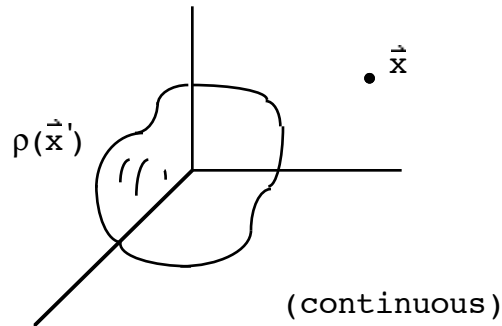


Chapter 4: Multipoles, Electrostatics of Macroscopic Media, Dielectrics

4.1 Cartesian and spherical multipole expansions

Let's say we have:



The potential is given by

$$\Phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}. \quad (4.1)$$

Choose origin w/i charge distribution. If $r = |\vec{x}|$ is large compared to the characteristic dimensions of the distribution, we get

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{(\vec{x}' \cdot \vec{\nabla}')^{\ell}}{\ell!} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \Big|_{\vec{x}'=0}. \quad (4.2)$$

Notice

$$\vec{\nabla}' \cdot \frac{1}{|\vec{x} - \vec{x}'|} = -\vec{\nabla}' \cdot \frac{1}{|\vec{x} - \vec{x}'|}, \quad (4.3)$$

so that

$$\frac{(\vec{x}' \cdot \vec{\nabla}')^\ell}{\ell!} \frac{1}{|\vec{x} - \vec{x}'|} \Big|_{\vec{x}'=0} = (-1)^\ell \frac{(\vec{x}' \cdot \vec{\nabla}')^\ell}{\ell!} \frac{1}{r}, \quad (4.4)$$

$$\Rightarrow \frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r} - \vec{x}' \cdot \vec{\nabla}' \frac{1}{r} + \frac{1}{2} (\vec{x}' \cdot \vec{\nabla}')^2 \frac{1}{r} - \dots \quad (r > r'). \quad (4.5)$$

Work these out:

$$(\vec{x}' \cdot \vec{\nabla}') \frac{1}{r} = \sum_i x'_i \nabla_i \frac{1}{\sqrt{x^2 + y^2 + z^2}} = - \sum_i \frac{x'_i x_i}{r^3} = - \frac{\vec{x}' \cdot \vec{x}}{r^3}, \quad (4.6)$$

$$(\vec{x}' \cdot \vec{\nabla}')^2 \frac{1}{r} = \sum_{i,j} x'_i x'_j \nabla_i \nabla_j \frac{1}{\sqrt{x^2 + y^2 + z^2}} = - \sum_{i,j} x'_i x'_j \nabla_i \frac{x_j}{r^3}, \quad (4.7)$$

or

$$(\vec{x}' \cdot \vec{\nabla}')^2 \frac{1}{r} = \sum_{i,j} x'_i x'_j \frac{(3x_i x_j - \delta_{ij} r^2)}{r^5}. \quad (4.8)$$

Therefore

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r} + \frac{\vec{x}' \cdot \vec{x}}{r^3} + \frac{1}{2} \sum_{i,j} \frac{x'_i x'_j}{r^5} (3x_i x_j - \delta_{ij} r^2) + \dots \quad (4.9)$$

Last term:

$$\begin{aligned} \sum_{i,j} \frac{x'_i x'_j}{r^5} (3x_i x_j - \delta_{ij} r^2) &= \sum_{i,j} \frac{3x'_i x'_j x_i x_j - r'^2 r^2}{r^5} \\ &= \sum_{i,j} \frac{x_i x_j}{r^5} (3x'_i x'_j - r'^2 \delta_{ij}). \end{aligned} \quad (4.10)$$

$$\Rightarrow \Phi(\vec{x}) = \int d^3x' \rho(\vec{x}') \left\{ \frac{1}{r} + \frac{\vec{x}' \cdot \vec{x}}{r^3} + \frac{1}{2} \sum_{i,j} \frac{x_i x_j}{r^5} (3x'_i x'_j - r'^2 \delta_{ij}) + \dots \right\}, \quad (4.11)$$

$$\Phi(\vec{x}) = \frac{q}{r} + \frac{\vec{x} \cdot \vec{p}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \dots, \quad (4.12)$$

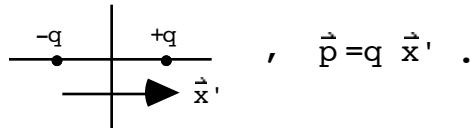
where

components

$$1 \quad q = \int d^3x' \rho(\vec{x}') \quad \text{charge (scalar)}$$

components

$$3 \quad \vec{p} \equiv \int d^3x' \vec{x}' \rho(\vec{x}') \quad \text{el. dipole (vector)}$$



$$5 \quad Q_{ij} \equiv \int d^3x' (3x'_i x'_j - \delta_{ij} r'^2) \rho(\vec{x}') \quad \text{quadrupole (tensor)}$$

↑ increases like $2\ell + 1$

Notice ("traceless")

$$\sum_i Q_{ii} = \int d^3x' (3r'^2 - 3r'^2) \rho(\vec{x}') = 0, \quad (4.13)$$

$$\Rightarrow Q_{11} + Q_{22} + Q_{33} = 0. \quad (4.14)$$

Fields: ($r \neq 0$)

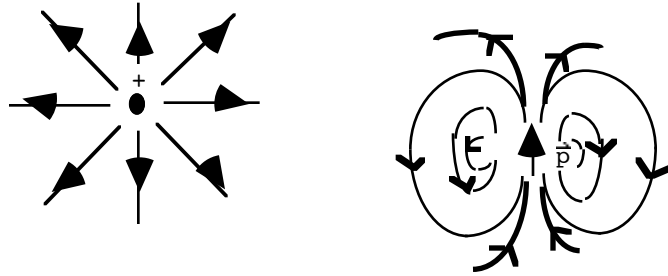
$$\vec{E} = -\vec{\nabla} \left(\frac{q}{r} \right) = \frac{q \hat{r}}{r^2} \quad (\text{point charge} \sim \frac{1}{r^2}), \quad (4.15)$$

$$\vec{E} = -\vec{\nabla} \left(\frac{\vec{x} \cdot \vec{p}}{r^3} \right) = \frac{3(\vec{x} \cdot \vec{p}) \vec{x} - \vec{p} r^2}{r^5} \quad (\text{point dipole} \sim \frac{1}{r^3}), \quad (4.16)$$

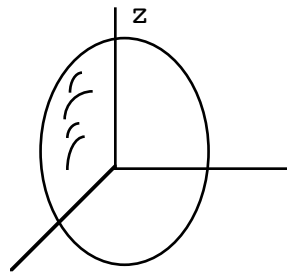
$$E_i = -\nabla_i \left(\frac{1}{2} \sum_{j,k} Q_{jk} \frac{x_j x_k}{r^5} \right), \text{ or}$$

$$E_i = \frac{1}{2} \left\{ \frac{5 \sum_{j,k} Q_{jk} x_j x_k x_i - 2r^2 \sum_k Q_{ik} x_k}{r^7} \right\} \quad (\text{point quadrupole} \sim \frac{1}{r^4}). \quad (4.17)$$

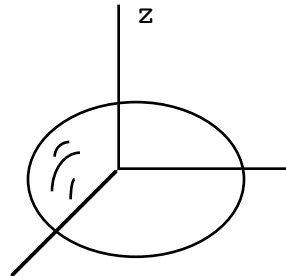
Charge & dipole fields:



Two types of quadrupole distributions are: (rot. invariant about z-axis)



"prolate", $Q_{33} > 0$



"oblate", $Q_{33} < 0$

Obviously, both the location and orientation of our axes (in general) affect the moments.

The problem with the above is that the number of indices increases with increasing ℓ . Possible to avoid this by expanding in spherical harmonics. Of course

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell, m} \frac{r'^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\theta, \phi) \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}^*(\theta', \phi'), \quad (4.18)$$

so

$$\Phi(\vec{x}) = \sum_{\ell, m} \frac{1}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\theta, \phi) \int d^3x' r'^{\ell} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}^*(\theta', \phi') \rho(\vec{x}'). \quad (4.19)$$

Introduce

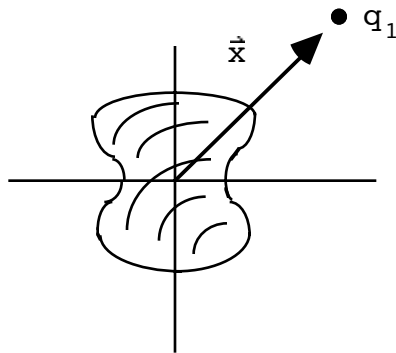
$$\rho_{\ell m} \equiv \int d^3x' r'^{\ell} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}^*(\theta', \phi') \rho(\vec{x}'), \quad (4.20)$$

$$\Rightarrow \Phi(\vec{x}') = \sum_{\ell, m} \frac{1}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\theta, \phi) \rho_{\ell m}. \quad (4.21)$$

4.2 Multipole energy expansions

Energy of interaction. Point charge, q_1 :

$$W = q_1 \Phi(\vec{x}) = \frac{q_1 q}{r} + q_1 \frac{\vec{x} \cdot \vec{p}}{r^3} + \frac{q_1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \dots \quad (4.22)$$



Introduce

$$\vec{E}^{(0)} = - \frac{q_1 \vec{x}}{r^3} \Rightarrow \frac{\partial E_j^{(0)}}{\partial x_i} = q_1 \left[\frac{3x_i x_j - \delta_{ij} r^2}{r^5} \right], \quad (4.23)$$

Then

$$W = q_1 \phi - \vec{p} \cdot \vec{E}^{(0)} + \frac{q_1}{6} \sum_{i,j} Q_{ij} \left[\frac{3x_i x_j - \delta_{ij} r^2}{r^5} \right], \quad (4.24)$$

$$\Rightarrow W = q_1 \phi - \vec{p} \cdot \vec{E}^{(0)} + \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial E_j^{(0)}}{\partial x_i} + \dots \quad (4.25)$$

Here, x_i is the distance from the multipole origin to the charge. Can now imagine integrating over many q_1 's: this gives us the energy of interaction of two arbitrary charge distributions. From this, we can read off various types of interaction:

Dipole - dipole:

$$W_{dd} = -\vec{p}_2 \cdot \vec{E}_1^d = \frac{-3(\vec{x} \cdot \vec{p}_1)(\vec{x} \cdot \vec{p}_2) + (\vec{p}_1 \cdot \vec{p}_2)r^2}{r^5}. \quad (4.26)$$

$\Uparrow (\vec{x} \rightarrow -\vec{x}$ in previous expression which had the dipole at the origin)

Dipole - quadrupole:

$$W_{dq} = -\vec{p}_2 \cdot \vec{E}^q = -\frac{1}{2} \sum_{i,j} Q_{ij} \left\{ \frac{2r^2 p_i x_j - 5x_i x_j (\vec{p} \cdot \vec{x})}{r^7} \right\}. \quad (4.27)$$

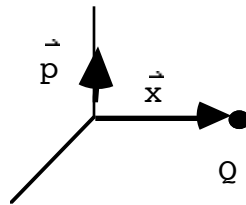
$\Uparrow (\vec{x} \rightarrow -\vec{x})$

Same as quadrupole - dipole?

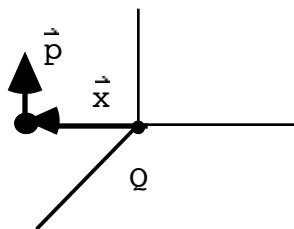
$$W_{qd} = \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial E_j^{(0)}}{\partial x_i} = \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial}{\partial x_i} \left\{ \frac{3(\vec{x} \cdot \vec{p})x_j - p_j r^2}{r^5} \right\},$$

$$W_{qd} = \frac{1}{2} \sum_{i,j} Q_{ij} \left\{ \frac{2r^2 p_i x_j - 5x_i x_j (\vec{p} \cdot \vec{x})}{r^7} \right\} = -W_{dq} \quad (4.28)$$

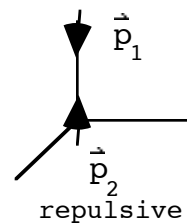
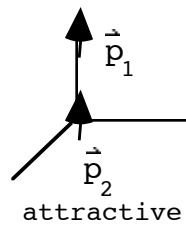
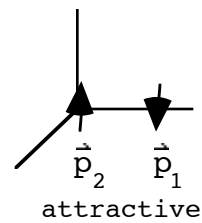
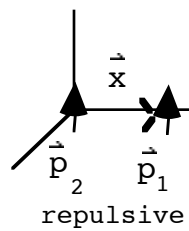
Difference in sign occurs because in the first case we have



while in the second

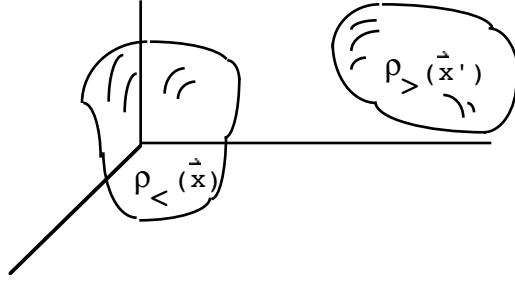


What we find in the dipole - dipole case is:



Explains a wealth of data having to do with forces between atoms in solids.

Above becomes more complicated for higher order interactions. Expansion in spherical harmonics helps here also. In the case of non-overlapping charge densities, as in



we have

$$W = \int d^3x d^3x' \frac{\rho_{<}(\vec{x}) \rho_{>}(\vec{x}')}{|\vec{x} - \vec{x}'|}. \quad (4.29)$$

But

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell, m} \frac{4\pi}{2\ell+1} \frac{r_{<}^{\ell}}{r_{>}^{\ell+m}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi), \quad (4.30)$$

(Above picture: $r_{>} = r'$ $r_{<} = r$)

$$\Rightarrow W = \sum_{\ell, m} \int d^3x' d^3x \rho_{<}(\vec{x}) \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta, \phi) \frac{r^{\ell}}{r'^{\ell+1}} Y_{\ell m}^*(\theta', \phi') \rho_{>}(\vec{x}'). \quad (4.31)$$

Define

$$\rho_{(>)}_{\ell m} \equiv \int d^3x \sqrt{\frac{4\pi}{2\ell+1}} r^{\ell} Y_{\ell m}(\theta, \phi) \rho_{>}(\vec{x}). \quad (4.32)$$

Then very simply:

$$W = \sum_{\ell, m} \rho_{<\ell m} \rho_{>\ell m}^*. \quad (4.33)$$

Can establish relationships between the different sets of expansion coefficients. For example

$$\vec{p} = \int d^3x' \rho(\vec{x}') r' [\sin\theta' \cos\phi' \hat{i} + \sin\theta' \sin\phi' \hat{j} + \cos\theta' \hat{k}], \quad (4.34)$$

$$\vec{p} = \sqrt{\frac{8\pi}{3}} \int d^3x' \rho(\vec{x}') r' \left[\frac{1}{2} [-Y_{11} + Y_{1-1}] \hat{i} \right]$$

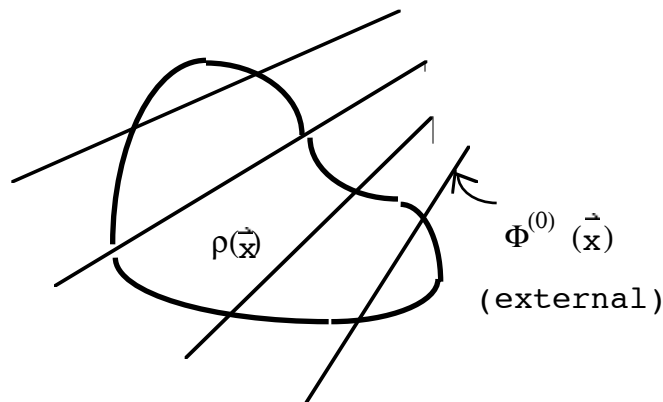
$$+ \frac{i}{2} [Y_{11} + Y_{1-1}] \hat{j} + \frac{1}{\sqrt{2}} Y_{10} \hat{k}], \quad (4.35)$$

$$\vec{p} = \frac{1}{\sqrt{2}} (-\rho_{11}^{<} + \rho_{1-1}^{<}) \hat{i} + \frac{i}{\sqrt{2}} (\rho_{11}^{<} + \rho_{1-1}^{<}) \hat{j} + \rho_{10}^{<} \hat{k}. \quad (4.36)$$

Can also see that if $q=0$, $\vec{p}$ is independent of origin. (True for higher moments as well as if all the lower ones vanish.)

