$\frac{r^\ell r'^\ell}{b^{2\ell+1}} = \frac{b}{r'} \frac{r^\ell}{\left(\frac{b^2}{r'}\right)^{\ell+1}} = \frac{b}{r'} \frac{r^\ell}{\bar{r}_b^{\ell+1}} \quad (\bar{r}_b > r), \quad (3.272)$$

so we can perform the sums by the above result

$$G(\vec{r}, \vec{r}') \rightarrow \frac{1}{|\vec{r} - \vec{r}'|} - \frac{b}{r'} \frac{1}{|\vec{r} - \vec{\bar{r}}_b|}, \quad (3.273)$$

where

$$\vec{\bar{r}}_b = \left(\frac{b^2}{r'}, \theta', \phi' \right). \quad (3.274)$$

locates the image charge. (outside sphere)

3. $b \rightarrow \infty$, a finite (outside sphere)

$$G(\vec{r}, \vec{r}') \rightarrow \sum_{\ell, m} \left(\frac{r_{<}^\ell}{r_{>}^{\ell+1}} - \frac{a^{2\ell+1}}{r^{\ell+1} r'^{\ell+1}} \right) \frac{4\pi}{2\ell+1} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi). \quad (3.275)$$

As before

$$\frac{a^{2\ell+1}}{r^{\ell+1} r'^{\ell+1}} = \frac{a}{r'} \frac{\left(\frac{a^2}{r'}\right)^\ell}{r^{\ell+1}} = \frac{a}{r'} \frac{\bar{r}_a^\ell}{r^{\ell+1}} \quad (\bar{r}_a < r), \quad (3.276)$$

$$G(\vec{r}, \vec{r}') \rightarrow \frac{1}{|\vec{r} - \vec{r}'|} - \frac{a}{r'} \frac{1}{|\vec{r} - \vec{r}_a|}, \quad (3.277)$$

where

$$\vec{r}_a = \left(\frac{a^2}{r'}, \theta', \phi' \right). \quad (3.278)$$

is the image charge (inside sphere).

We can see the surface delta function in the expansion,

$$\begin{aligned} \frac{\partial G}{\partial n'} \Big|_{r'=a} = - \frac{\partial G}{\partial r'} \Big|_{r'=a} = - \sum_{\ell, m} \left(\frac{\ell a^{\ell-1}}{r^{\ell+1}} + (\ell+1) \frac{a^{2\ell+1}}{r^{\ell+1} a^{\ell+2}} \right) \\ \times \frac{4\pi}{2\ell+1} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi), \end{aligned} \quad (3.279)$$

$$= -4\pi \sum_{\ell, m} \frac{a^{\ell-1}}{r^{\ell+1}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi), \quad (3.280)$$

$$\Rightarrow \lim_{r \rightarrow a^+} - \frac{1}{4\pi} \frac{\partial G}{\partial n'} \Big|_{r'=a} = a^{-2} \delta(\cos\theta - \cos\theta') \delta(\phi - \phi'). \quad (3.281)$$

We now see it explicitly (compare with Eq.(1.89)).

3.13 Conducting sphere in a uniform field

Of course, can do B.V. problems in spherical coordinates as well. Do the same problem as in Ch.2: sphere in a uniform field. Start:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0. \quad (3.282)$$

Assume

$$\Phi = R(r) P(\theta) Q(\phi). \quad (3.283)$$

Get

$$PQ \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{RQ}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta} \right) + \frac{RP}{\sin^2\theta} \frac{d^2 Q}{d\phi^2} = 0. \quad (3.284)$$

Multiply by $\sin^2\theta/\Phi$:

$$\frac{\sin^2\theta}{R} \left[\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) \right] + \frac{\sin\theta}{P} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta} \right) + \frac{1}{Q} \frac{d^2 Q}{d\phi^2} = 0, \quad (3.285)$$

$$\Rightarrow \frac{1}{Q} \frac{d^2 Q}{d\phi^2} = -m^2, \quad (3.286)$$

$$\Rightarrow Q = e^{\pm im\phi}. \quad (3.287)$$

If full range $0 \leq \phi \leq 2\pi$ is allowed, must have $m=0, \pm 1, \pm 2, \dots$. Now have:

$$\underbrace{\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right)}_{r \text{ only}} + \underbrace{\frac{1}{P \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta} \right) - \frac{m^2}{\sin^2\theta}}_{\theta \text{ only}} = 0 \quad (3.288)$$

$$\Rightarrow \frac{1}{P \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta} \right) - \frac{m^2}{\sin^2\theta} = -\ell(\ell+1)$$

or

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta} \right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta} \right] P = 0, \quad (3.289)$$

and

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = \ell(\ell+1),$$

$$\Rightarrow \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - \ell(\ell+1)R = 0. \quad (3.290)$$

Solutions:

"Legendre funct. of the 2nd kind";
infinite at end points, $\theta=0, \pi$.