4.3 External fields and forces on multipole distributions

Find the force on an arbitrary set of multipoles in an external field:



$$d\vec{F}(\vec{x}) = dq (\vec{E}^{(0)}(\vec{x}) + \vec{E}^r(\vec{x})), \quad (4.37)$$

$$\Rightarrow \vec{F} = \int dq \vec{E}^{(0)}(\vec{x}) = \int d^3x \rho(\vec{x}) \vec{E}^{(0)}(\vec{x}) \quad (4.38)$$

$(\vec{E}^r(\vec{x}))$ is the remainder field of the rest of $\rho(\vec{x})$ and will not contribute to the total force; see similar discussion beginning at Eq.(4.190).) Trick: introduce two independent sets of coordinates x_i and x'_i , measured from the same origin. Look at each component:

$$\vec{E}^{(0)} = \vec{E}^{(0)}(0) + (\vec{x} \cdot \vec{\nabla}') \vec{E}^{(0)}(\vec{x}') \Big|_{x'=0} + \frac{1}{2} (\vec{x} \cdot \vec{\nabla}')^2 \vec{E}^{(0)}(\vec{x}') \Big|_{x'=0} + \dots \quad (4.39)$$

3rd term really means:

$$\begin{aligned}
(\vec{x} \cdot \vec{\nabla}')^2 \vec{E}^{(0)}(\vec{x}') \Big|_{x'=0} &= \sum_{i,j} x_i \frac{\partial}{\partial x_i'} x_j \frac{\partial}{\partial x_j'} \vec{E}^{(0)}(\vec{x}') \Big|_{x'=0} \\
&= \sum_{i,j} x_i x_j \frac{\partial^2 \vec{E}^{(0)}(\vec{x}')}{\partial x_i' \partial x_j'} \Big|_{x'=0}. \quad (4.40)
\end{aligned}$$

A useful vector identity is

$$\vec{\nabla}'(\vec{a} \cdot \vec{b}) = (\vec{a} \cdot \vec{\nabla}')\vec{b} + (\vec{b} \cdot \vec{\nabla}')\vec{a} + \vec{a} \times (\vec{\nabla}' \times \vec{b}) + \vec{b} \times (\vec{\nabla}' \times \vec{a}). \quad (4.41)$$

Therefore

$$\vec{\nabla}'(\vec{x} \cdot \vec{E}^{(0)}(\vec{x}')) = (\vec{x} \cdot \vec{\nabla}')\vec{E}^{(0)}(\vec{x}'), \quad \left(\begin{array}{l} \text{use electrostatics:} \\ \vec{\nabla}' \times \vec{E}^{(0)} = 0 \end{array} \right) \quad (4.42)$$

$$\Rightarrow \vec{\nabla}'(\vec{x} \cdot (\vec{x} \cdot \vec{\nabla}')\vec{E}^{(0)}(\vec{x}')) = (\vec{x} \cdot \vec{\nabla}')(\vec{x} \cdot \vec{\nabla}')\vec{E}^{(0)}(\vec{x}'). \quad (4.43)$$

[In the above we have to use:

$$\vec{\nabla}' \times [(\vec{x} \cdot \vec{\nabla}')\vec{E}^{(0)}(\vec{x}')] = \vec{\nabla}' \times [\vec{\nabla}'(\vec{x} \cdot \vec{E}^{(0)}(\vec{x}'))] = 0.]$$

Thus

$$\begin{aligned}
\vec{E}^{(0)}(\vec{x}) &= \vec{E}^{(0)}(0) + \vec{\nabla}'(\vec{x} \cdot \vec{E}^{(0)}(\vec{x}')) \Big|_{x'=0} \\
&\quad + \frac{1}{2} \vec{\nabla}' \{ \vec{x} \cdot [(\vec{x} \cdot \vec{\nabla}')\vec{E}^{(0)}(\vec{x}')]] \} \Big|_{x'=0} + \dots \quad (4.44)
\end{aligned}$$

More explicitly on the last term:

$$\begin{aligned}
\vec{E}^{(0)}(\vec{x}) &= \vec{E}^{(0)}(0) + \vec{\nabla}'(\vec{x} \cdot \vec{E}^{(0)}(\vec{x}')) \Big|_{x'=0} \\
&\quad + \frac{1}{2} \vec{\nabla}' \sum_{i,j} x_i x_j \frac{\partial}{\partial x_j'} E_i^{(0)}(\vec{x}') \Big|_{x'=0} + \dots \quad (4.45)
\end{aligned}$$

$\vec{\nabla}' \cdot \vec{E}^{(0)} = 0$ for the external field in the region of interest,
so we may add

$$\begin{aligned}
-\frac{1}{6} r^2 \vec{\nabla}' \cdot \vec{E}^{(0)}(\vec{x}') \Big|_{x'=0} &= -\frac{1}{6} \sum_i r^2 \frac{\partial \vec{E}_i^{(0)}}{\partial x_i'} \Big|_{x'=0} \\
&= -\frac{1}{6} \sum_{i,j} r^2 \delta_{ij} \frac{\partial \vec{E}_i^{(0)}}{\partial x_j'} \Big|_{x'=0}. \quad (4.46)
\end{aligned}$$

We now find that

$$\begin{aligned}
\vec{E}^{(0)}(\vec{x}) &= \vec{E}^{(0)}(0) + \vec{\nabla}' (\vec{x} \cdot \vec{E}^{(0)}(\vec{x}')) \Big|_{x'=0} \\
&\quad + \frac{1}{6} \vec{\nabla}' \sum_{i,j} (3x_i x_j - r^2 \delta_{ij}) \frac{\partial \vec{E}_i^{(0)}}{\partial x_j'} \Big|_{x'=0} + \dots \quad (4.47)
\end{aligned}$$

Thus, for the force, $\vec{F}$, from (4.38):

$$\begin{aligned}
\vec{F} &= \int d^3x \rho(\vec{x}) \left[\vec{E}^{(0)}(0) + \vec{\nabla}' (\vec{x} \cdot \vec{E}^{(0)}(\vec{x}')) \Big|_{x'=0} \right. \\
&\quad \left. + \frac{1}{6} \vec{\nabla}' \sum_{i,j} (3x_i x_j - r^2 \delta_{ij}) \frac{\partial \vec{E}_i^{(0)}}{\partial x_j'} \Big|_{x'=0} + \dots \right]. \quad (4.48)
\end{aligned}$$

Using our def^{ns} of q , $\vec{p}$ and Q_{ij} , this then shows that ($\vec{x}' \rightarrow \vec{x}$ now)

$$\begin{aligned}
\vec{F} &= q \vec{E}^{(0)}(0) + \left\{ \vec{\nabla} (\vec{p} \cdot \vec{E}^{(0)}(\vec{x})) \right\} \Big|_{x=0} \\
&\quad + \left\{ \vec{\nabla} \left[\frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial \vec{E}_i^{(0)}}{\partial x_j} \right] \right\} \Big|_{x=0} + \dots \quad (4.49)
\end{aligned}$$

An alternate form is given by

$$\begin{aligned}
\vec{F} &= q \vec{E}^{(0)}(0) + (\vec{p} \cdot \vec{\nabla}) \vec{E}^{(0)}(\vec{x}) \Big|_{x=0} \\
&\quad + \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \vec{E}^{(0)}}{\partial x_i \partial x_j} \Big|_{x=0} + \dots \quad (4.50)
\end{aligned}$$

Different assumptions are necessary to reach these two force expressions; see prob. 4.3.1(a).

4.4 Introduction of the "electric polarization" and "displacement field"

We can think of the above as applying to an atom. We want to deal with a macroscopic description instead of dealing with individual atoms. Let us therefore integrate over the density, $n(\vec{x})$, of such atoms. ($q = 0$ for neutral atoms)

$$\vec{F}_{\text{bulk}} \equiv \sum_{i=\text{atoms}} \vec{F}_i = \sum_i (\vec{p}_i \cdot \vec{\nabla}) \vec{E}^{(0)} \Big|_{\vec{x}_i}, \quad (4.51)$$

$$\vec{F}_{\text{bulk}} \approx \int d^3x \, n(\vec{x}) \left(\vec{p}(\vec{x}) \cdot \vec{\nabla} \right) \vec{E}^{(0)}(\vec{x}). \quad (4.52)$$

↑ possible space dependence

Let us define

$$\vec{P}(\vec{x}) = n(\vec{x}) \vec{p}(\vec{x}) \quad \left[\text{Units: } \frac{\text{dipole strength}}{\text{volume}} \sim \frac{q\ell}{\ell^3} \right] \quad (4.53)$$

(same units as $\vec{E}$) as the "electric polarization". Clear that $\vec{P}$ can have $\vec{x}$ dependence either from the density or some intrinsic change in $\vec{p}$ from atom to atom. Now, integrate by parts (original integral over entire sample):

$$\vec{F}_{\text{bulk}} \approx \int_V d^3x (-\vec{\nabla} \cdot \vec{P}(\vec{x})) \vec{E}^{(0)}(\vec{x}) + \int_S d\vec{a} (\vec{P} \cdot \hat{n}) \vec{E}^{(0)}(\vec{x}). \quad (4.54)$$

"V" and "S" now refer to the idealized sample volume, surface. (In this form the surface, S, is not in the volume, V.) Compare with $\int d^3x \, \rho_{\text{eff}}(\vec{x}) \vec{E}^{(0)}(\vec{x})$ to identify

$$\left. \begin{aligned} \rho_{\text{eff}}^d(\vec{x}) &= -\vec{\nabla} \cdot \vec{P}, \\ \sigma_{\text{eff}}^d(\vec{x}) &= \vec{P} \cdot \hat{n}. \end{aligned} \right\} \quad (4.55)$$

[Can also show (prob. 4.3.1(b))

$$\rho_{\text{eff}}^q(\vec{x}) = \frac{1}{6} \sum_{i,j} \frac{\partial^2 q_{ij}(\vec{x})}{\partial x_i \partial x_j}, \quad (4.56)$$

where $q_{ij}(\vec{x}) = n(\vec{x})Q_{ij}(\vec{x})$, which can also be written as a contribution to $\vec{P}$.]

What is the meaning of ρ_{eff} ? It is the effective charge density contributed by all charges bound in the atoms. Effectively, for bulk material

$$\vec{\nabla} \cdot \vec{E} = 4\pi [\rho_{\text{free}} + \rho_{\text{bound}}]. \quad (4.57)$$

If we identify

$$\rho_{\text{bound}} = \rho_{\text{eff}}^d = -\vec{\nabla} \cdot \vec{P}, \quad (4.58)$$

then

$$\vec{\nabla} \cdot \vec{E} = 4\pi [\rho_{\text{free}} - \vec{\nabla} \cdot \vec{P}]. \quad (4.59)$$

Define ("displacement field")

$$\vec{D} = \vec{E} + 4\pi \vec{P}, \quad (4.60)$$

$$\Rightarrow \vec{\nabla} \cdot \vec{D} = 4\pi \rho_{\text{free}}. \quad (4.61)$$

Will usually assume ("linear & isotropic")

$$\vec{P} = \chi(\vec{x}) \vec{E}, \quad (4.62)$$

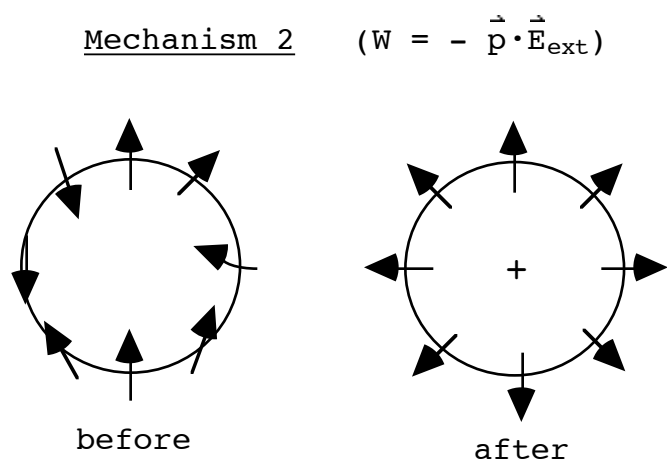
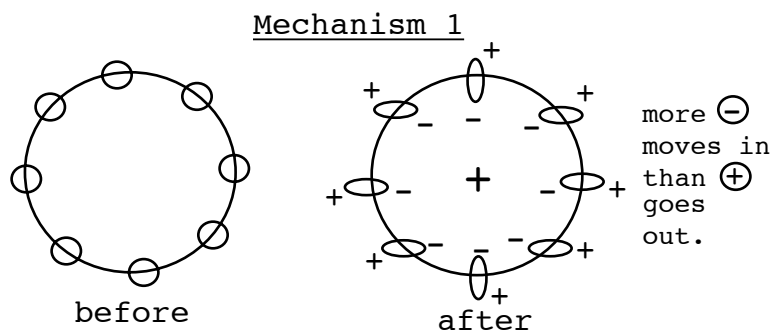
$$\Rightarrow \vec{D} = \epsilon(\vec{x}) \vec{E}, \quad \epsilon(\vec{x}) = 1 + 4\pi \chi(\vec{x}). \quad (4.63)$$

$\Uparrow$ dielectric constant

If $\epsilon = \text{constant}$ in the material, then

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi \rho_{\text{free}}}{\epsilon}. \quad (4.64)$$

One expects that electric fields are reduced so that $\epsilon > 1$. This is understandable in that dipoles shield charge.



This mechanism is temperature dependent (see section 4.6).

Mechanism 1: induced polarization

Mechanism 2: orientation polarization ("polar" substances)

4.5 Green functions in the presence of linear dielectrics

Go back to how we derived Green functions. Now have that

$$\vec{\nabla} \cdot [\epsilon(\vec{x}) \vec{\nabla} \Phi(\vec{x})] = -4\pi \rho(\vec{x}). \quad (4.65)$$

(ρ understood to be free charge.) Now let us assume the Green function solves

$$\vec{\nabla} \cdot [\epsilon(\vec{x}) \vec{\nabla} G(\vec{x}', \vec{x})] = -4\pi \delta(\vec{x} - \vec{x}'). \quad (4.66)$$

Still represents the electric field of a + unit charge. Go through old song and dance:

$$\begin{aligned} \int d^3x' \{ G(\vec{x}, \vec{x}') \vec{\nabla}' \cdot [\epsilon(\vec{x}') \vec{\nabla}' \Phi(\vec{x}')] - \Phi(\vec{x}') \vec{\nabla}' \cdot [\epsilon(\vec{x}') \vec{\nabla}' G(\vec{x}, \vec{x}')] \} \\ = -4\pi \int d^3x' \{ G(\vec{x}, \vec{x}') \rho(\vec{x}') - \Phi(\vec{x}') \delta(\vec{x} - \vec{x}') \}. \end{aligned} \quad (4.67)$$

$$\text{RHS} = 4\pi \Phi(\vec{x}) - 4\pi \int d^3x' G(\vec{x}, \vec{x}') \rho(\vec{x}'). \quad (4.68)$$

$$\begin{aligned} \text{LHS} &= \int d^3x' \vec{\nabla}' \cdot [G(\vec{x}, \vec{x}') \epsilon(\vec{x}') \vec{\nabla}' \Phi(\vec{x}')] - \vec{\nabla}' G(\vec{x}, \vec{x}') \epsilon(\vec{x}') \Phi(\vec{x}') \\ &= \oint da' [\epsilon(\vec{x}') G(\vec{x}, \vec{x}') \frac{\partial \Phi}{\partial n'} - \epsilon(\vec{x}') \Phi(\vec{x}') \frac{\partial G}{\partial n'}(\vec{x}, \vec{x}')] \}. \end{aligned} \quad (4.69)$$

Choice of BC on $G(\vec{x}, \vec{x}')$ now. Choose

$$\begin{aligned} G_D(\vec{x}, \vec{x}') \Big|_{x' \text{ on } s} = 0, \\ \Rightarrow \Phi(\vec{x}) = \int d^3x' G_D(\vec{x}, \vec{x}') \rho(\vec{x}') - \frac{1}{4\pi} \oint da' \epsilon(\vec{x}') \Phi(\vec{x}') \frac{\partial G_D}{\partial n'}. \end{aligned} \quad (4.70)$$

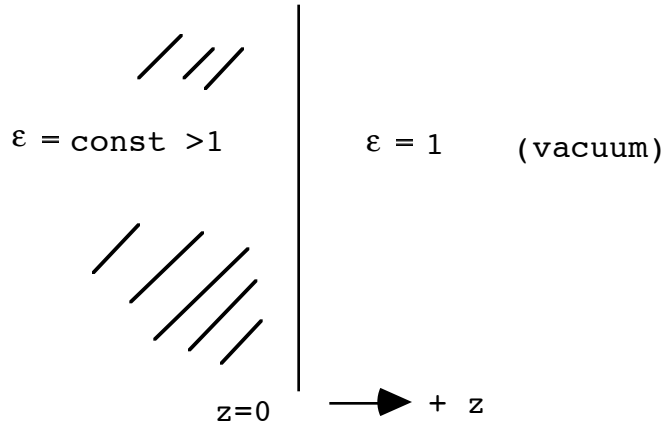
Can show that

$$G_D(\vec{x}, \vec{x}') = G_D(\vec{x}', \vec{x}), \quad (4.71)$$

as before.

4.6 Green function for the dielectric slab

Now apply our knowlege to:



Get $G_D(\vec{x}, \vec{x}')$ for $z' > 0$. Must solve

$$z > 0: \quad \nabla^2 G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}'), \quad (4.72)$$

$$z < 0: \quad \nabla^2 G(\vec{x}, \vec{x}') = 0. \quad (4.73)$$

As usual, use

$$4\pi \delta(\vec{x} - \vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \delta(z - z'), \quad (4.74)$$

$$G(\vec{x}, \vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} g(z, z'), \quad (4.75)$$

$$-\nabla^2 G(\vec{x}, \vec{x}') = 4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \left[k^2 - \frac{\partial^2}{\partial z^2} \right] g(z, z'). \quad (4.76)$$

So we get ($z' > 0$)

$$z > 0: \quad \left[-\frac{\partial^2}{\partial z^2} + k^2 \right] g(z, z') = \delta(z - z'), \quad (4.77)$$

$$z < 0: \quad \left[-\frac{\partial^2}{\partial z^2} + k^2 \right] g(z, z') = 0. \quad (4.78)$$

Our B.C.'s are

$$g \Big|_{0^-}^{0^+} = 0, \quad (\Phi \text{ is continuous} \Rightarrow \vec{E}_{||} \text{ is cont.}) \quad (4.79)$$

$$\varepsilon \frac{\partial}{\partial z} g \Big|_{0-} = \frac{\partial}{\partial z} g \Big|_{0+} . \quad (D_n \text{ is cont.}) \quad (4.80)$$

The solutions in the various regions are:

$$z < 0: \quad g = Ae^{kz}, \quad (\text{finite as } z \rightarrow -\infty) \quad (4.81)$$

$$0 < z < z': \quad g = Be^{kz} + Ce^{-kz}, \quad (4.82)$$

$$z' < z: \quad g = De^{-kz}. \quad (\text{finite as } z \rightarrow +\infty) \quad (4.83)$$

The above BC's now require that

$$(4.79) \quad \Rightarrow \quad A = B+C, \quad (4.84)$$

$$(4.80) \quad \Rightarrow \quad \varepsilon k A = k(B-C), \quad (4.85)$$

from which we find

$$B = \frac{\varepsilon+1}{2} A, \quad C = \frac{1-\varepsilon}{2} A. \quad (4.86)$$

As z approaches z' , we have

$$g \Big|_{z'-}^{z'+} = 0, \quad (4.87)$$

$$- \frac{\partial}{\partial z} g \Big|_{z'-}^{z'+} = 1, \quad (4.88)$$

which imply

$$De^{-kz'} = Be^{kz'} + Ce^{-kz'}, \quad (4.89)$$

$$kDe^{-kz'} + k(Be^{kz'} - Ce^{-kz'}) = 1. \quad (4.90)$$

Just give the solution. (Can check it for yourselves):

$$A = \frac{2}{\varepsilon + 1} \frac{1}{2k} e^{-kz'}, \quad (4.91)$$

$$B = \frac{1}{2k} e^{-kz'}, \quad (4.92)$$

$$C = - \frac{\varepsilon - 1}{\varepsilon + 1} \frac{1}{2k} e^{-kz'}, \quad (4.93)$$

$$D = - \frac{\varepsilon - 1}{\varepsilon + 1} \frac{1}{2k} e^{-kz'} + \frac{1}{2k} e^{kz'}. \quad (4.94)$$

Putting these back, we find that

$$z < 0: \quad g = \frac{2}{\varepsilon + 1} \frac{1}{2k} e^{-k(z' - z)} \quad \left(= \frac{2}{\varepsilon + 1} \frac{1}{2k} e^{-k|z' - z|} \right), \quad (4.95)$$

$$0 < z < z': \quad g = \frac{1}{2k} \left[e^{-k(z' - z)} - \frac{\varepsilon - 1}{\varepsilon + 1} e^{-k(z + z')} \right], \quad (4.96)$$

$$z' < z: \quad g = \frac{1}{2k} \left[e^{-k(z - z')} - \frac{\varepsilon - 1}{\varepsilon + 1} e^{-k(z + z')} \right]. \quad (4.97)$$

Notice the last two combine as

$$z > 0: \quad g = \frac{1}{2k} \left[e^{-k|z - z'|} - \frac{\varepsilon - 1}{\varepsilon + 1} e^{-k(z + z')} \right]. \quad (4.98)$$

Old result:

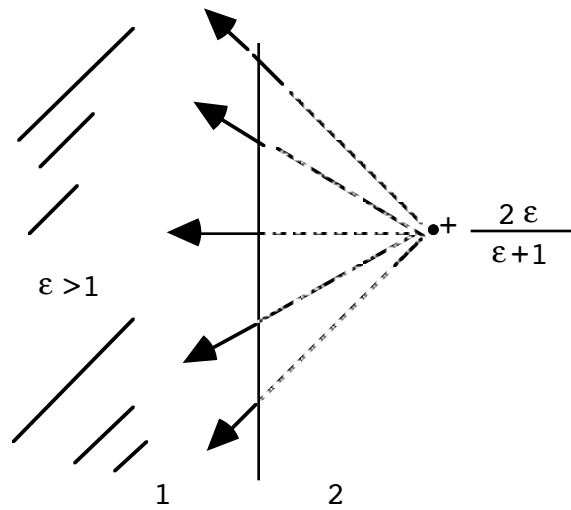
$$4\pi \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')_{\perp}} \frac{1}{2k} e^{-k|z - z'|} = \frac{1}{|\vec{x} - \vec{x}'|}. \quad (4.99)$$

Therefore ($z' > 0$)

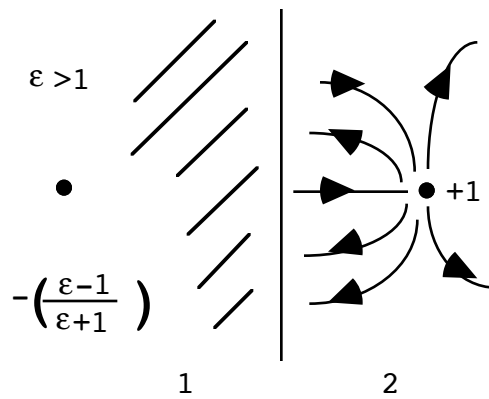
$$z < 0: \quad G(\vec{x}, \vec{x}') = \frac{1}{\varepsilon} \frac{2\varepsilon}{\varepsilon + 1} \frac{1}{|\vec{x} - \vec{x}'|}, \quad (4.100)$$

$$z > 0: \quad G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{\varepsilon - 1}{\varepsilon + 1} \frac{1}{|\vec{x} - \vec{x}''|}. \quad (4.101)$$

where $\vec{x}'' = (x', y', -z')$. Also gives $z' < 0$, $z > 0$ solution from symmetry of G : $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$. (Eq.(4.64) is the reason we are writing (4.100) in the above form.) Interpretation: ($z < 0$)



$z > 0$:



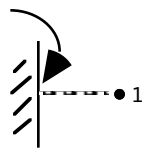
Put them both together for final solution.
Charge on the interface?

$$-\vec{\nabla} \cdot \vec{P} = \rho_{\text{bound}}, \quad (4.102)$$

$$\Rightarrow -(\vec{P}_2 - \vec{P}_1) \cdot \hat{n}_{21} = \sigma_{\text{bound}}. \quad (4.103)$$

$$\vec{P}_1 = \frac{\epsilon-1}{4\pi} \vec{E}_1, \quad \vec{P}_2 = 0, \quad (4.104)$$

$$\Rightarrow \sigma_{\text{bound}} = \frac{-1}{2\pi} \frac{\epsilon-1}{\epsilon+1} \frac{z'}{(\rho^2+z'^2)^{3/2}}. \quad (4.105)$$



($\rho^2=x^2+y^2$, as measured from .) For what it's worth, there is a surface delta function here, as we have seen before for a conductor:

$$\lim_{z' \rightarrow 0^+} \frac{-2z'}{[\rho^2 + z'^2]^{3/2}} \rightarrow -4\pi\delta(\vec{x}_{\perp} - \vec{x}'_{\perp}), \quad (4.106)$$

$$\Rightarrow \sigma_{\text{bound}} \rightarrow -\frac{\epsilon-1}{\epsilon+1} \delta(\vec{x}_{\perp} - \vec{x}'_{\perp}). \quad (4.107)$$

Look at special cases: $\epsilon \rightarrow \infty$ (perfect conductor)

$$z < 0: G(\vec{x}, \vec{x}') = 0. \text{ (no } \vec{E} \text{ field in conductor)} \quad (4.108)$$

$$z > 0: G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}''|}. \quad (4.109)$$

Neumann B.C. given as $\epsilon \rightarrow 0$. ($\vec{D}$ instead of $\vec{E}$ vanishes for $z < 0$.)

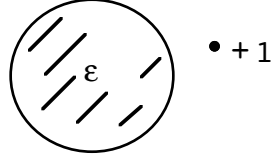
Trivial case, $\epsilon \rightarrow 1$:

$$\text{all } z: G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|}.$$

(Can do all this also by method of images.)

4.7 Green function for the dielectric sphere

Next problem: (dielectric sphere with unit charge outside)



Need to solve:

$$r > a: \quad -\nabla^2 G(\vec{x}, \vec{x}') = 4\pi\delta(\vec{x} - \vec{x}'), \quad (4.110)$$

$$r < a: \quad -\vec{\nabla} \cdot [\epsilon \vec{\nabla} G(\vec{x}, \vec{x}')] = 0. \quad (4.111)$$

Assume

$$G(\vec{x}, \vec{x}') = 4\pi \sum_{\ell, m} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) g_{\ell}(r, r'). \quad (4.112)$$

As usual, get

$$-\frac{\partial}{\partial r} \left(r^2 \frac{dg_{\ell}}{\partial r} \right) + \ell(\ell + 1)g_{\ell} = \delta(r - r'), \quad r > a \quad (4.113)$$

$$\epsilon \left[-\frac{d}{dr} \left(r^2 \frac{dg_{\ell}}{dr} \right) + \ell(\ell + 1)g_{\ell} \right] = 0, \quad r < a. \quad (4.114)$$

BC are

$$g_{\ell} \Big|_{a-}^{a+} = 0. \quad (\Phi \text{ is cont.} \Rightarrow \vec{E}_{||} \text{ cont.}) \quad (4.115)$$

$$\epsilon \frac{\partial g_{\ell}}{\partial r} \Big|_{a-} = \frac{\partial g_{\ell}}{\partial r} \Big|_{a+}. \quad (D_n \text{ is continuous}) \quad (4.116)$$

The solutions are ($r' > a$)

$$r < a: \quad g_{\ell} = A_{\ell} r^{\ell}, \quad (4.117)$$

$$a < r < r': \quad g_{\ell} = B_{\ell} r^{\ell} + C_{\ell} r^{-\ell-1}, \quad (4.118)$$

$$r' < r: \quad g_{\ell} = D_{\ell} r^{-\ell-1}. \quad (4.119)$$

The above BC requires

$$(4.115) \Rightarrow A_{\ell} a = B_{\ell} a + C_{\ell} a^{-\ell-1}, \quad (4.120)$$

$$(4.116) \Rightarrow \varepsilon A_{\ell} \ell a^{\ell-1} = \ell B_{\ell} a^{\ell-1} - (\ell+1) C_{\ell} a^{-\ell-2}. \quad (4.121)$$

from which we find (leave it to you again)

$$B_{\ell} = \frac{\ell(1+\varepsilon)+1}{2\ell+1} A_{\ell}, \quad (4.122)$$

$$C_{\ell} = \frac{\ell(1-\varepsilon)}{2\ell+1} a^{2\ell+1} A_{\ell}. \quad (4.123)$$

Other conditions at $r = r'$ are:

$$g_{\ell} \Big|_{r'^{-}}^{r'^{+}} = 0, \quad (4.124)$$

$$-r'^2 \frac{\partial}{\partial r} g_{\ell} \Big|_{r'^{-}}^{r'^{+}} = 1. \quad (4.125)$$

which give

$$B_{\ell} r'^{\ell} + C_{\ell} r'^{-\ell-1} = D_{\ell} r'^{-\ell-1}, \quad (4.126)$$

$$-r'^2 [-(\ell+1) D_{\ell} r'^{-\ell-2} - (\ell B_{\ell} r'^{\ell-1} - (\ell+1) C_{\ell} r'^{-\ell-2})] = 1. \quad (4.127)$$

Again, I'll just give the solution:

$$A_{\ell} = \frac{1}{\ell(1+\varepsilon)+1} \frac{1}{r'^{\ell+1}}, \quad (4.128)$$

$$B_{\ell} = \frac{1}{2\ell+1} \frac{1}{r'^{\ell+1}}, \quad (4.129)$$

$$C_{\ell} = \frac{-(\varepsilon-1)\ell}{\ell(1+\varepsilon)+1} \frac{1}{2\ell+1} \frac{a^{2\ell+1}}{r'^{\ell+1}}, \quad (4.130)$$

$$D_{\ell} = C_{\ell} + \frac{r'^{\ell}}{2\ell+1}. \quad (4.131)$$

The Green function is now given by: ($r' > a$)

$$r < a: \quad G(\vec{x}, \vec{x}') = \sum_{\ell, m} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \frac{4\pi}{\ell(\ell+1)+1} \frac{r^\ell}{r'^{\ell+1}}, \quad (4.132)$$

$$r > a: \quad G(\vec{x}, \vec{x}') = \sum_{\ell, m} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \frac{4\pi}{2\ell+1} \frac{r^{<\ell}}{r^{>\ell+1}} \\ - \sum_{\ell, m} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \frac{4\pi}{2\ell+1} \frac{\ell(\ell-1)}{\ell(\ell+1)+1} \frac{a^{2\ell+1}}{(rr')^{\ell+1}}. \quad (4.133)$$

$$\text{Can write as } (P_\ell = \sum_m \frac{4\pi}{2\ell+1} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi))$$

$$r < a: \quad G(\vec{x}, \vec{x}') = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{\ell(\ell+1)+1} \frac{r^\ell}{r'^{\ell+1}} P_\ell(\cos\gamma), \quad (\text{also gives } r' < a, \\ r > a \text{ form}) \quad (4.134)$$

$$r > a: \quad G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \sum_{\ell=1}^{\infty} \frac{(\ell-1)\ell}{\ell(\ell+1)+1} \frac{a^{2\ell+1}}{(rr')^{\ell+1}} P_\ell(\cos\gamma). \quad (4.135)$$

No more image charge interpretation (that I know of) except when $\varepsilon \rightarrow \infty$. In this limit we almost recover Eq.(3.275), except for the $\ell=0$ term. This is an explicit realization of the comments regarding this limit in Ch.1. (The $\varepsilon \rightarrow 0$ limit should connect to prob. 3.12.1.)