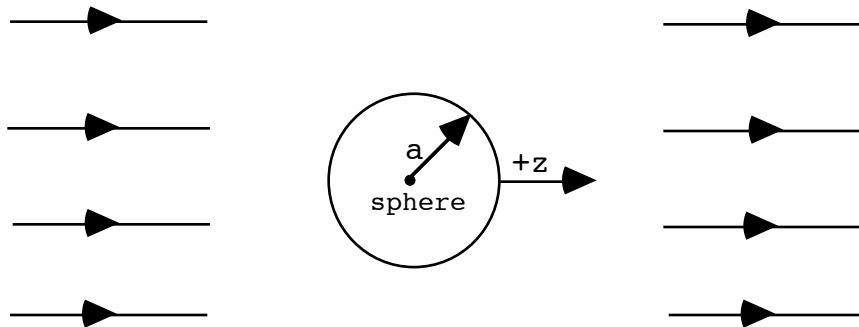
↙

$P \sim P_{\ell}^m(\cos\theta), Q_{\ell}^m(\cos\theta)$

$$R(r) \sim r^{\ell}, r^{-\ell-1}.$$

Both with $\ell = 0, 1, 2, \dots$

Simple problem:



General solution:

$$\Phi(r, \theta) = \sum_{\ell=0}^{\infty} A_{\ell} r^{\ell} P_{\ell}(\cos\theta) + \sum_{\ell=0}^{\infty} B_{\ell} r^{-\ell-1} P_{\ell}(\cos\theta). \quad (3.291)$$

B.C.: uniform electric field in +z direction of magnitude E_0 .
No ϕ dep. $\Rightarrow m=0$ only

$$\text{B.C.: } 1. \Phi(r, \theta) \Big|_{r \rightarrow \infty} = -E_0 z, \quad (3.292)$$

$$2. \Phi(r, \theta) \Big|_{r=a} = V_0, \quad (V_0=0 \text{ if grounded}) \quad (3.293)$$

$$P_0(\cos\theta) = 1, \quad P_1(\cos\theta) = \cos\theta,$$

$$1. \Rightarrow A_1 = -E_0, \quad A_{\ell} = 0, \quad \ell > 1.$$

Charge on sphere:

$$Q = \int d\Omega \sigma = -\frac{1}{4\pi} \oint \left. \frac{\partial \Phi}{\partial r} \right|_{r=a} d\Omega = \frac{(\ell+1)}{4\pi} \sum_{\ell=0}^{\infty} B_{\ell} a^{-\ell-2} \underbrace{\int d\Omega P_{\ell}}_{4\pi \delta_{\ell 0}}$$

$$\Rightarrow B_0 = Q.$$

$$2. \Rightarrow A_0 - E_0 a P_1(\cos\theta) + \sum_{\ell=0}^{\infty} B_{\ell} a^{-\ell-1} P_{\ell}(\cos\theta) = V_0,$$

so

$$A_0 = V_0 - \frac{Q}{a}, \quad \ell=0,$$

$$B_1 = E_0 a^3, \quad \ell=1,$$

$$B_{\ell} = 0, \quad \ell > 1.$$

Determines

$$\Phi(r, \theta) = V_0 - \frac{Q}{a} + \frac{Q}{r} - E_0 \left(r - \frac{a^3}{r^2} \right) \cos\theta. \quad (3.294)$$

Notice that V_0 and Q are independent parameters here. A grounded sphere ($V_0 = 0$) in this case does not necessarily have zero charge because the potential is not required to vanish at infinity.

3.14 Method of last resort: eigenfunction expansions

Method of last resort: Eigenfunction expansion. Consider a more general eqⁿ than we usually solve.

$$\nabla^2 \psi(\mathbf{x}) + k^2 \psi(\mathbf{x}) = 0. \quad (3.295)$$

(# of dims. is arbitrary.) Given a certain confined geometry, allows only discrete values of k^2 :

$$\nabla^2 \psi_n(\mathbf{x}) + k_n^2 \psi_n(\mathbf{x}) = 0. \quad (3.296)$$

Normalized as

$$\int_V \psi_n^* \psi_m d^3x = \delta_{nm}. \quad (3.297)$$

Assume these are complete. Thus

$$G(\vec{x}, \vec{x}') = \sum_n a_n \psi_n^*(\vec{x}') \psi_n(\vec{x}), \quad (3.298)$$

where we have enforced the symmetry $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$ as usual. $G(\vec{x}, \vec{x}')$ solves

$$\vec{\nabla}^2 G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}'), \quad (3.299)$$

so

$$-\sum_n a_n \psi_n^*(\vec{x}') \psi_n(\vec{x}) k_n^2 = -4\pi \delta(\vec{x} - \vec{x}'). \quad (3.300)$$