Look at $r > a$ solution when $r' \gg a$. We have

$$G(\vec{x}, \vec{x}') \approx \frac{1}{|\vec{x} - \vec{x}'|} - \frac{\varepsilon-1}{\varepsilon+2} \frac{a^3}{r^2 r'^2} \cos\gamma, \quad (4.136) \\ (\cos\gamma = \frac{\vec{x} \cdot \vec{x}'}{rr'})$$

$$\Rightarrow G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + \frac{\vec{x} \cdot \vec{p}}{r^3}, \quad \vec{p} = \frac{\varepsilon-1}{\varepsilon+2} a^3 \left(-\frac{\vec{x}'}{r'^3} \right). \quad (4.137) \\ \uparrow \\ \vec{E}'(0)$$

$\vec{p}$ is the induced dipole moment of the sphere due to the positive charge. Says that the dipole moment induced in a sphere of radius a by a uniform electric field is

$$\vec{p} = \frac{\epsilon-1}{\epsilon+2} a^3 \vec{E}_{\text{const.}} \quad (4.138)$$

Other case ($r \ll a$):

$$G(\vec{x}, \vec{x}') \approx \frac{1}{r'} + \frac{3}{\epsilon+2} \frac{r}{r'^2} \cos\gamma, \quad (4.139)$$

$$G(\vec{x}, \vec{x}') = \frac{1}{r'} + \frac{3}{\epsilon+2} \frac{\vec{x} \cdot \vec{x}'}{r'^2}, \quad (4.140)$$

$$G(\vec{x}, \vec{x}') = \frac{1}{r'} - \frac{3}{\epsilon+2} \vec{x} \cdot \vec{E}'(0). \quad (4.141)$$

↑ point charge's electric
field at origin

Says, for the positive charge very far away, the electric field in the sphere is approximately

$$\vec{E} = -\vec{\nabla} G(\vec{x}, \vec{x}') = \frac{3}{\epsilon+2} \vec{E}'(0), \quad (4.142)$$

which is less than $\vec{E}'(0)$ ($\epsilon > 1$).

4.8 Field energy and dielectrics

In the absence of any constitutive relation between $\vec{D}$ and $\vec{E}$, all we know for a given material is

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho, \quad (4.143)$$

$$\vec{\nabla} \times \vec{E} = 0. \quad (4.144)$$

The second eqⁿ implies we may still take

$$\vec{E} = -\vec{\nabla}\Phi. \quad (4.145)$$

We continue to require $\vec{F} = q\vec{E}$ so that Φ as usual has the meaning of potential energy. Energy to move an infinitesimal charge:

$$\delta W_1 = \int_A^B (-\vec{F}_1) \cdot d\vec{\ell} = -\delta q_1 \int_A^B \vec{E} \cdot d\vec{\ell}, \quad (4.146)$$

$\uparrow$ work on the charge

$$\Rightarrow \delta W_1 = \delta q_1 (\Phi_B - \Phi_A). \quad (4.147)$$

Take A to be our reference point (can be at ∞ or any other point)

$$\Phi_A = 0, \quad \Phi_B \rightarrow \Phi(\vec{x}_1). \quad (4.148)$$

Move another charge from A to B':

$$\delta W_2 = \delta q_2 \Phi(\vec{x}_2) + \mathcal{O}(\delta q_1 \delta q_2). \quad (4.149)$$

Add them up:

$$\delta W = \sum_i \delta W_i = \sum_i \delta q_i \Phi(\vec{x}_i), \quad (4.150)$$

$$\delta q_i \rightarrow d^3x \delta \rho(\vec{x}), \quad (4.151)$$

$$\delta W = \int d^3x \delta \rho(\vec{x}) \Phi(\vec{x}) \quad \underline{\text{always true}}. \quad (4.152)$$

($W = \frac{1}{2} \int d^3x \rho(\vec{x}) \Phi(\vec{x})$ is not, however, necessarily implied by this.) Now since

$$\rho = \frac{1}{4\pi} \vec{\nabla} \cdot \vec{D}, \quad (4.153)$$

$$\Rightarrow \delta \rho = \frac{1}{4\pi} \vec{\nabla} \cdot \delta \vec{D}, \quad (4.154)$$

$$\Rightarrow \delta W = \frac{1}{4\pi} \int d^3x \vec{\nabla} \cdot \delta \vec{D} \Phi(\vec{x}), \quad (4.155)$$

$$\Rightarrow \delta W = \frac{1}{4\pi} \int d^3x \left[\vec{\nabla} \cdot (\delta \vec{D} \Phi) - \delta \vec{D} \cdot \nabla \Phi \right], \quad (4.156)$$

$$\Rightarrow \delta W = \frac{1}{4\pi} \int d^3x \vec{E} \cdot \delta \vec{D} + \frac{1}{4\pi} \int_S (\Phi \delta \vec{D}) \cdot \hat{n} da. \quad (4.157)$$

Surface term vanishes for localized charge distribution:

$$\frac{1}{4\pi} \int_S (\delta \vec{D} \cdot \hat{n}) \Phi da \xrightarrow{R \rightarrow \infty} \frac{V}{4\pi} \int_S \delta \vec{D} \cdot \hat{n} da = V \delta q, \quad (4.158)$$

But at large distances $V \sim \frac{Q_{\text{tot}}}{R} \rightarrow 0$ as $R \rightarrow \infty$,

$$\Rightarrow \delta W = \frac{1}{4\pi} \int d^3x \vec{E} \cdot \delta \vec{D}. \quad (4.159)$$

As far as we can go unless we can write this as a perfect differential. Assuming relations of the form

$$D_\alpha = \sum_\beta \epsilon_{\alpha\beta} E_\beta, \quad (4.160)$$

we have ($\epsilon_{\alpha\beta}$ are assumed independent of variations in the charge density)

$$\delta D_\alpha = \sum_\beta \epsilon_{\alpha\beta} \delta E_\beta, \quad (4.161)$$

$$\Rightarrow \sum_\alpha E_\alpha \delta D_\alpha = \sum_{\alpha, \beta} \epsilon_{\alpha\beta} \delta E_\beta E_\alpha. \quad (4.162)$$

Also

$$\sum_\alpha D_\alpha \delta E_\alpha = \sum_{\alpha, \beta} \epsilon_{\alpha\beta} E_\beta \delta E_\alpha, \quad (4.163)$$

$$\Rightarrow \vec{D} \cdot \delta \vec{E} = \vec{E} \cdot \delta \vec{D} \quad \text{if } \epsilon_{\alpha\beta} = \epsilon_{\beta\alpha}, \quad (4.164)$$

$$\Rightarrow \vec{E} \cdot \delta \vec{D} = \frac{1}{2} \delta(\vec{E} \cdot \vec{D}). \quad (4.165)$$

[The above certainly includes the case where $\vec{D} = \epsilon \vec{E}$, $\epsilon = \epsilon(\vec{x})$. It excludes, however, for example, a situation where

$$\vec{D} = \epsilon(\vec{E}^2) \vec{E}$$

for then

$$\vec{E} \cdot \delta \vec{D} = \vec{D} \cdot \delta \vec{E} + \delta \epsilon(\vec{E}^2) \vec{E}^2.]$$

So, for a certain class of constitutive relations, we get

$$W = \frac{1}{8\pi} \int d^3x \vec{E} \cdot \vec{D} = \frac{1}{8\pi} \int d^3x \epsilon(\vec{x}) \vec{E}^2(\vec{x}). \quad (4.166)$$

↑
isotropic

Other expressions for W can be developed. In particular since we know all static properties are in $G(\vec{x}, \vec{x}')$, should be able to relate it to W . Since $\vec{E} = -\vec{\nabla} \Phi$, above says of course (localized dist. again)

$$W = \frac{1}{2} \int d^3x \rho(\vec{x}) \Phi(\vec{x}). \quad (4.167)$$

↑↑ tip off that self-energies included

Now remember (Dirichlet)

$$\Phi(\vec{x}) = \int d^3x' \rho(\vec{x}') G_D(\vec{x}, \vec{x}') - \frac{1}{4\pi} \oint_S da' \Phi(\vec{x}') \epsilon(\vec{x}') \frac{\partial G_D}{\partial n'}. \quad (4.168)$$

The surface, S , being referred to in (4.168) are surfaces of the entire volume where fields are defined, not the surfaces of dielectrics. Let's say that $\Phi|_S = 0$ (Certainly true for free space). Then another expression for the energy is

$$W = \frac{1}{2} \int d^3x \rho(\vec{x}) \Phi(\vec{x}) = \frac{1}{2} \int d^3x \int d^3x' \rho(\vec{x}) G_D(\vec{x}, \vec{x}') \rho(\vec{x}'). \quad (4.169)$$

This is just a generalization of: (Ch.1)

$$W = \frac{1}{2} \int d^3x d^3x' \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}. \quad (4.170)$$

Now instead of introducing charge, think of introducing a dielectric. Amount of energy to do this?

$$\Delta W \equiv W - W_0, \quad (4.171)$$

$$\Rightarrow \Delta W = \frac{1}{2} \int d^3x d^3x' \rho(\vec{x}) [G_D(\vec{x}, \vec{x}') - G_D^0(\vec{x}, \vec{x}')] \rho(\vec{x}'), \quad (4.172)$$

$$\begin{aligned} \Delta W &= \frac{1}{2} \int d^3x \rho(\vec{x}) [\Phi(\vec{x}) - \Phi_0(\vec{x})] \\ &= -\frac{1}{8\pi} \int d^3x \vec{D} \cdot \vec{\nabla} [\Phi(\vec{x}) - \Phi_0(\vec{x})] \\ &= \frac{1}{8\pi} \int d^3x \vec{D} \cdot (\vec{E} - \vec{E}_0). \end{aligned} \quad (4.173)$$

Now consider

$$\begin{aligned} \int d^3x \vec{E} \cdot \vec{D} &= \int d^3x \vec{E} \cdot \vec{D}_0 + \underbrace{\int d^3x \vec{E} \cdot (\vec{D} - \vec{D}_0)}_{\int d^3x \Phi \vec{\nabla} \cdot (\vec{D} - \vec{D}_0) = 0} \quad (\text{by parts}). \end{aligned} \quad (4.174)$$

Therefore

$$\Delta W = \frac{1}{8\pi} \int d^3x [\vec{E} \cdot \vec{D}_0 - \vec{D} \cdot \vec{E}_0]. \quad (4.175)$$

If

$$\begin{aligned} &\text{some dielectrics already} \\ &\Downarrow \text{present} \\ \vec{D} &= \epsilon(\mathbf{x}) \vec{E}, \quad \vec{D}_0 = \epsilon_0 \vec{E}_0 \end{aligned} \quad (4.176)$$

$$\begin{aligned} &(\text{integration effectively} \\ &\Downarrow \text{over volume of dielectric}) \\ \Rightarrow \Delta W &= \frac{1}{8\pi} \int d^3x (\epsilon_0 - \epsilon) \vec{E} \cdot \vec{E}_0, \end{aligned} \quad (4.177)$$

or if $\epsilon_0 = 1$

$$\Delta W = \int d^3x \left(-\frac{1}{2} \vec{P} \cdot \vec{E}_0 \right). \quad (4.178)$$

$\Uparrow$ not a perma. dipole

4.9 Bulk forces on dielectrics: theory

By using (4.178) or other means of finding an appropriate energy expression, we may find the total force from

$$\vec{F} \cdot \delta \vec{x} = -\delta_Q W, \quad (4.179)$$

$\Uparrow$ fixed charges

$$\Rightarrow \vec{F}_Q = - \left(\frac{\partial W}{\partial \vec{x}} \right)_Q \quad (4.180)$$

On the other hand, consider the movement of a dielectric in the presence of conductors kept at fixed voltage. (Assume all free charges are on surface of conductors.) Now we expect

$$\vec{F}_V = - \frac{\delta}{\delta \vec{x}} (W + W_b)_V. \quad (4.181)$$

$\Uparrow \quad \Uparrow$ battery energy
field energy

But

$$W = \frac{1}{2} \sum_i \int d\sigma_i(\vec{x}) V_i \quad (4.182)$$

$$= \frac{1}{2} \sum_i Q_i V_i, \quad (4.183)$$

$$\Rightarrow \delta_V W = \frac{1}{2} \sum_i \delta Q_i V_i. \quad (4.184)$$

On the other hand, the battery's change in energy is

$$\delta_V W_b = \sum_i \delta \bar{Q}_i V_i. \quad (4.185)$$

But

$$\delta \bar{Q}_i = -\delta Q_i, \quad (4.186)$$

$$\Rightarrow \delta_V \bar{W}_b = -2\delta_V W, \quad (4.187)$$

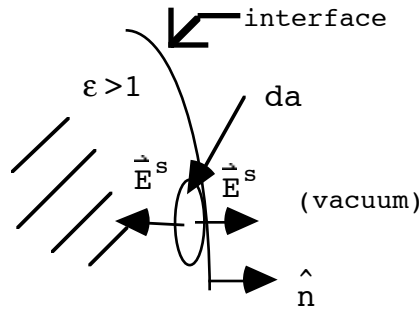
$$\Rightarrow \vec{F}_V = + \left(\frac{\partial W}{\partial \vec{x}} \right)_V. \quad (4.188)$$

It is important to realize that in a given static situation that we must have $\vec{F}_V = \vec{F}_Q$ in spite of the minus sign differences in (4.180) and (4.188).

Energy methods as discussed above are helpful, but they give you no idea of where the forces originate (although they are usually simpler). Go back to our $\vec{F}_{\text{bulk}}$ (from (4.54)):

$$\vec{F}_{\text{bulk}} = \int_S da (\vec{P} \cdot \hat{n}) \vec{E}^{(0)}(\vec{x}). \quad (4.189)$$

Consider a small surface element da :



Near the surface of each da :

$$\vec{E} = \vec{E}^{(0)} + \vec{E}^r + \vec{E}^s, \quad (4.190)$$

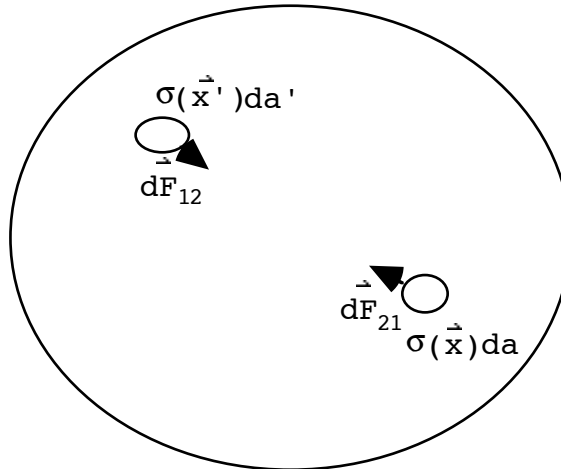
where $\vec{E}^s$ is the self-field ($\vec{E}^s = \pm 2\pi\sigma\hat{n}$), $\vec{E}^r$ is from the rest of the surface and $\vec{E}^{(0)}$ is external. Therefore the average field at the interface is

$$\frac{\vec{E}_1 + \vec{E}_2}{2} = \vec{E}^{(0)} + \vec{E}^r. \quad (4.191)$$

From (4.189) and (4.191)

$$\vec{F}_{\text{bulk}} = \int_S da (\vec{P} \cdot \hat{n}) \left[\frac{\vec{E}_1 + \vec{E}_2}{2} - \vec{E}^r(\vec{x}) \right]. \quad (4.192)$$

Let's consider:



Newton's third law tells us that $d\vec{F}_{12} = -d\vec{F}_{21}$. This implies that the second term in (4.192) is zero when the integration is over the entire surface. This means we can always use $\vec{E}^{(0)}$ or $\frac{\vec{E}_1 + \vec{E}_2}{2}$ in such expressions. However, this is not to say that there are not self-forces or stresses w/i a given material; one only has to recall the outward pressure on the surfaces of a conductor, $2\pi\sigma^2$, we found in Ch. 2. to realize this.

4.10 Nonlinear dielectric example: a phenomenological quark confinement model

Before I go on, I want to develop one model where there is a nonlinear relation between $\vec{E}$ and $\vec{D}$. General expression:

$$\delta W = \frac{1}{4\pi} \int d^3x \vec{E} \cdot \delta \vec{D}. \quad (4.193)$$

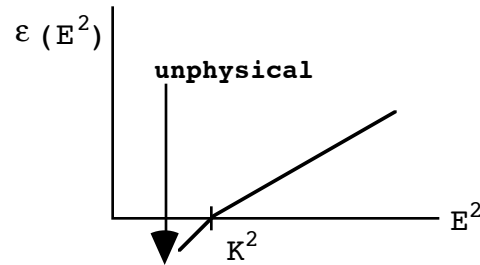
Model:

$$\vec{E} = -\vec{\nabla}\Phi, \quad (4.194)$$

$$\vec{D} = \epsilon \vec{E}, \quad (4.195)$$

$$\text{but } \epsilon = 2\alpha \ln\left(\frac{E^2}{K^2}\right) \quad (E^2 = \vec{E} \cdot \vec{E}, \alpha > 0). \quad (4.196)$$

Extremely nonlinear. Picture:



Fix this up by saying that $D=0$ (or $\epsilon=0$) outside the region where $E^2 > K^2$. Our only hope for getting an expression for W is if we can write the above δW as a perfect differential.

Consider

$$\delta \left[\frac{1}{2} \vec{E} \cdot \vec{D} + \alpha E^2 \right] = \frac{1}{2} \vec{D} \cdot \delta \vec{E} + \frac{1}{2} \vec{E} \cdot \delta \vec{D} + 2\alpha \vec{E} \cdot \delta \vec{E}. \quad (4.197)$$

Now

$$\delta \vec{D} = \delta(\epsilon \vec{E}) = \left(2\alpha \ln \frac{E^2}{K^2} \right) \delta \vec{E} + \frac{4\alpha}{E^2} (\vec{E} \cdot \delta \vec{E}) \vec{E}, \quad (4.198)$$

$$\Rightarrow \vec{E} \cdot \delta \vec{D} = \left(2\alpha \ln \frac{E^2}{K^2} \right) \vec{E} \cdot \delta \vec{E} + 4\alpha \vec{E} \cdot \delta \vec{E}, \quad (4.199)$$

$$= \vec{D} \cdot \delta \vec{E} + 4\alpha \vec{E} \cdot \delta \vec{E}, \quad (4.200)$$

$$\Rightarrow \delta \left[\frac{1}{2} \vec{E} \cdot \vec{D} + \alpha E^2 \right] = \frac{1}{2} \vec{E} \cdot \delta \vec{D} - 2\alpha \vec{E} \cdot \delta \vec{E} + 2\alpha \vec{E} \cdot \delta \vec{E} + \frac{1}{2} \vec{E} \cdot \delta \vec{D}, \quad (4.201)$$

$$\delta \left[\frac{1}{2} \vec{E} \cdot \vec{D} + \alpha E^2 \right] = \vec{E} \cdot \delta \vec{D} ! \quad (4.202)$$

Therefore

$$W = \frac{1}{4\pi} \int d^3x \left[\frac{1}{2} \vec{E} \cdot \vec{D} + \alpha \vec{E}^2 \right]. \quad (4.203)$$

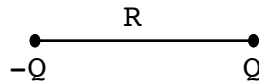
If we write

$$E = |\vec{E}| \equiv Kf(D), \quad (4.204)$$

then from the above we have that

$$f(D) \geq 1 \quad (4.205)$$

whenever $E^2 \geq K^2$. This can be used to give a lower bound on the energy of certain charge configurations. Given



then

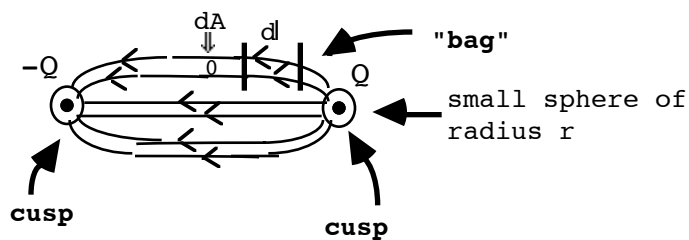
$$W > \underbrace{\frac{1}{8\pi} \int d^3x \vec{E} \cdot \vec{D}}_{ED} = \frac{K}{8\pi} \int d^3x f(D) D, \quad (4.206)$$

$$\Rightarrow W > \frac{K}{8\pi} \int d^3x D. \quad (\text{a pos. def. number}). \quad (4.207)$$

Choose

$$d^3x = dl dA. \quad (4.208)$$

Picture:



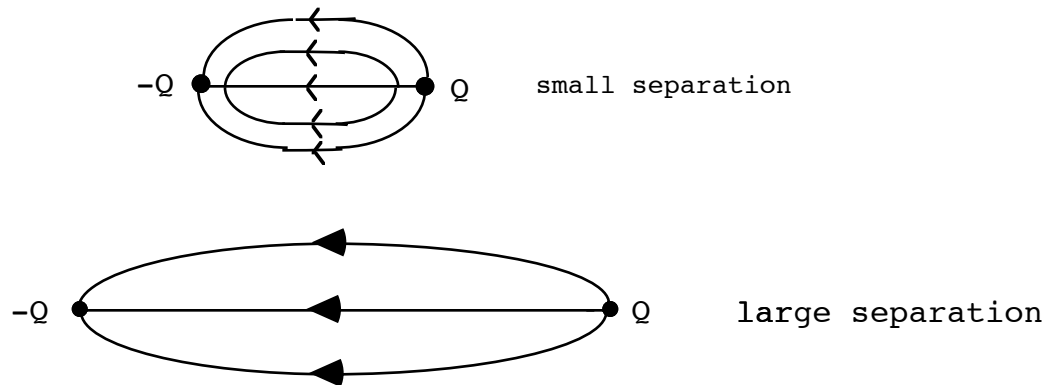
$$W > \frac{K}{8\pi} \int d\ell dA \quad D > \frac{K\ell_{\min}}{8\pi} \int d\vec{A} \cdot \vec{D}, \quad (4.209)$$

$$\underbrace{4\pi |Q|}$$

$$(\ell_{\min} = R - 2r)$$

$$\Rightarrow W > \frac{1}{2} K(R - 2r) |Q|. \quad (\text{only for } R \gg r) \quad (4.210)$$

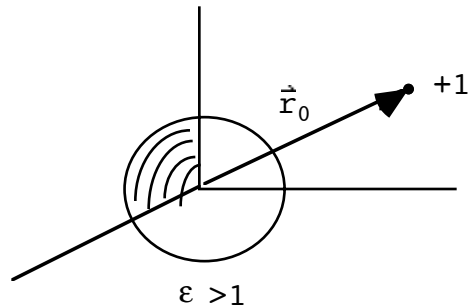
$\Rightarrow$ Potential grows at least linearly at large R . By computer simulation, can show that it in fact saturates the lower limit for large R . What does this describe?



System becomes string-like and confined. This is a phenomenological quark model for mesons due to S.L. Adler. The three quark (baryon) equations, which have an effective $U(1) \times U(1)$ symmetry, may also be derived (Milton, Wilcox, and Pinsky).

4.11 Bulk forces on dielectrics: examples

Finish this up with two problems as examples of forces on dielectrics. First problem (similar to one assigned):



Get the force between the dielectric and the positive unit charge. Do it 3 ways. (I must be mad.) First way (easiest):

$$\Delta W = \frac{1}{2} \int d^3x d^3x' \rho(\vec{x}) [G_D(\vec{x}, \vec{x}') - G_D^0(\vec{x}, \vec{x}')] \rho(\vec{x}'). \quad (4.211)$$

Use

$$\rho(\vec{x}) = \delta(\vec{x} - \vec{r}_0). \quad (4.212)$$

Then (use $r > a$ form of $G_D(\vec{x}, \vec{x}')$)

$$\Delta W = \frac{1}{2} [G_D(\vec{x}, \vec{x}') - G_D^0(\vec{x}, \vec{x}')] \Big|_{\vec{x}, \vec{x}' = \vec{r}_0}. \quad (4.213)$$

be careful!

(must be done as a limit since $G_D(\vec{x}, \vec{x}), G_D^0(\vec{x}, \vec{x}) = \infty$.) This gives

$$\Delta W = - \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{(\epsilon-1)\ell}{\ell(\epsilon+1)+1} \frac{a^{2\ell+1}}{r_0^{2\ell+2}} P_{\ell}(1), \quad (4.214)$$

=1

$$\Rightarrow \Delta W = - \frac{\epsilon-1}{2r_0} \sum_{\ell=1}^{\infty} \frac{\ell}{\ell(1+\epsilon)+1} \left(\frac{a}{r_0}\right)^{2\ell+1}. \quad (4.215)$$

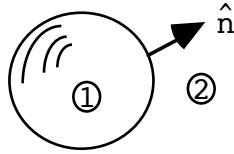
At large r_0 :

$$\Delta W \approx - \left(\frac{\epsilon-1}{\epsilon+2}\right) \frac{a^3}{2r_0^4} \Rightarrow F_r = - \left(\frac{\epsilon-1}{\epsilon+2}\right) \frac{2a^3}{r_0^5}. \quad (4.216)$$

↑ force on charge (take origin on sphere)

Always pulled toward charge ($\epsilon > 1$). Force is different from grounded conducting sphere which is inverse cube at large r_0 (but the same for neutral, isolated sphere). Same problem using explicit force expression: (hardest way)

$$F_z = \int da \ (\vec{P} \cdot \hat{n}) \left(\frac{E_{2z} + E_{1z}}{2} \right). \quad (4.217)$$



$$(\vec{P}_1 \cdot \hat{n}) = \frac{1}{4\pi} (E_{2r} \Big|_a - E_{1r} \Big|_a), \quad (4.218)$$

$$G_1 \approx \underbrace{\frac{1}{r'} + \frac{3}{2+\epsilon} \frac{r}{r'^2} \cos\theta}_{P_1} + \underbrace{\frac{5}{3+2\epsilon} \frac{r^2}{r'^3} \frac{1}{2} (3\cos^2\theta - 1)}_{P_2}, \quad (4.219)$$

$$\begin{aligned} & \text{(taking z-axis along } \vec{r}' = \vec{r}_0) \\ G_2 \approx & \frac{1}{r'} + \frac{r}{r'^2} \cos\theta + \frac{r^2}{r'^3} \frac{1}{2} (3\cos^2\theta - 1) - \frac{(\epsilon-1)}{2+\epsilon} \frac{a^3}{r^2 r'^2} \cos\theta \\ & - \frac{2(\epsilon-1)}{3+2\epsilon} \frac{a^5}{r^3 r'^3} \frac{1}{2} (3\cos^2\theta - 1) + \dots \end{aligned} \quad (4.220)$$

$$E_{1r} = - \frac{\partial}{\partial r} G_1, \quad E_{2r} = - \frac{\partial}{\partial r} G_2, \quad (4.221)$$

$$\Rightarrow E_{1r} \Big|_a \approx - \frac{3}{2+\epsilon} \frac{1}{r'^2} \cos\theta - \frac{5}{3+2\epsilon} \frac{a}{r'^3} (3\cos^2\theta - 1), \quad (4.222)$$

$$E_{2r} \Big|_a \approx - \frac{3\epsilon}{2+\epsilon} \frac{1}{r'^2} \cos\theta - \frac{5\epsilon}{3+2\epsilon} \frac{a}{r'^3} (3\cos^2\theta - 1), \quad (4.223)$$

$$\Rightarrow \vec{P} \cdot \hat{n} \Big|_a \approx \frac{1}{4\pi} \left[\frac{3(1-\epsilon)}{2+\epsilon} \frac{1}{r'^2} \cos\theta + \frac{5(1-\epsilon)}{3+2\epsilon} \frac{a}{r'^3} (3\cos^2\theta - 1) \right]. \quad (4.224)$$

Likewise

$$E_{1z} = - \frac{\partial}{\partial z} G_1, \quad (4.225)$$

$$E_{1z} = - \frac{\partial}{\partial z} \left(\frac{3}{2+\epsilon} \frac{z}{r'^2} + \frac{5\epsilon}{3+2\epsilon} \frac{1}{r'^3} \frac{1}{2} (3z^2 - r^2) + \dots \right). \quad (4.226)$$

$$\frac{\partial}{\partial z} \left(\frac{1}{r} \right) = - \frac{z}{r^3}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$E_{1z} \Big|_a = - \frac{3}{2+\epsilon} \frac{1}{r'^2} - \frac{10}{3+2\epsilon} \frac{a}{r'^3} \cos\theta, \quad (4.227)$$

$$E_{2z} \Big|_a = - \frac{3}{2+\epsilon} \frac{1}{r'^2} - \frac{3(\epsilon-1)}{2+\epsilon} \frac{1}{r'^2} \cos^2\theta - \frac{2a}{r'^3} \cos\theta \\ + \frac{(\epsilon-1)}{3+2\epsilon} \frac{a}{r'^3} [-15 \cos^3\theta + 9 \cos\theta]. \quad (4.228)$$

$$\Rightarrow \frac{E_{2z}+E_{1z}}{2} \Big|_a = - \frac{3}{2+\epsilon} \frac{1}{r'^2} - \frac{3}{2} \frac{(\epsilon-1)}{2+\epsilon} \frac{1}{r'^2} \cos^2\theta \\ - \left(\frac{8+2\epsilon}{3+2\epsilon} \right) \frac{a}{r'^3} \cos\theta + \frac{(\epsilon-1)}{3+2\epsilon} \frac{a}{r'^3} \left[- \frac{15}{2} \cos^3\theta + \frac{9}{2} \cos\theta \right]. \quad (4.229)$$