Mult. by ψ_m^* and integrate over V :

$$a_n \psi_n^*(\vec{x}') k_n^2 = 4\pi \psi_n^*(\vec{x}'),$$

$$\Rightarrow a_n = \frac{4\pi}{k_n^2}, \quad (3.301)$$

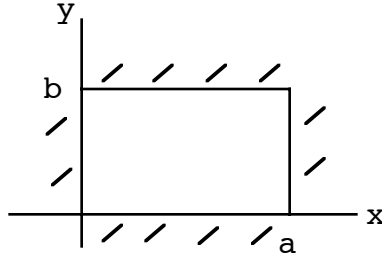
$$\Rightarrow G(\vec{x}, \vec{x}') = 4\pi \sum_n \frac{\psi_n^*(\vec{x}') \psi_n(\vec{x})}{k_n^2}. \quad (3.302)$$

Trivial example. Solve

$$\vec{\nabla}^2 \psi(\mathbf{x}) + k^2 \psi(\mathbf{x}) = 0, \quad (3.303)$$

in a 2D box with $\psi|_s = 0$. (Will give us the Dirichlet G.F.)
Geometry:

3.46



Solve

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \psi + k^2 \psi = 0. \quad (3.304)$$

The B.C. are

$$\psi(0, y) = 0, \quad (3.305)$$

$$\psi(x, 0) = 0. \quad (3.306)$$

Obviously separable. Get $(\psi(x, y) = \psi_x(x) \psi_y(y))$

$$\frac{1}{\psi_x} \frac{d^2 \psi_x}{dx^2} + \frac{1}{\psi_y} \frac{d^2 \psi_y}{dy^2} = -k^2, \quad (3.307)$$

$$\Rightarrow \frac{d^2 \psi_x}{dx^2} = -k_x^2 \psi_x, \quad (3.308)$$

$$\frac{d^2 \psi_y}{dy^2} = -k_y^2 \psi_y. \quad (3.309)$$

where

$$k_x^2 + k_y^2 = k^2. \quad (3.310)$$

Look at the x-direction,

$$\psi(x) = C_1 \sin k_x x + C_2 \cos k_x x,$$

$$\text{B.C.: } 0 = C_1 \sin k_x a,$$

$$\Rightarrow k_x a = \ell \pi, \ell=1,2,3,\dots$$

Similarly

$$k_y b = m \pi, m=1,2,3,\dots$$

Correctly normalized eigenfunction is

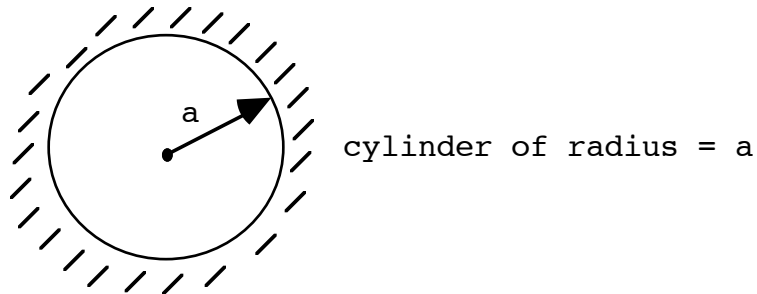
$$\psi_{nm}(\vec{x}) = \sqrt{\frac{4}{ab}} \sin\left(\frac{\ell \pi x}{a}\right) \sin\left(\frac{m \pi y}{b}\right). \quad (3.311)$$

Plug in:

$$G(\vec{x}, \vec{x}') = \frac{16}{\pi ab} \sum_{\ell, m=1}^{\infty} \frac{\sin\left(\frac{\ell \pi x}{a}\right) \sin\left(\frac{\ell \pi x'}{a}\right) \sin\left(\frac{m \pi y}{b}\right) \sin\left(\frac{m \pi y'}{b}\right)}{\left\{ \frac{\ell^2}{a^2} + \frac{m^2}{b^2} \right\}} \quad (3.312)$$

A double infinite series! (i.e., almost useless, except for theorists!)

One more problem:



Want to solve

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + k^2 \psi = 0. \quad (3.313)$$

The B.C. is $(\psi = \psi(\rho, \phi))$

$$\psi(a, \phi) = 0. \quad (3.314)$$

(Also that $\psi(\rho, \phi)$ is single-valued in ϕ .) Assume separable,
get ($\psi = \psi_\rho \psi_\phi$; mult. by $\frac{\rho^2}{\psi}$)

$$\rho^2 \left[\frac{1}{\psi_\rho \rho} \frac{d}{d\rho} \left(\rho \frac{d\psi_\rho}{d\rho} \right) + k^2 \right] + \underbrace{\frac{1}{\psi_\phi} \frac{d^2 \psi_\phi}{d\phi^2}}_{-m^2} = 0, \quad (3.315)$$

$$\Rightarrow \frac{d^2 \psi_\phi}{d\phi^2} = -m^2 \psi_\phi,$$

$$\psi_\phi = \frac{1}{\sqrt{2\pi}} e^{im\phi},$$

Also

$$\Rightarrow \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d\psi_\rho}{d\rho} \right) + \left(k^2 - \frac{m^2}{\rho^2} \right) \psi_\rho = 0. \quad (3.316)$$