Notice that $\frac{1}{r'^4}$ terms go like $\int_{-1}^1 d\cos\theta \left(\frac{\cos\theta}{\cos^3\theta} \right) = 0$. Lowest order terms:

$$F_z = \frac{2\pi a^3}{4\pi r'^5} \int_{-1}^1 dx \left\{ \frac{3(1-\epsilon)}{2+\epsilon} \frac{(\epsilon-1)}{3+2\epsilon} \left[- \frac{15}{2} x^4 + \frac{9}{2} x^2 \right] - \frac{3(1-\epsilon)}{2+\epsilon} \left(\frac{8+2\epsilon}{3+2\epsilon} \right) x^2 \right. \\ \left. + \frac{5(1-\epsilon)}{3+2\epsilon} (3x^2-1) \left[- \frac{3}{2+\epsilon} - \frac{3}{2} \frac{(\epsilon-1)}{2+\epsilon} x^2 \right] \right\}, \quad (4.230)$$

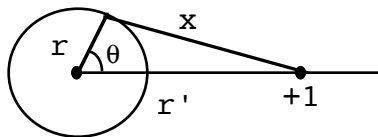
◦
◦ (much algebra)
◦

$$F_z = \frac{2a^3}{r'^5} \left(\frac{\epsilon-1}{2+\epsilon} \right). \quad (4.231)$$

Can also calculate the force explicitly using the external field:

$$F_z = \int da (\vec{P} \cdot \hat{n}) E_z^0. \quad (4.232)$$

$\Uparrow$ external field



$$\Phi_0 = \frac{1}{x} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{r'^{\ell+1}} P_\ell(\cos\theta), \quad (4.233)$$

$$\Phi_0 \approx \frac{1}{r'} + \frac{r}{r'^2} \cos\theta + \frac{r^2}{r'^3} \frac{1}{2} (3\cos^2\theta - 1) + \dots, \quad (4.234)$$

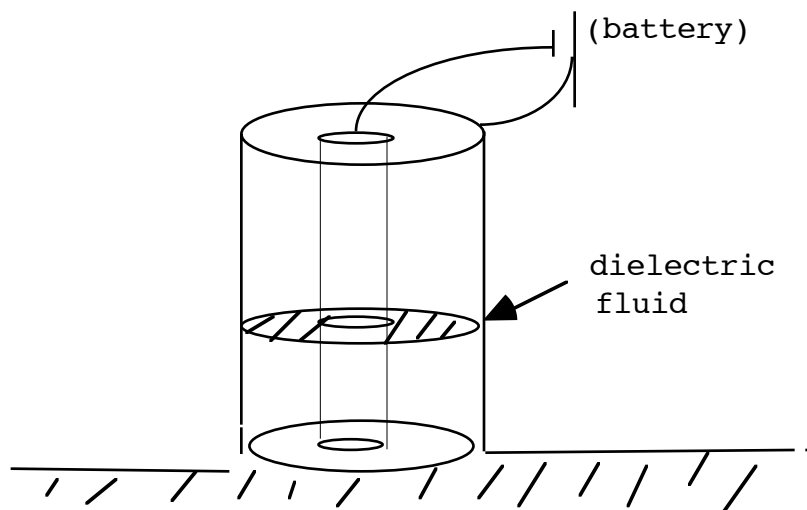
$$\Rightarrow E_z^0 \Big|_a = - \frac{\partial}{\partial z} \Phi_0 \Big|_a \approx - \frac{1}{r'^2} - 2 \frac{a}{r'^3} \cos\theta. \quad (4.235)$$

(Compare with complicated expression for $\frac{E_{2z} + E_{1z}}{2}$ above)

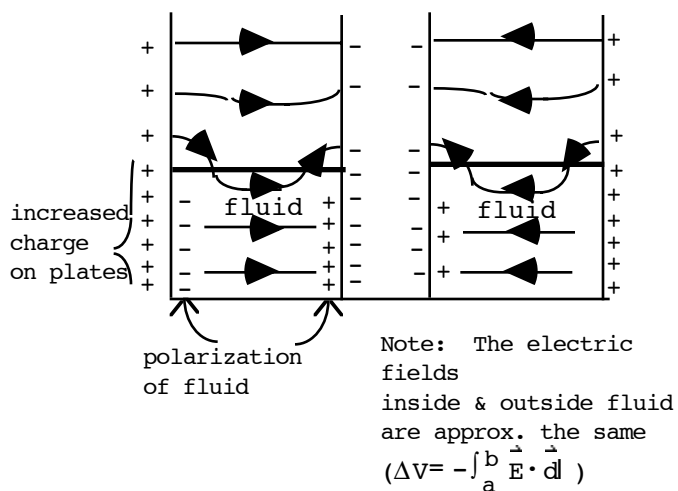
$$F_z \approx 2\pi a^2 \frac{1}{4\pi} \frac{a}{r'^5} \int_{-1}^1 dx \left\{ -\frac{6(1-\epsilon)}{2+\epsilon} x^2 - \frac{5(1-\epsilon)}{3+2\epsilon} (3x^2 - 1) \right\}. \quad (4.236)$$

$$F_z \approx \frac{2a^3}{r'^5} \frac{\epsilon - 1}{2 + \epsilon}. \quad (4.237)$$

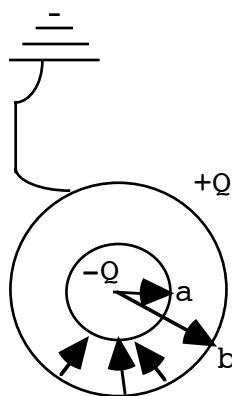
Another problem: Find the fluid level of a dielectric fluid inside a cylindrically shaped capacitor.



What's going on: (cross section of cylindrical tank)



Top view:



Use an energy method-easier. Inside or outside the fluid:

$$E_r \approx - \frac{V}{\ell \ln \frac{b}{a}} \frac{1}{r}. \quad (4.238)$$

↑↑ inward if $V > 0$

$$4\pi Q_{\text{plates}} = \int \vec{D} \cdot \hat{n} \, da \Rightarrow \Delta Q_{\text{plates}} = \frac{V \Delta z}{2 \ell \ln b/a} (\epsilon - 1), \quad (4.239)$$

$$= \frac{2\pi V \Delta z}{\ell \ln b/a} \chi. \quad (4.240)$$

battery supplies energy ↓↓

$$\Delta W_{\text{battery}} = V \Delta Q_{\text{battery}} = - V \Delta Q_{\text{plates}} = - \frac{2\pi \chi \Delta z V^2}{\ell \ln b/a}. \quad (4.241)$$

Add up energies:

$$\Delta W^{\text{tot}} = \Delta W^{\text{gravity}} + \Delta W^{\text{field}} + \Delta W^{\text{battery}} = 0, \quad (4.242)$$

$$\Delta W^{\text{gravity}} = \rho g (\Delta z)^2 \pi (b^2 - a^2), \quad (4.243)$$

$$\Delta W^{\text{field}} = \frac{1}{2} V \Delta Q^{\text{plates}}, \quad (= -\frac{1}{2} \Delta W^{\text{battery}}) \quad (4.244)$$

$$\Rightarrow 0 = \rho g (\Delta z)^2 \pi (b^2 - a^2) - \frac{\pi \chi \Delta z V^2}{\epsilon \ln b/a}, \quad (4.245)$$

$$\Rightarrow \chi \approx \frac{(b^2 - a^2) \rho g \Delta z \ln(b/a)}{V^2}. \quad (4.246)$$

Problems

4.1.1 Theorem: It is always possible to find an origin such that the dipole moments, $\vec{p}$, vanish for a charge distribution whose total charge, q , is nonvanishing.

Either prove this theorem or give a counter-example.

4.1.2 A charge distribution has multipole moments q , $\vec{p}$, Q_{ij} with respect to one set of coordinate axes, and moments q' , $\vec{p}'$ and Q'_{ij} with respect to another set whose origin is located at the point $\vec{R} = (X, Y, Z)$ relative to the first. (The axes are parallel.) Determine explicitly the connections between the monopole, dipole and quadrupole moments in the two coordinate frames.

4.1.3 The center of a cubical volume with sides L is placed at the coordinate origin. It's sides are aligned perpendicularly with the x, y, z axes. Within the volume is a charge density $\rho(x, y, z) = Kx$, where K is a constant.

(a) Calculate the monopole, dipole and all the quadrupole moments of this charge distribution.

(b) What is the leading form of the electric field far away from the cube?

4.1.4 Work out the form of the next pole element, R_{ijk} , in the expansion (see Eq.(4.12) of the text),

$$\Phi(\vec{x}) = \frac{q}{r} + \frac{\vec{x} \cdot \vec{p}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \frac{1}{6} \sum_{i,j,k} R_{ijk} \frac{x_i x_j x_k}{r^7} + \dots$$

Show that R_{ijk} has only 7 independent elements.

4.2.1 For a cylindrically symmetric quadrupole ($Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$, all other Q_{ij} 's = 0) in an external potential, $\Phi(\vec{x})$, show that the energy of interaction between the field and the quadrupole is

$$W = \frac{1}{4} Q_{33} \frac{\partial^2 \Phi}{\partial z^2},$$

and the resulting force on the quadrupole is

$$\vec{F} = \frac{1}{4} Q_{33} \frac{\partial^2 \vec{E}}{\partial z^2}.$$

4.3.1 (a) Show that an alternate form for the force we found in (4.49) of the script is

$$\vec{F} = q \vec{E}^{(0)}(0) + (\vec{p} \cdot \vec{\nabla}) \vec{E}^{(0)}(\vec{x}) \Big|_{\vec{x}=0} + \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \vec{E}^{(0)}(\vec{x})}{\partial x_i \partial x_j} \Big|_{\vec{x}=0} + \dots$$

Which form is more general and why?

(b) Use the third term on the right above to argue that atoms with a nonzero quadrupole moment density, $q_{ij}(\vec{x}) \equiv n(\vec{x}) Q_{ij}(\vec{x})$, contribute a bulk effective charge density,

$$\rho_{\text{eff}}^0(\vec{x}) = \frac{1}{6} \sum_{i,j} \frac{\partial^2 q_{ij}(\vec{x})}{\partial x_i \partial x_j}.$$

(c) Show that the above term can also be written as an effective contribution to $\vec{p}$.

4.4.1 (a) Show that in the volume of a linear, isotropic material with dielectric constant ϵ , the free charge and bound charge densities are related by

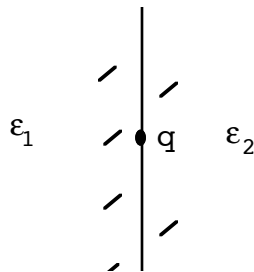
$$\rho_{\text{bound}} = \left(\frac{1-\epsilon}{\epsilon} \right) \rho_{\text{free}}.$$

(b) Show that (a) implies that the total bound surface charge for arbitrary geometry is given as

$$\int da \sigma_{\text{bound}} = \left(\frac{\epsilon-1}{\epsilon} \right) Q_{\text{free}},$$

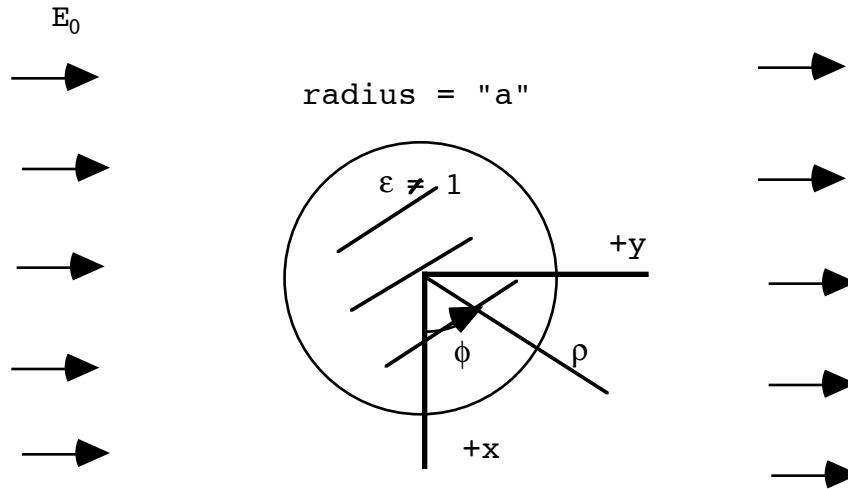
where Q_{free} is the total free charge in the volume.

4.4.2 A point free charge, q , is located directly at the plane interface of two infinite dielectric slabs, as shown. The dielectric constant on the left is ϵ_1 , on the right, ϵ_2 .



Starting from first principles, find the $\vec{E}$ and $\vec{D}$ fields everywhere.

4.4.3 An infinitely long cylinder of dielectric material is placed in an initially uniform electric field of magnitude E_0 pointing in the $+y$ direction as shown.



Treating this as a boundary value problem, show that the potentials inside and outside the cylinder are given by

$$\Phi_{\text{in}} = - \frac{2E_0}{\epsilon+1} \rho \sin \phi,$$

$$\Phi_{\text{out}} = - E_0 \rho \sin \phi + E_0 \frac{\epsilon-1}{\epsilon+1} a^2 \rho^{-1} \sin \phi .$$

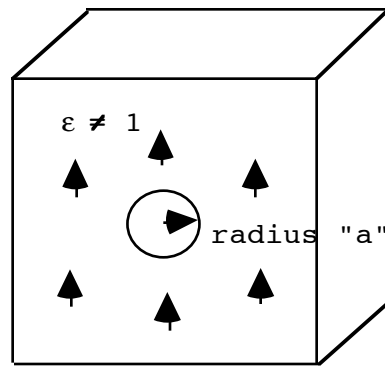
4.4.4 (a) Given a material with a space dependent polarization, $\vec{P}$, show that the potential at an arbitrary point is given by

$$\Phi(\vec{x}) = \int d^3x' \frac{\vec{P}(\vec{x}') \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3},$$

$$\Phi(\vec{x}) = \int ds' \frac{\vec{P} \cdot \hat{n}'}{|\vec{x} - \vec{x}'|} - \int d^3x' \frac{\vec{\nabla}' \cdot \vec{P}(\vec{x}')}{|\vec{x} - \vec{x}'|}.$$

where $\hat{n}'$ is the volume outward normal. [Note: In the second form the volume V' is considered not to include the surface, S' .]

(b) Apply part (a) to a spherical bubble of vacuum with radius "a" enclosed in a semi-infinite dielectric slab.