$$\psi_\rho \sim J_{1m}(k_{mn}\rho), \quad N_m(k_{mn}\rho),$$

$$\left(J_{1m}(k_{mn}\rho) = \frac{\sqrt{2}}{a} \frac{J_m(k_{mn}\rho)}{J_{m+1}(x_{mn})} \right)$$

where

$$k_{mn} = \frac{x_{mn}}{a}, \quad J_m(x_{mn}) = 0. \quad (3.317)$$

Correctly normalized eigenfunction with our B.C.'s::

$$\psi_{mn}(\vec{x}) = J_{1m}(k_{mn}\rho) \frac{1}{\sqrt{2\pi}} e^{im\phi}. \quad (3.318)$$

$$(n=1, 2, 3, \dots \text{ and } m=0, \pm 1, \pm 2, \pm 3, \dots)$$

Plug in:

$$G_D(\vec{x}, \vec{x}') = \frac{a^2}{\pi} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{J_{1m}(k_{mn}\rho) J_{1m}(k_{mn}\rho') e^{im(\phi - \phi')}}{x_{mn}^2}. \quad (3.319)$$

Comparison of such eigenfunction solutions with other known solutions allows interesting summation formulas to be derived. For example, comparison of (3.312) with a reduced Green function solution gives a formula for one of the sums there, whereas comparison of (3.319) with an image solution for the same problem (prob. 2.17) gives a formula for the sum there.

Problems

3.1.1 By evaluating the expression (the “ $\perp$ ” notation indicates the x and y components of a vector; “ α ” is the cylindrical angle associated with $\vec{k}$; see below),

$$\int_0^{2\pi} \frac{d\alpha}{(2\pi)^2} e^{i\vec{k}_\perp \cdot (\vec{x} - \vec{x}')_\perp}$$

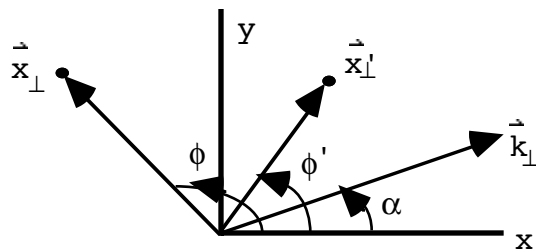
two different ways (or any other way you can think of), show the Bessel function “addition theorem”,

$$J_0(kD) = \sum_m J_m(k\rho) J_m(k\rho') e^{im(\phi - \phi')},$$

where

$$D = |(\vec{x} - \vec{x}')_\perp|, \quad \rho = |\vec{x}_\perp|, \quad \rho' = |\vec{x}'_\perp|.$$

The picture for this is:



3.1.2 Using the generating function for Bessel functions,

$$e^{it \cos \phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(t),$$

prove the sum rules:

$$(a) \sum_{m=-\infty}^{\infty} J_m^2(t) = 1, \quad (b) J_0(2t) = \sum_{m=-\infty}^{\infty} (-1)^m J_m^2(t).$$

3.1.3 Show that

$$(a) \quad \int_0^a d\rho \, \rho^{m+1} J_m(k\rho) = \frac{1}{k} a^{m+1} J_{m+1}(ka),$$

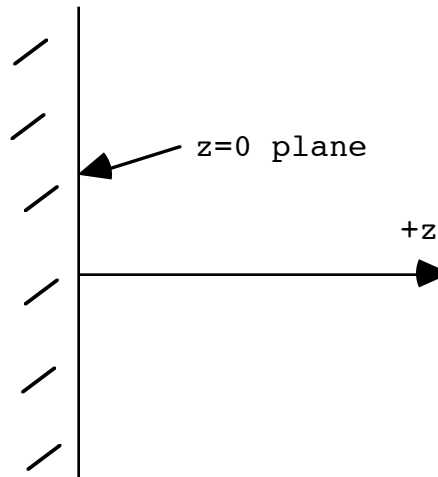
$$(b) \quad \int_0^\infty \frac{dk k}{(2\pi)} J_0(kD) = \delta(x-x') \delta(y-y'), \quad (D^2 \equiv (x-x')^2 + (y-y')^2).$$

3.3.1 Show that for zeros of derivatives of Bessel functions ($\bar{k} \equiv k_{mn}$, $\bar{y} \equiv y_{mn} = \bar{k}a$)

$$\int_0^{\bar{a}} d\rho \rho J_m^2(\bar{k}\rho) = \frac{a^2}{2} \left(1 - \frac{m^2}{\bar{y}^2}\right) J_m^2(\bar{y}).$$

Use the method described in the notes after (3.56).

3.4.1 Using the reduced Green function technique, show that the Dirichlet Green function for the infinite conducting wall problem,



may be solved in cylindrical coordinates to give

3.52

$$G(\vec{x}, \vec{x}') = 4 \int_0^\infty dk k g(z, z') \left[\frac{1}{2} J_0(k\rho) J_0(k\rho') + \sum_{m=1}^\infty J_m(k\rho) J_m(k\rho') \cos(m(\phi - \phi')) \right].$$

Find $g(z, z')$.

3.4.2 Find the Dirichlet Green's function for an infinite, flat conducting plate in infinite space. *Take the surface of the conductor as the z - x plane.* Show that the Green function may be written in cylindrical coordinates (ρ, z, ϕ) as

$$G_D(\vec{x}, \vec{x}') = 4 \int_0^\infty dk \sum_{m=1}^\infty J_m(k\rho) J_m(k\rho') \sin(m\phi) \sin(m\phi') e^{k(z_{<} - z_{>})},$$

where $z_{<}$ ($z_{>}$) is the lesser (greater) of z and z' .

3.7.1 Show directly (using, say, the differential equations and large x expansions) the Wronskian relation

$$W[I_m(x), K_m(x)] = -\frac{1}{x}.$$

3.7.2 Evaluate the Wronskian (ν is a positive non-integer):

$$W[J_\nu(t), J_{-\nu}(t)] = ?$$

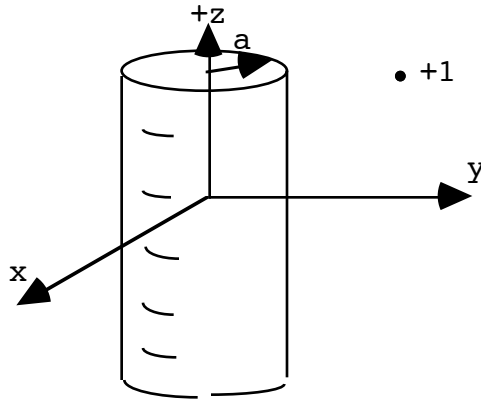
[Hints: The differential equation satisfied by both $J_\nu(t)$ and $J_{-\nu}(t)$ is,

$$\left[\frac{d^2}{dt^2} + \frac{1}{t} \frac{d}{dt} - \frac{\nu^2}{t^2} + 1 \right] J_{\nu, -\nu}(t) = 0.$$

The large t expansion of $J_\nu(t)$ is

$$J_\nu(t) = \sqrt{\frac{2}{\pi t}} \cos\left(t + \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

3.7.3 Consider an infinitely long conducting cylinder of radius a , with the z -axis of coordinates along the axis of symmetry of the cylinder.



Assuming the form for the Dirichlet Green function,

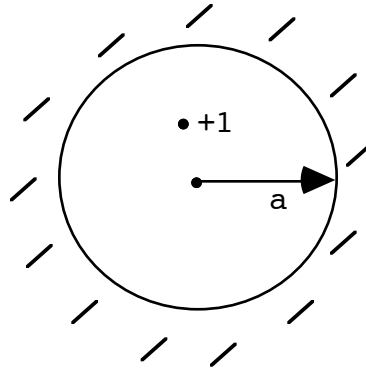
$$G_D(\vec{x}, \vec{x}') = 4\pi \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \frac{1}{\pi} \int_0^{\infty} dk \cos[k(z-z')] g_m(\rho, \rho'),$$

and with

$$4\pi\delta(\vec{x}-\vec{x}') = 4\pi \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \frac{1}{\pi} \int_0^{\infty} dk \cos[k(z-z')] \frac{1}{\rho} \delta(\rho-\rho'),$$

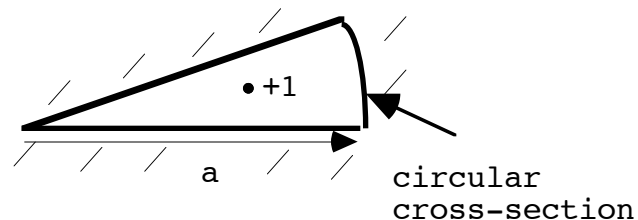
find the reduced Green function, $g_m(\rho, \rho')$, for the volume exterior to the cylinder.

3.7.4 Consider again the infinite conducting cylinder in three dimensions, with the z -axis of coordinates along the axis of symmetry of the cylinder.



Assuming the same form for the Dirichlet Green function and delta function as above, find the reduced Green function, $g_m(\rho, \rho')$, for the *interior* solution. This may be done either using the Wronskian technique or as an eigenfunction expansion.

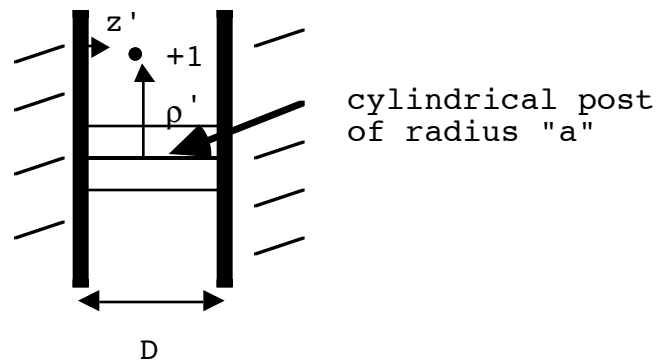
3.7.5 Consider a infinitely long vacuum-filled wedge-shaped hole (wedge radius "a") inside a conductor in three dimensions. (We are working with a point charge, not a line charge.)