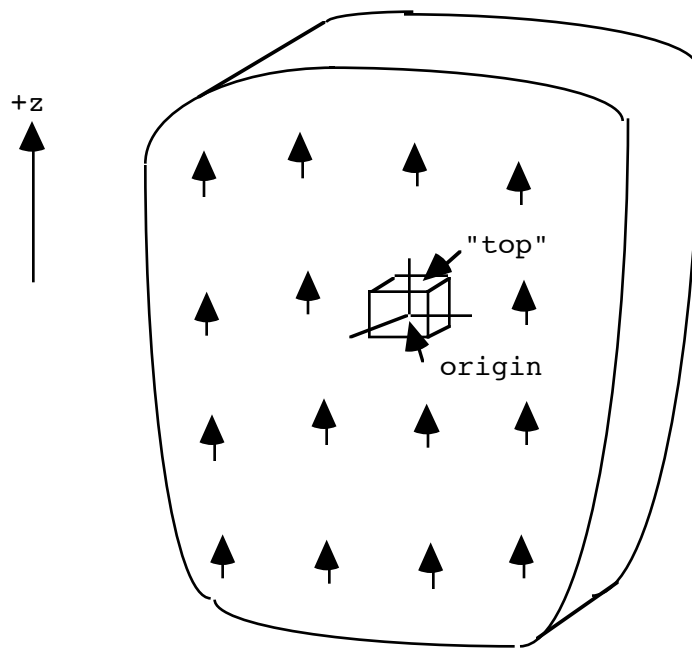


$$\vec{P} = P_0 \hat{k}$$

Assume the slab has a uniform polarization, $\vec{P} = P_0 \hat{z}$, outside the sphere. Show that the electric field everywhere inside the sphere is given by

$$\vec{E} = \frac{4\pi}{3} P_0 \hat{z}.$$

4.4.5 There is a cubic hole of vacuum within a piece of material which has a uniform electric polarization, $\vec{P} = P_0 \hat{z}$:



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(a) Taking the origin of coordinates at the center of the cubic hole, show that the electric field at the origin can be expressed as

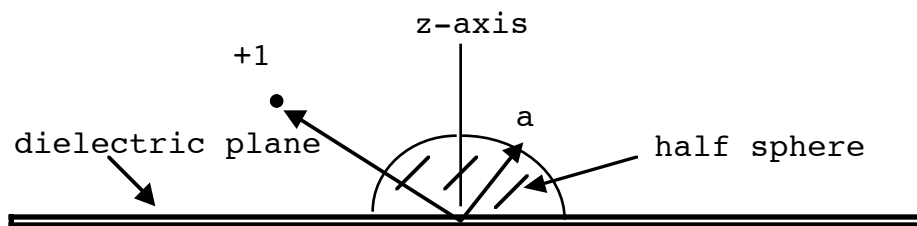
$$\vec{E}(0) = 2P_0 \int_{\text{"top"}} \frac{\vec{x}' da'}{r'^3},$$

where the integration is over the "top" of the cube.

(b) Using symmetry and the concept of solid angle, argue that this expression reduces to

$$\vec{E}(0) = \frac{4\pi}{3} P_0 \hat{z}.$$

4.5.1 Find the Green function for a *conducting* half sphere of radius "a" sphere sitting on top of an infinite *dielectric* plane. The charge is on the side shown.



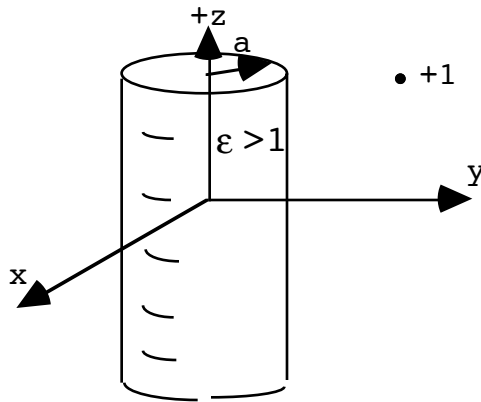
4.5.2 Find the *two dimensional* Green function,

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi \delta(\mathbf{x} - \mathbf{x}'),$$

for a long cylinder of dielectric material with dielectric constant ϵ and radius "a". Consider the case of the charge *outside* the cylinder. [Ans:

$$G(\mathbf{x}, \mathbf{x}') = 4\pi \sum_m \frac{e^{im(\phi - \phi')}}{2\pi} g_m(\rho, \rho'), \quad g_m(\rho, \rho') = \begin{cases} \frac{1}{|m|(1+\epsilon)} \left(\frac{\rho}{\rho'}\right)^{|m|}, & \rho < a \\ \frac{1}{2|m|} \left(\frac{\rho_{<}}{\rho_{>}}\right)^{|m|} \left[1 + \frac{1-\epsilon}{1+\epsilon} \left(\frac{a}{\rho_{<}}\right)^{|m|}\right], & \rho > a \end{cases} \quad .]$$

4.5.3 (a) Consider an infinitely long dielectric ($\epsilon > 1$) cylinder of radius a (3-dimensions), with the z -axis of coordinates along the axis of symmetry of the cylinder.



Assuming the reduced form for the Green function,

$$G(\vec{\mathbf{x}}, \vec{\mathbf{x}}') = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \int_0^{\infty} dk \cos[k(z - z')] g_m(\rho, \rho'),$$

and with

$$4\pi\delta(\vec{\mathbf{x}} - \vec{\mathbf{x}}') = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \int_0^{\infty} dk \cos[k(z - z')] \frac{1}{\rho} \delta(\rho - \rho'),$$

establish the differential equation satisfied by $g_m(\rho, \rho')$ for the case of the point charge exterior to the cylinder.

(b) Given the general form of solutions in the three regions,

$$g_m(\rho, \rho') = \begin{cases} A_m I_m(k\rho), & \rho < a, \\ B_m I_m(k\rho) + C_m K_m(k\rho), & a < \rho < \rho', \\ D_m K_m(k\rho), & \rho' < \rho. \end{cases}$$

where $K_m(k\rho)$, $I_m(k\rho)$ are (linearly independent) Bessel functions of imaginary argument, apply the continuity and boundary conditions to get the equations which determine the coefficients.

4.5.4 Get the actual solution for the coefficients in prob. 4.5.3 above. Ans. (I think!):

$$A_m = \frac{K_m(k\rho')}{ka(\epsilon I_m'(ka)K_m(ka) - I_m(ka)K_m'(ka))},$$

$$B_m = K_m(k\rho'),$$

$$C_m = \frac{I_m(ka)}{K_m(ka)} K_m(k\rho') \left(\frac{1}{ka(\epsilon I_m'(ka)K_m(ka) - I_m(ka)K_m'(ka))} - 1 \right),$$

$$D_m = I_m(k\rho') + C_m.$$

This solution may be written in the $\rho_>, \rho_<$ notation as

$$g_m = \frac{K_m(k\rho_>)I_m(k\rho_<)}{ka(\epsilon I_m'(ka)K_m(ka) - I_m(ka)K_m'(ka))},$$

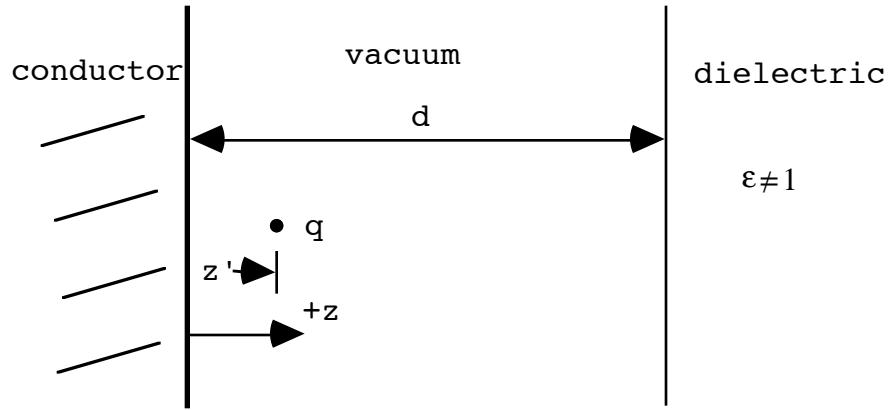
for $\rho < a$, and

$$g_m = K_m(k\rho_>)I_m(k\rho_<) +$$

$$\frac{K_m(k\rho_>)K_m(k\rho_<)}{K_m(ka)} I_m(ka) \left(\frac{1}{ka(\epsilon I_m'(ka)K_m(ka) - I_m(ka)K_m'(ka))} - 1 \right),$$

for $\rho > a$.

4.6.1 A semi-infinite planar dielectric slab, with dielectric constant $\epsilon \neq 1$, is placed parallel to and a distance d above the surface of a perfectly conducting plane.



Take the surface of the conducting plane to be $z = 0$ and the surface of the dielectric slab to be at $z = d > 0$. The Green function, $G_D(\vec{x}, \vec{x}')$, for a positive unit charge in the region between the plate and slab will satisfy the differential equation,

$$-\vec{\nabla}^2 G_D(\vec{x}, \vec{x}') = 4\pi \delta(\vec{x} - \vec{x}'),$$

where $\delta(\vec{x} - \vec{x}')$ is a Dirac delta function. Given the Bessel function expansions,

$$\delta(\vec{x} - \vec{x}') = \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \int_0^{\infty} dk k J_m(k\rho) J_m(k\rho') \delta(z - z'),$$

$$G_D(\vec{x}, \vec{x}') = 4\pi \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \int_0^{\infty} dk k J_m(k\rho) J_m(k\rho') g(z, z'),$$

show that, when boundary conditions at $z = 0, d$ are supplied, this gives

$$g(z, z') = \begin{cases} f(z_{>}) \sinh(kz_{<}) & , z < d \\ K(z') e^{k(2d-z)} & , z > d \end{cases}$$

where $z_{<}(z_{>})$ is the lesser (greater) of z and z' and

$$f(z_{>}) = \frac{e^{kz_{>}}}{k} \left(\frac{1 + \left(\frac{1+\epsilon}{1-\epsilon} \right) e^{2k(d-z_{>})}}{1 + \left(\frac{1+\epsilon}{1-\epsilon} \right) e^{2kd}} \right),$$

$$K(z') = \left(\frac{2}{1-\epsilon} \right) \frac{\sinh(kz')}{k \left(1 + \left(\frac{1+\epsilon}{1-\epsilon} \right) e^{2kd} \right)}.$$

[Note that the radial Laplacian in cylindrical coordinates is

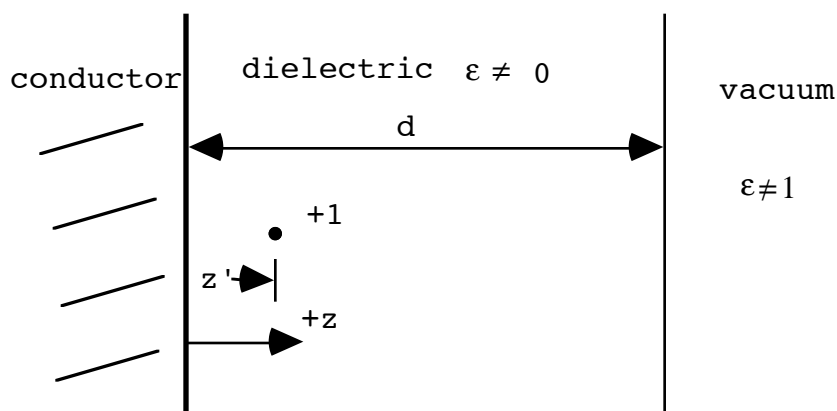
$$\nabla^2 = \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right],$$

and the differential equation $J_m(k\rho)$ satisfies is

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2 \right] J_m(k\rho) = 0.]$$

4.6.2 Let's build on prob. 4.6.1 above. This problem consisted of parallel conductor and dielectric surfaces, as shown above.

Find a simple substitution of parameters which changes the above Dirichlet Green function to the Dirichlet Green function for the following situation:



Write down the Green function, $G^{\text{new}}(\vec{x}, \vec{x}')$, as completely as possible.

4.6.3 Return once again to prob. 4.6.1. (No need to repeat the figure at this point.) The Dirichlet Green function, $G_D(\vec{x}, \vec{x}')$, for a positive unit charge in the region between the plate and slab ($0 < z' < d$) was given by

$$G_D(\vec{x}, \vec{x}') = 4\pi \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \int_0^{\infty} dk k J_m(k\rho) J_m(k\rho') g(z, z'),$$

where $g(z, z')$ satisfied

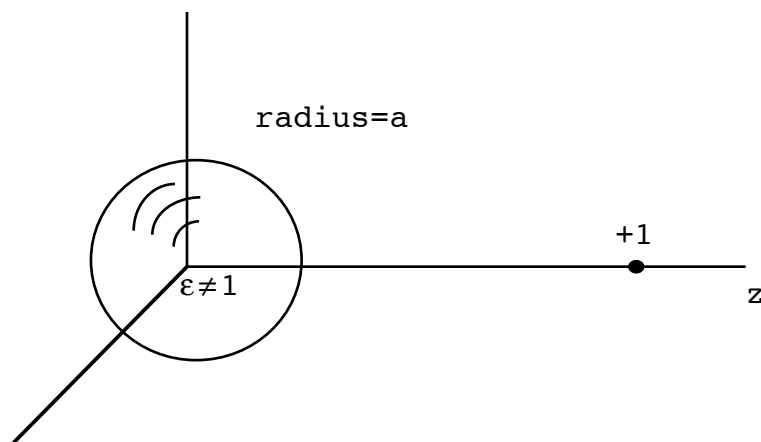
$$\left(k^2 - \frac{\partial^2}{\partial z^2} \right) g(z, z') = \delta(z - z').$$

We found the forms (the explicit forms of $f(z_>)$ and $K(z')$ are not important here)

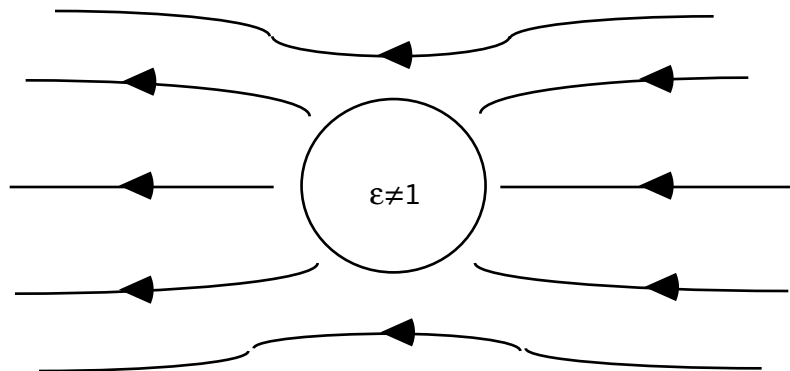
$$g(z, z') = \begin{cases} f(z_>) \sinh(kz_<) & , z < d \\ K(z') e^{k(2d-z)} & , z > d \end{cases}$$

where $z_<(>)$ is the lesser (greater) of z and z' . Now determine $g(z, z')$ for the $z' > d$ branches (the unit charge is in the dielectric). Give the functional z -dependence of each branch. *You need only write the equations which define the new coefficients.*

4.7.1 From the Green function solution in the notes for a positive unit charge in the presence of a spherical dielectric,



generate the potential, both inside and outside, for a spherical dielectric placed in an initially uniform electric field:



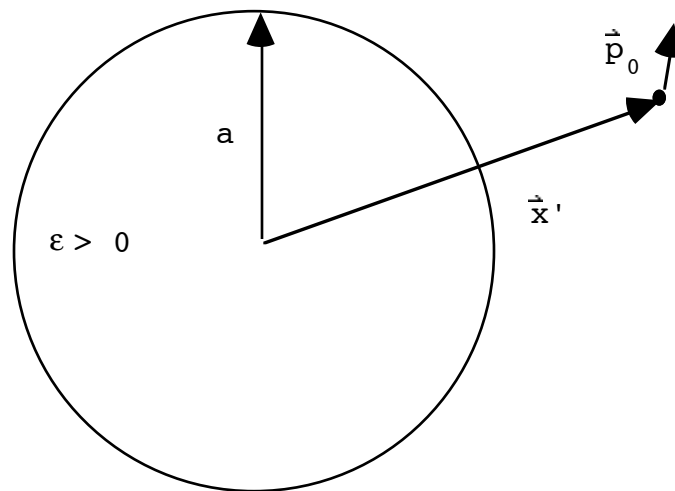
4.7.2 Write down the Green function for a dielectric sphere when $r' < a$ (The unit charge is inside the sphere. Part of the solution can be gotten directly by using the symmetry $G_D(\vec{x}', \vec{x}) = G_D(\vec{x}, \vec{x}')$ and the solution $r' > a$ given in the text.) In the limit $r \gg a$, identify an effective dipole moment of the system and evaluate the total polarization charge on the surface. (Compare with prob. 4.4.1 above.)