Use the reduced Green function method to derive a form for the Dirichlet Green function, $G_D(\rho, \phi, z; \rho', \phi', z')$, in cylindrical coordinates. It is possible to derive a differential equation in either the ρ or z variables. [Hint: Starting out with a correct form of the three-dimensional delta function is the key to doing this problem.]

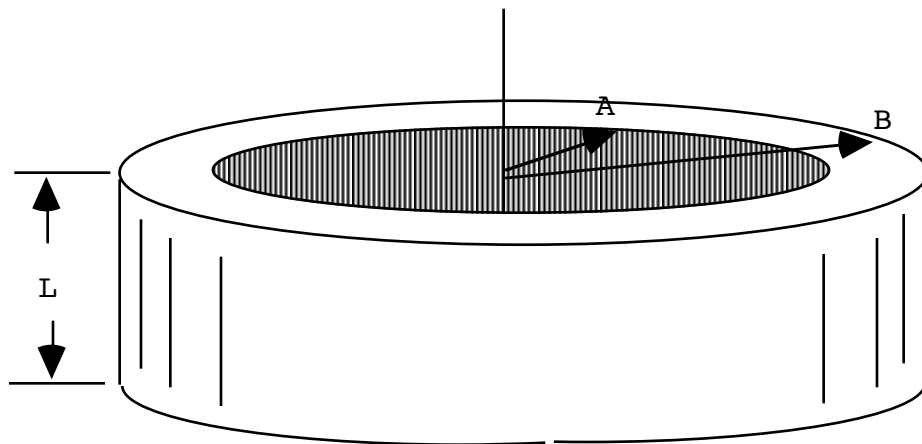
3.7.6 Consider the region between flat parallel conducting plates and outside of a conducting cylindrical post of radius

"a" connecting the plates. The distance between the plates is "D".



Find the Green function for the region between the plates and outside of the cylindrical post. [Hint: Choose the non-oscillatory direction for the Green function to be ρ .]

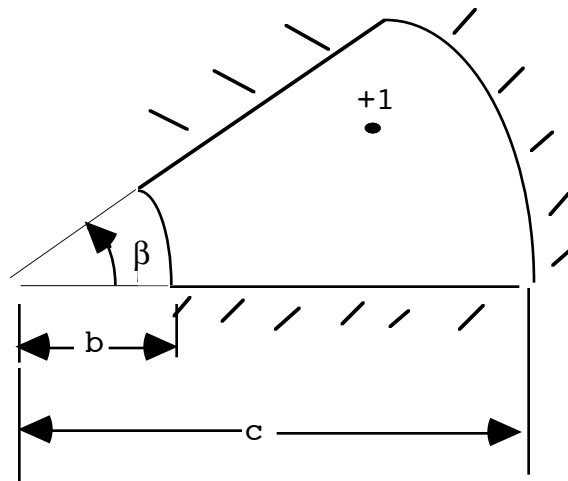
3.7.7 Consider the interior region of a cylindrical toroid of rectangular cross section. The height is L , and the inner and outer radii are A and B , respectively:



Construct the Dirichlet Green function in cylindrical coordinates (ρ, z, ϕ) for the interior region of the cylinder. [Hint: The key as always is an appropriate representation of the Dirac delta function in the interior. There are actually two ways of proceeding here,

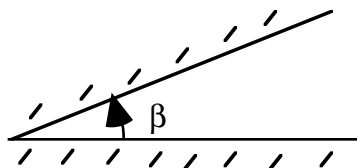
depending on whether one chooses the z or ρ variable as oscillatory.]

3.8.1 Consider the closed conducting wedge shown. It opens with angle β and is bounded by circular conducting radii b and c ($b < c$).



Find the Dirichlet Green function for a *line charge* of unit charge density in this geometry by the reduced Green function method.

3.8.2 In Eq.(3.166) of the script we found that the Green's function for a line charge in the volume bounded by conducting walls intersecting at arbitrary angle β ,



could be written as the infinite sum,

$$G(\vec{x}, \vec{x}') = \sum_{n=1}^{\infty} \frac{4}{n} \left(\frac{\rho_{<}}{\rho_{>}} \right)^{\gamma} \sin\left(\frac{n\pi\phi}{\beta}\right) \sin\left(\frac{n\pi\phi'}{\beta}\right),$$

where $\rho_< (\rho_>)$ is the lesser (greater) of ρ and ρ' and $\gamma = \frac{n\pi}{\beta}$.

Using the summation method of Ch.2, show that this expression can be summed exactly into the closed form

$$G(\vec{x}, \vec{x}') = \ln \left[\frac{1 + \left(\frac{\rho_<}{\rho_>}\right)^{2\pi/\beta} - 2 \left(\frac{\rho_<}{\rho_>}\right)^{\pi/\beta} \cos\left(\frac{\pi}{\beta}(\phi + \phi')\right)}{1 + \left(\frac{\rho_<}{\rho_>}\right)^{2\pi/\beta} - 2 \left(\frac{\rho_<}{\rho_>}\right)^{\pi/\beta} \cos\left(\frac{\pi}{\beta}(\phi - \phi')\right)} \right].$$

[Hints:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} .]$$

3.8.3 By using a reduced Green function technique or other means, show that the two-dimensional Green function for Dirichlet boundary conditions in the cylindrical region $0 \leq \rho \leq a$ has the expansion,

$$G = -\ln(\rho_>^2/a^2) + 2 \sum_{m=1}^{\infty} \frac{\cos[m(\phi - \phi')]}{m} (\rho_<^m) \left(\frac{1}{\rho_>^m} - \frac{\rho_>^m}{a^{2m}} \right),$$

where $\rho_> (\rho_<)$ is the greater (lesser) of ρ and ρ' . [Comparing with (3.319) of the text gives some interesting sums over Bessel functions.]

3.9.1 With the help of the generating function

$$G(x, t) \equiv \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x),$$

for the Legendre polynomials, $P_n(x)$, obtain the relation

$$x \frac{d}{dx} P_{n+1} - \frac{d}{dx} P_n = (n+1)P_{n+1}.$$

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3.9.2 Show by direct substitution that

$$P_n(x) \equiv \frac{1}{2^n n!} \left(\frac{d}{dx} \right)^n (x^2 - 1)^n$$

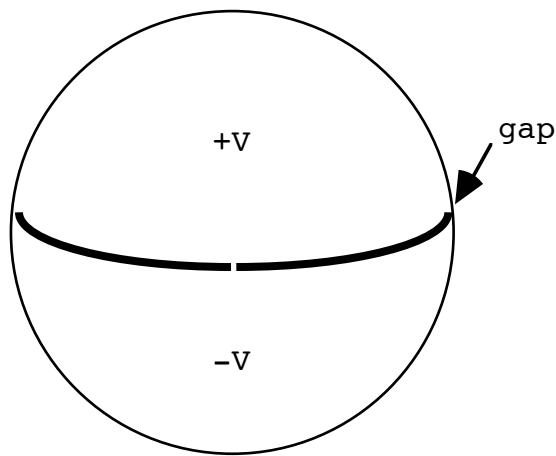
satisfies

$$\frac{dP_{\ell+1}(x)}{dx} = (2\ell+1)P_{\ell}(x) + \frac{dP_{\ell-1}(x)}{dx}.$$

(b) Give an argument which shows that

$$P_n(1) = 1.$$

3.12.1 A capacitor consists of two thin hemispherical conducting caps at potentials V and $-V$ separated by a thin gap, as shown. the radius of the sphere is "a".



(a) Define the capacitance as

$$C \equiv \frac{Q}{V},$$

where Q is the charge on the upper cap. Treating this either as a boundary value problem or using the Green function for this geometry, show that

$$C = \frac{a}{2} \sum_{j=1}^{\infty} [P_{2j-2}(0) - P_{2j}(0)]^2,$$

where $P_j(0)$ is the value of the Legendre polynomial at $x=0$.

(b) Given that

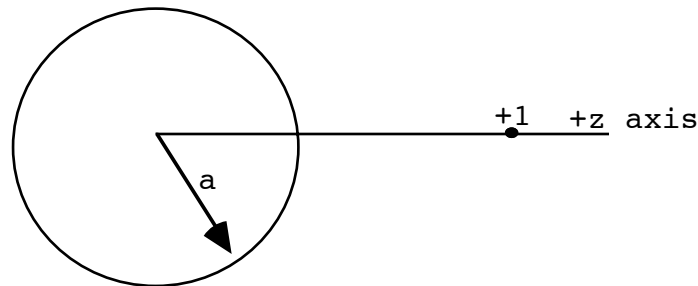
$$\frac{1}{\sqrt{1+\mu^2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \mu^{2n},$$

and comparing with the Coulomb expansion, show that

$$C = \frac{a}{2\pi} \sum_{j=1}^{\infty} 4(j-\frac{1}{4})^2 \left(\frac{\Gamma(j-\frac{1}{2})}{j!} \right)^2$$

where $\Gamma(j-\frac{1}{2})$ is the gamma function with argument $j-\frac{1}{2}$.

3.12.2 (a) Find the electrostatic Neumann Green function, $G(\vec{x}, \vec{x}')$, for the outside region of a sphere of radius "a". For simplicity, take the +1 charge on the +z axis:



The "Coulomb expansion" for $\frac{1}{|\vec{x}-\vec{x}'|}$ should be helpful here.