4.7.3 (a) Find the Green function for a spherical bubble of vacuum of radius a , embedded in a dielectric medium with a dielectric constant, ϵ . Consider the case of the free unit charge *outside* the sphere. [Hint: Compare the boundary

conditions with that of a dielectric sphere in vacuum on ps.4.20,4.21 of the notes.]

(b) The bubble exists in a dielectric fluid. Find the force between the bubble and a positive unit free charge in the fluid. Attractive or repulsive?

4.7.4 (a) An electric dipole of moment $\vec{p}_0$ (pointing in an arbitrary direction) is located at a distance $\vec{x}'$ away from the center of a dielectric sphere of radius "a".



Find the leading (nontrivial) form of the electric potential, $\Phi(\vec{x})$:

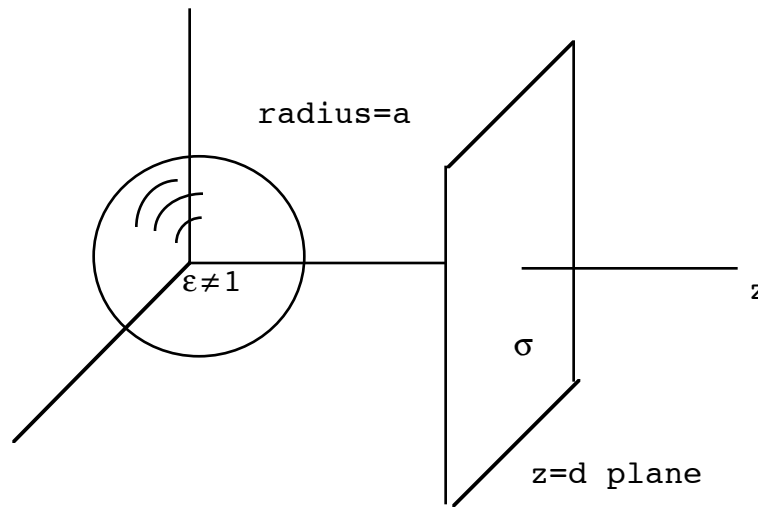
- (a) far away from the sphere and dipole ($r \gg a, r'$)
- (b) close to the sphere's origin ($r \ll a$).

4.7.5 Consider a solid dielectric sphere of radius "a" located inside a conducting sphere of inner radius "b". The centers of the two spheres coincide. Assuming a Green function solution of the form (4.112) of the text with the point charge located at $a < r' < b$, give the functional forms of the function $g_l(r)$ for $r < b$ and the boundary conditions

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which determine the unknown coefficients. (You don't need to solve for the coefficients.)

4.7.6 Use the Green function solution in the notes to generate the inside electric field, $r < a$, for a spherical dielectric placed in the field of a charged plane with surface charge density, σ , located at $z=d$ on the z -axis.



4.8.1 Show that (integrations over all space):

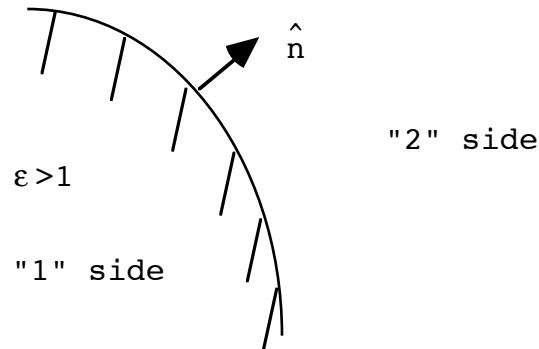
(a) $\int d^3x \vec{E} \cdot \vec{D} = 0$ for a piece of matter with intrinsic polarization, $\vec{P}$ (and no free charges).

(b) Using (a) or other means, present a simple argument that electrostatic field energy, $W^f = \frac{1}{2} \int d^3x \Phi \rho$, in the case of intrinsic polarization is given by $-\frac{1}{2} \int d^3x \vec{P} \cdot \vec{E}$.

4.9.1 (a) Show that the normal force per unit area, $\vec{\mathcal{F}} \cdot \hat{n}$, ($\hat{n}$ directed outward from the dielectric) on an arbitrary dielectric surface is given by

$$\vec{\mathcal{F}} \cdot \hat{n} = \frac{1}{8\pi} (E_{2n}^2 - E_{1n}^2),$$

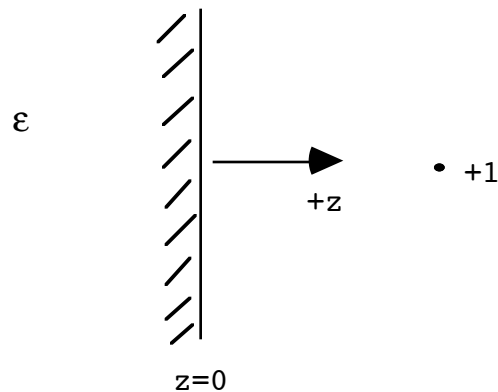
where E_{2n} , E_{1n} are surface normal components of $\vec{E}_2$ and $\vec{E}_1$.



(b) When only bound charge, σ_b , is present show that

$$\vec{\mathcal{F}} \cdot \hat{n} = 2\pi \sigma_b^2 \left(\frac{1+\epsilon}{\epsilon-1} \right).$$

4.9.2 Calculate the force on the half-infinite dielectric plane,



due to the presence of the unit charge at $z' > 0$. Do it:

(a) By evaluating the force between the unit charge and the image charge.

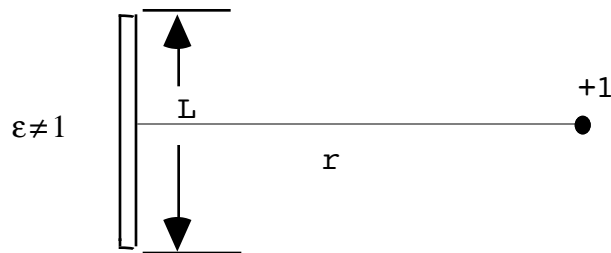
(b) By using an energy method explained in the text:

$$\Delta W = \frac{1}{2} \int d^3x d^3x' \rho(\vec{x}) [G_D(\vec{x}, \vec{x}') - G_D^0(\vec{x}, \vec{x}')] \rho(\vec{x}').$$

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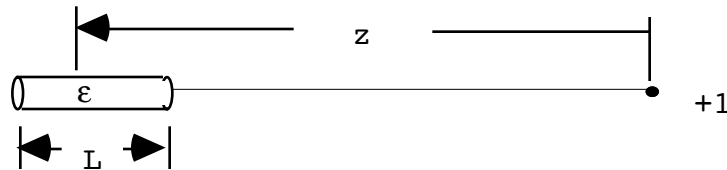
(c) By explicitly evaluating the force on the induced surface charge density. (Be careful which electric field you use, as there is a discontinuity in $\vec{E}$ at $z=0$.)

4.9.3 Find the approximate force (attraction, repulsion?) between a dielectric rod of length L and radius a ($L \gg a$) and a positive unit point charge located a distance $r \gg L, a$ from the rod. The point charge is located perpendicular to the rod's axis, on the rod's midpoint plane:



[Hint: The point charge's $\vec{E}$ field at the rod's position will be approximately uniform. Prob. 4.4.3 result may be useful.]

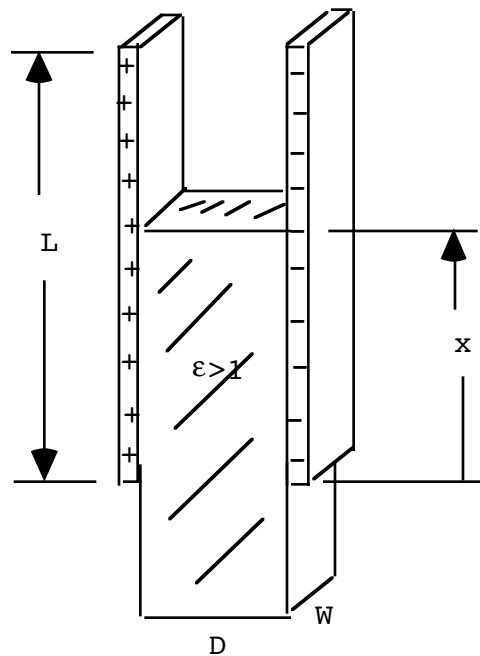
4.9.4 The middle of a long, thin cylinder of radius a and length L , with dielectric constant ϵ , is located a large distance z from a positive point charge ($z \gg L \gg a$). It is oriented with its lengthwise dimension pointed toward the charge, as shown.



Find the force of the charge on the dielectric. Are they attracted or repelled for $\epsilon > 1$?

4.9.5 (a) Two parallel conducting plates of a capacitor of length L and width W are separated by a distance D . The

region between the plates is filled to a distance x with a dielectric material with constant ϵ . If the plates are maintained at a constant potential V by connection to a battery, calculate the force, F_x , on the dielectric block. Neglect edge effects. Is the block pulled in or pushed out of the capacitor?



(b) The dielectric block has been withdrawn and a fixed charge $\pm Q$ has been placed on the plates. The magnitude of this charge is given by $\frac{V}{D} = \frac{4\pi Q}{LW}$ so that the potential in (a) is established when the dielectric block is not yet inserted. The block is reinserted a distance x so that it partially fills the space between the plates. Again neglecting edge effects, calculate the force, F_x , on the block.

4.9.6 Referring to prob. 4.6.1 above, show that the z -direction force on the dielectric surface may be written exactly as

$$F_z = 4 \left(\frac{1+\epsilon}{1-\epsilon} \right) \int_0^\infty dk k \frac{e^{2kd} (\sinh(kz'))^2}{\left(1 + \left(\frac{1+\epsilon}{1-\epsilon} \right) e^{2kd} \right)^2}.$$

4.9.7 (a) As a special case of prob. 4.9.6, show that this force may be evaluated as

$$F_z = \frac{1}{4} \left(\frac{1-\epsilon}{1+\epsilon} \right) \left(\frac{1}{(d+z')^2} + \frac{1}{(d-z')^2} - \frac{2}{d^2} \right),$$

for weak dielectrics, $\epsilon \approx 1$.

(b) Using an image method, confirm this result.

4.9.8 (a) As another special case of prob. 4.9.6, now show that the force (attractive) on one plate of a parallel plate capacitor dielectric (in the $\epsilon \rightarrow \infty$ limit of a dielectric) when a +1 charge is a distance z' from the other plate and the distance between the plates is D , is given by (the minus sign means attractive)

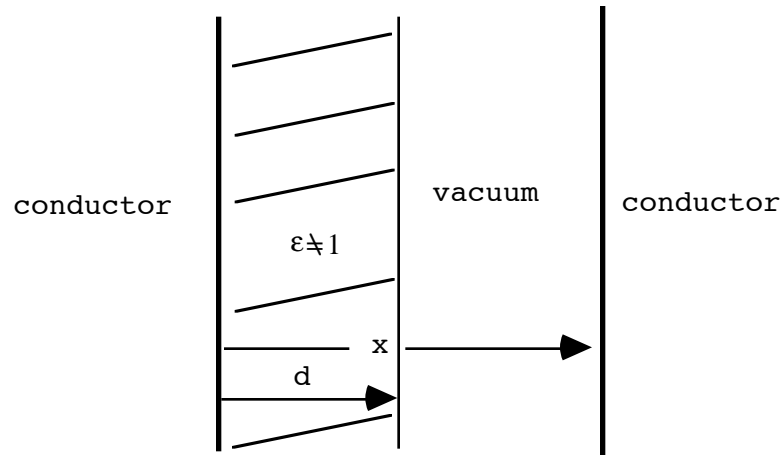
$$F_z = - \int_0^\infty dk \, k \left(\frac{\sinh kz'}{\sinh kD} \right)^2.$$

(b) Jackson in prob. 3.19 says that the charge density on the formerly dielectric surface is

$$\sigma(\rho) = - \frac{q}{2\pi} \int_0^\infty dk \, k \left(\frac{\sinh kz'}{\sinh kD} \right) J_0(k\rho).$$

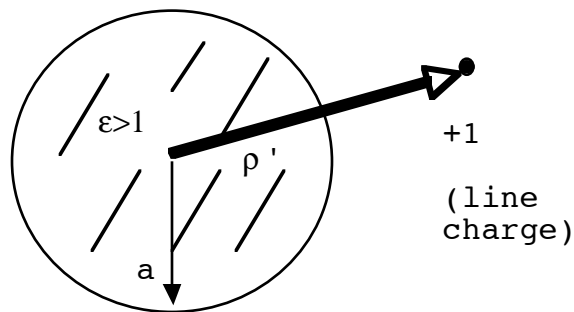
Show that the above force also follows from Jackson's expression for the surface charge density for $q=1$.

4.9.9 A capacitor, which consists of a dielectric layer ($\epsilon \neq 1$) of width d between conducting plates, is in the process of being constructed. The distance between the metallic plates is $x > d$ and the dielectric is attached to one of the plates. Find the force/area on one of the metallic surfaces if:



- (a) A battery establishes a constant potential difference, ΔV , between the plates. Is the force attractive or repulsive?
- (b) The battery is disconnected and the metallic plates have a constant free surface charge density, $\pm \sigma$. Again, is the force attractive or repulsive?

4.9.10 In Prob. 4.5.2, we examined the boundary and source conditions for the two dimensional Green function for the region outside of a cylinder of dielectric material, with dielectric constant $\epsilon > 1$.



Given the above, find the induced force per unit length between the cylindrical post and the line charge. Is it attractive or repulsive? [**+2 points** for correctly summing the resulting series.]

4.10.1 In the notes I showed for the dielectric quark confinement model that with

$$\begin{aligned}\vec{E} &= -\vec{\nabla}\Phi, \\ \vec{D} &= \epsilon\vec{E}, \\ \epsilon &= 2\alpha \ln\left(\frac{E^2}{K^2}\right) \quad (\alpha > 0),\end{aligned}$$

$$W = \frac{1}{4\pi} \int d^3x \left[\frac{1}{2} \vec{E} \cdot \vec{D} + \alpha E^2 \right],$$

under a first variation

$$\vec{E} \Rightarrow \vec{E} + \delta\vec{E},$$

we get

$$\delta W^{(1)} = \frac{1}{4\pi} \int d^3x \vec{E} \cdot \delta\vec{D}.$$

If we require that

$$\vec{\nabla} \cdot \delta\vec{D} = 0,$$

(equivalent to $\delta\rho = 0$ since $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$) argue that $\delta W^{(1)} = 0$.

(b) Consider a second variation of W . Keeping terms of second order in $\delta\vec{E}$, show that

$$\delta W^{(2)} > 0,$$

implying that the extremum found is a stable minimum.

4.11.1 A piece of linear dielectric material is brought slowly into a region where an initial electric field, $\vec{E}_0(\vec{x})$, has already been established. The change in energy of the system is

$$W - W_0 = - \frac{1}{2} \int_V d^3x \vec{P}(\vec{x}) \cdot \vec{E}_0(\vec{x}),$$

where $\vec{P}(\vec{x})$ is the polarization vector and the integration is only over the volume, V , of the introduced dielectric. Using the energy expression above and Eq.(4.141), find the force between a dielectric sphere of radius "a" and a charge, Q , located a distance, d , from the center of the sphere when $d \gg R$. [This is worked out three other ways in Ch.4!]

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