(b) What is the dipole charge density, $D(\vec{x})$, on the sphere's surface?

3.12.3 (a) Solve the Neumann interior problem for the region inside two concentric spheres of interior radius "a" and

exterior radius "b". Show that the most general form of the solution is (the P_ℓ are the Legendre polynomials)

$$\nabla^2 G_N(\vec{x}', \vec{x}) = -4\pi\delta(\vec{x} - \vec{x}'),$$

$$G_N(\vec{x}', \vec{x}) = \sum_{\ell} g_{\ell}(r, r') P_{\ell}(\cos \gamma) + \sum_{\ell} P_{\ell}(\cos \gamma) (\alpha_{\ell} r'^{\ell} + \beta_{\ell} r^{-(\ell+1)}),$$

where ($\ell > 0$ only)

$$g_{\ell}(r, r') = \frac{\ell(\ell+1)}{b^{2\ell+1} - a^{2\ell+1}} \left(\frac{r_{<}^{\ell}}{\ell} + \frac{a^{2\ell+1}}{(\ell+1)r_{<}^{\ell+1}} \right) \left(\frac{r_{>}^{\ell}}{\ell} + \frac{b^{2\ell+1}}{(\ell+1)r_{>}^{\ell+1}} \right),$$

and where $r_{<}$ ($r_{>}$) is the lesser (greater) of r and r' . Notice $g_0(r, r')$ is not specified: can you determine it? Is it unique? What are the values of α_{ℓ} and β_{ℓ} ? Be sure to refresh yourself on the unusual properties of the interior Neumann Green function from Ch.1. (When $b \rightarrow \infty$ we have an outside problem. Will your results necessarily agree with prob. 3.12.2(a) above in this limit?)

(b) Verify that the surface delta functions discussed in Section 1.7, Eq.(1.78), are present in the final answer of prob. 3.12.3.

3.12.4 (a) Using the Green function for a sphere of radius "a" or other means, show that the outside potential, $V(\theta, \phi)$, from a given potential, $\Phi(\vec{x})$, on the surface of a sphere can be written as

$$\Phi(\vec{x}) = \sum_{\ell, m} \left(\frac{a}{r} \right)^{\ell+1} Y_{\ell m}(\theta, \phi) B_{\ell m},$$

where

$$B_{\ell m} = \oint_S d\Omega' V(\theta', \phi') Y_{\ell m}^*(\theta', \phi').$$

What is the inside for for $\Phi(\vec{x})$?

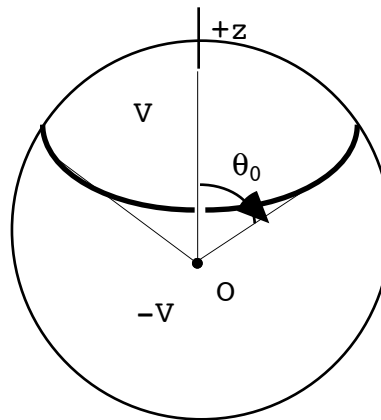
(b) Show that this is also equivalent to (+ for inside, - for outside)

$$\Phi(\vec{x}) = \pm \frac{a(a^2 - r^2)}{4\pi} \oint_s d\Omega' \frac{V(\theta', \phi')}{(a^2 + r^2 - 2ar \cos \gamma)^{3/2}},$$

where

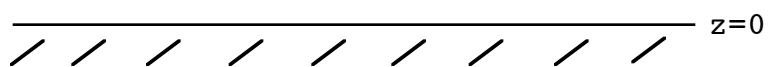
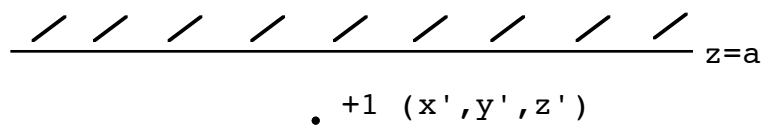
$$\cos \gamma \equiv \frac{\vec{r} \cdot \vec{r}'}{rr'}.$$

3.13.1 A conducting sphere of radius "a" is split into two unequal pieces which are insulated from one another. The "top" piece subtends a uniform polar angle, θ_0 , as measured from from the center of the sphere and is raised to potential V . The rest of the sphere has potential $-V$.



Find the electric field, $\vec{E}$, at the center of the sphere, O .

3.14.1 Using the method of eigenfunction expansions or other means, show that another expression for the Dirichlet Green function for the infinite parallel plate capacitor problem,



is

$$G_D(\vec{x}, \vec{x}') = \frac{4}{a} \int_0^\infty dk k J_0(kD) \sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi z}{a}) \sin(\frac{n\pi z'}{a})}{k^2 + (\frac{n\pi}{a})^2},$$

where $D \equiv |\vec{x} - \vec{x}'|$ and J_0 is the Bessel function of order zero.